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A Classes of Variational Inequality Problems Involving Multivalued Mappings

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Abstract: The main objective of the Variational inequality problem is to study some functional analytