

Taylor Series and Getting the General Solutions for the Equations of Motion Using Poisson Bracket Relations

Ibtisam Frhan Al-Maaitah

Applied Physics Department, Faculty of Science, Tafila Technical University, Tafila, Jordan

Email address:

ibtrand@yahoo.com

To cite this article:

Ibtisam Frhan Al-Maaitah. Taylor Series and Getting the General Solutions for the Equations of Motion Using Poisson Bracket Relations. *World Journal of Applied Physics*. Vol. 6, No. 3, 2021, pp. 47-51. doi: 10.11648/j.wjap.20210603.12

Received: June 26, 2021; **Accepted:** July 28, 2021; **Published:** August 4, 2021

Abstract: In mathematics and classical mechanics, the Poisson bracket is an important binary operation in Hamiltonian mechanics, playing a central role in Hamilton's equations of motion, which govern the time evolution of a Hamiltonian dynamical system. The Poisson bracket also distinguishes a certain class of coordinate transformations, called canonical transformations, which map canonical coordinate systems into canonical coordinate systems. In this work we study some examples from the classical mechanics of particles and apply mathematical method for building the equation of motion. In the present paper Poisson Brackets and their properties are presented, by using Poisson brackets and their properties we calculate some brackets. We use the Poisson bracket with Hamiltonians to express the time dependence of a function $u(t)$, the main idea Taylor series is taken as the required solution for equation of motion using the properties of the Poisson Brackets, We have examined examples from the classical mechanics to illustrate the idea such as motion with a constant acceleration, simple harmonic oscillator, freely falling particle. The solutions are compatible with what is known in classical mechanics. The work is fundamental and sheds new light onto classical mechanics. Poisson brackets are a powerful and sophisticated tool in the Hamiltonian formalism of Classical Mechanics. They also happen to provide a direct link between classical and quantum mechanics.

Keywords: Poisson Brackets, JACOBI Identity, Taylor Series, Leibniz rule, Harmonic Oscillator, Hamilton Function, Generalized Coordinates, Generalized Momenta

1. Introduction

Poisson brackets are of great importance in physics: An important binary operation in Hamiltonian mechanics, play a central role in Hamilton's equations of motion, distinguishes a class of coordinate transformations (canonical transformations), very useful tool in quantum mechanics and field theory. [1, 2]. A technique that uses the Poisson bracket is supposed to allow us to derive all the differential equations of motion of a system from the just one piece of information, namely from the expression of the total energy of the system, i.e., from its Hamiltonian [3-5].

The Poisson brackets definition for the dynamics of a position-dependent mass particle was establishing, which is laid out in harmony with the classical mathematical portrait of analytical mechanics [6]. The Poisson bracket was used to find the integrals of motion. A numerical and analytical method was suggested to solve Navier-Stokes equations in

Helmholtz form [7]. In mathematics, the Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point. Taylor's series are named after Brook Taylor, who introduced them in 1715 [8-10].

The Poisson bracket appears often in classical mechanics and translates itself to quantum mechanics as the commutator. The total time derivative of a function u that depends on the canonical variables q and p , and can also depend on the time t [1, 2]:

$$\frac{du}{dt} = \sum \left(\frac{\partial u}{\partial q} \dot{q} + \frac{\partial u}{\partial p} \dot{p} \right) + \frac{\partial u}{\partial t} \quad (1)$$

$$\text{As we know that } \dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q}$$

Using the canonical equations of motion, the total time derivative becomes

$$\frac{du}{dt} = \sum \left(\frac{\partial u}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial H}{\partial q} \right) + \frac{\partial u}{\partial t} \quad (2)$$

which can be written using the following shorthand notation

$$\frac{du}{dt} = \{ u, H \} + \frac{\partial u}{\partial t} \quad (3)$$

where we define the Poisson bracket as

$$\{ u, H \} = \sum \left(\frac{\partial u}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial H}{\partial q} \right) \quad (4)$$

In the present paper Poisson Brackets and their properties are presented. The main idea Taylor series is taken as the required solution for equation of motion using the properties of the Poisson Brackets, We have discussed physical applications and treated some examples with Taylor series using Poisson brackets.

2. Poisson brackets

For any two functions on phase space, $f(q, p)$ and $g(q, p)$. The Poisson bracket associates to any such pair a third function, denoted $\{f, g\}$, which can be evaluated usually faster by the following formula: [1, 2]

$$\{f(q, p), g(q, p)\} = \sum \left(\frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \right) \quad (5)$$

Consider a particle (or system of particles) with generalized coordinates q and generalized momenta p . To motivate the idea of Poisson brackets, let us use Hamilton's equations, the time derivatives of coordinates and momenta may be expressed in terms of partial derivatives of the Hamiltonian [11, 12]

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}$$

H isn't explicitly dependent on time, then time does not appear explicitly on the RHS of Hamilton's equations. Then, we have

$$\dot{q}_i = \{ q_i, H \} \quad (6)$$

$$\dot{p}_i = \{ p_i, H \}$$

These are Hamilton's equations by using Poisson brackets; that is, the time derivative of the coordinates is the Poisson bracket for the coordinates and Hamilton function and the time derivative of the momenta is the Poisson bracket for the momenta and Hamilton function.

Properties of Poisson brackets

Let us begin by recording some fundamental properties of the Poisson bracket: Given two functions f and g that depend on phase space and time, their Poisson bracket $\{f, g\}$ is another function that depends on phase space and time. The following rules hold for any three functions f , g and h of phase space and time [13-15]:

(a) anti-symmetric: the Poisson bracket is anti-symmetric in the dynamical variables f

$$\{f, g\} = -\{g, f\} \quad (7)$$

In particular, the Poisson bracket of any observable with itself vanishes $\{f, f\} = 0$

(b) linearity:

$$\{\alpha f + \beta g, h\} = \alpha \{f, h\} + \beta \{g, h\} \quad (8)$$

Where α and β are constants.

(c) Leibniz rule: since the above formula for the Poisson bracket involves only first order derivatives of f , the Poisson bracket satisfies the Leibnitz/product rule of differential calculus.

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \quad (9)$$

which follows from the chain rule in differentiation

(d) Jacobi identity: More generally, Poisson's theorem is a consequence of the Jacobi identity.

For any three dynamical variables f , g and h , the following cyclic sum of 'double' Poisson brackets vanishes:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad (10)$$

Using anti-symmetry we could write the Jacobi identity also as

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \quad (11)$$

What we've seen above is that the Poisson bracket $\{, \}$ satisfies the same algebraic structure as matrix commutators $[,]$ and the differentiation operator. This is related to Heisenberg's and Schrodinger's viewpoints of quantum mechanics respectively.

(e) Also, if a function k is constant over phase space (but may depend on time), then $\{f, k\} = 0$ for any f .

(f) The fundamental Poisson brackets are between the basic dynamical variables, namely coordinates and momenta. The above formulae give for one degree of freedom

$$\begin{aligned} \{q_i, q_j\} &= 0, & \{p_i, p_j\} &= 0, \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \quad (12)$$

where δ_{ij} is the Kronecker delta.

The three properties of linearity, anti-symmetric and the Jacobi identity play such a fundamental role in many areas

of mathematics that they have been given a name: an algebraic structure involving a “product” that is bilinear, anti-symmetric and satisfies the Jacobi identity is called a Lie algebra.

Example:

By using Poisson brackets and their properties we can calculate the following brackets:

(a) $\{q, p\}$

$$\{q, p\} = q\{q, p\} + \{q, p\}q$$

$$\{q, p\} = 2q$$

(b) $\{q+p, q+p\}$

$$\{q+p, q+p\} = \{q, q+p\} + \{p, q+p\}$$

$$= \{q, q\} + \{q, p\} + \{p, q\} + \{p, p\}$$

$$= 0+1-1+0$$

$$= 0$$

(c) $\{qp, q^2\}$

$$\{qp, q^2\} = q\{p, q^2\} + \{q, q^2\}p$$

$$= q\{p, q\}q + q^2\{p, q\} + 0$$

$$= -q^2 - q^2$$

$$= -2q^2$$

3. Taylor Series and Getting the General Solutions for the Equations of Motion

We can also use the Poisson bracket with Hamiltonians to express the time dependence of a function $u(t)$, since H describes an infinitesimal translation in time. We first expand $u(t)$ around $t=0$ using the Taylor series [16], we have

$$u(t) = u_0 + t \left. \frac{du}{dt} \right|_0 + \frac{t^2}{2!} \left. \frac{d^2u}{dt^2} \right|_0 + \frac{t^3}{3!} \left. \frac{d^3u}{dt^3} \right|_0 + \dots \quad (13)$$

Now, express the time derivatives as Poisson brackets, and we find that:

$$\frac{du}{dt} = \{u_i, H_j\}$$

then

$$\frac{d^2u}{dt^2} = \frac{d}{dt} \{u_i, H_j\} = \{ \{u, H\}, H \}$$

and

$$\frac{d^3u}{dt^3} = \frac{d}{dt} \{ \{u, H\}, H \} = \{ \{ \{u, H\}, H \}, H \}$$

Substituting these three terms and similar ones into our Taylor’s expansion for $u(t)$ Eq. (13), we have

$$u(t) = u_0 + t \{u, H\}_0 + \frac{t^2}{2!} \{ \{u, H\}, H \}_0$$

$$+ \frac{t^3}{3!} \{ \{ \{u, H\}, H \}, H \}_0 + \dots \quad (14)$$

So, one can formally write down the time evolution of $u(t)$ as a series solution in terms of the Poisson brackets evaluated at $t = 0$

The Hamiltonian is the generator of the system’s motion in time [17].

4. Illustrative Examples

4.1. Motion with a Constant Acceleration

A particle of mass m is moving in a straight line with a constant acceleration, the Hamiltonian’s function is

$$H = \frac{p^2}{2m} - \max \quad (15)$$

By using the Eq. (14)

$$x(t) = x_0 + t \{x, H\}_0 + \frac{t^2}{2!} \{ \{x, H\}, H \}_0$$

$$+ \frac{t^3}{3!} \{ \{ \{x, H\}, H \}, H \}_0 + \dots \quad (16)$$

$$\{x, H\} = \left\{ x, \frac{p^2}{2m} - \max \right\} = \frac{p}{m}, \quad (17)$$

$$\{x, H\}_0 = \frac{p_0}{m}$$

$$\{ \{x, H\}, H \} = \left\{ \frac{p}{m}, \frac{p^2}{2m} - \max \right\} = a, \quad (18)$$

$$\{ \{x, H\}, H \}_0 = a$$

$$\{ \{ \{x, H\}, H \}, H \} = \{ a, H \} = 0 \quad (19)$$

The rest of the terms are equal to zero, we will get

$$x(t) = x_0 + \frac{p_0}{m}t + \frac{1}{2}at^2 + 0 + 0 + \dots$$

But,

$$\frac{p_0}{m} = v_0,$$

Hence;

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{20}$$

It is the known solution to this problem

4.2. Simple Harmonic Oscillator

Considering the Hamiltonian for a harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2 \tag{21}$$

The equation of motion

$$\dot{q} = \{ q, H \} \tag{22}$$

$$\dot{q} = \left\{ q, \frac{p^2}{2m} + \frac{1}{2} k q^2 \right\}$$

By using the properties for Poisson brackets :

$$\begin{aligned} \dot{q} &= \frac{1}{2m} \{ q, p^2 \} + \frac{1}{2} k \{ q, q^2 \} \\ &= \frac{1}{2m} 2p \{ q, p \} = \frac{p}{m} \end{aligned} \tag{23}$$

Also,

$$\begin{aligned} \dot{p}_i &= \{ p_i, H \} \\ &= \left\{ p, \frac{p^2}{2m} + \frac{1}{2} k q^2 \right\} \\ &= \frac{1}{2} k \{ p, q^2 \} = kq \{ p, q \} = -kq \end{aligned} \tag{24}$$

Then the displacement is

$$q(t) = q_0 \cos wt + \frac{p_0}{\sqrt{mk}} \sin wt \tag{25}$$

Where,

$$w = \sqrt{\frac{k}{m}}$$

4.3. A freely Falling Particle

A simple example in applying to a freely falling particle:
u=z

The familiar Hamiltonian for this system is

$$H = \frac{p^2}{2m} + mgz \tag{26}$$

We can solve for z(t) using Poisson brackets to demonstrate that this method will give a familiar result.

First, by using Taylor series, we note that

$$\begin{aligned} z(t) &= z(0) + t \{ z, H \}_0 + \frac{t^2}{2!} \{ \{ z, H \}, H \}_0 \\ &\quad + \frac{t^3}{3!} \{ \{ \{ z, H \}, H \}, H \}_0 + \dots \end{aligned} \tag{27}$$

Now, we evaluate the different terms:

$$\{ z, H \} = \frac{\partial z}{\partial z} \frac{\partial H}{\partial p} - \frac{\partial z}{\partial p} \frac{\partial H}{\partial z}$$

$$\{ z, H \} = \frac{p}{m} \tag{28}$$

$$\{ \{ z, H \}, H \} = \frac{\partial \{ z, H \}}{\partial z} \frac{\partial H}{\partial p} - \frac{\partial \{ z, H \}}{\partial p} \frac{\partial H}{\partial z}$$

$$\{ \{ z, H \}, H \} = -\frac{1}{m} \cdot mg$$

$$\{ \{ z, H \}, H \} = -g \tag{29}$$

At t=0, we have initial conditions:

$$z(0) = z_0,$$

$$p(0) = p_0$$

$$\{ z, H \}_0 = \frac{p_0}{m},$$

$$\{ \{ z, H \}, H \}_0 = -g$$

$$\{ \{ \{ z, H \}, H \}, H \}_0 = 0$$

Then inserting this into the Taylor expansion Eq. (27), we see that

$$z(t) = z_0 + \frac{p_0}{m} t - \frac{g t^2}{2} \tag{30}$$

5. Conclusion

In this work, we are presented the definition of Poisson brackets and their properties and used Taylor series to get the general solutions for the equations of motion using Poisson bracket relations. We are examined examples to illustrate the idea, such as motion with a constant acceleration, simple harmonic oscillator, freely falling particle. The solutions are compatible with what is known in classical mechanics.

References

- [1] H. Goldstein. *Classical Mechanics* (2nd edition). Addison-Wesley, Reading- Massachusetts (1980).
- [2] L. D. Landau; E. M. Lifshitz, *Course of Theoretical Physics, Vol. 1–Mechanics*. Franklin Book Company. (1972). ISBN978-0-08-016739-8.
- [3] Edward A. Desloge, *Classical Mechanics, Volume 1*, John Wiley and sons, (1982).
- [4] Atamp. Arya, *Introduction to classical Mechanics*, Allyn and Bacon, (1990).
- [5] Dare A. Wells, *Lagrangian Dynamics*, New York McGraw–Hill, (1967).
- [6] Casetta, L. Poisson brackets formulation for the dynamics of a position-dependent mass particle. *Acta Mech* 228, 4491–4496 (2017).
- [7] N. M. Bodunov and V. I. Khaliulin, *Journal of Physics* 1679 (2020) 032086.
- [8] S. G. Dani (2012). "Ancient Indian Mathematics–A Conspectus". *Resonance*. 17 (3): 236–246.
- [9] M. Abramowitz and I. A. Stegun, (Eds.). *Hand book of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, 9th printing. New York: Dover, p. 880, 1972.
- [10] G. Arfken, "Taylor's Expansion." 5.6 in *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, pp. 303-313, 1985.
- [11] E. N. Moore, *Theoretical Mechanics*, New York, John Wiley and sons, (1983).
- [12] R. a. Becker *Introduction to Theoretical Mechanics*, New York: McGraw-Hill, (1945).
- [13] Jerry B. Marion, *Classical Dynamics of Particles and Systems*, 2nd ed., Academic press. (1970).
- [14] Keth R. Symon, *Mechanics* 3rd ed. Addison–Wesley Publishing cob, (1980).
- [15] Grant R. Fowles and George L. Cassiday, *Analytical Mechanics* 5th ed. Saunders Golden Sunburst series, (1993).
- [16] E. F. Taylor, *Introductory Mechanics*, New York, John Wiley and sons, (1963).
- [17] Kittel, Knight, Ruderman, Helmholtz and Moyer, *Mechanics* (Berkeley Physics Course Vo. 11).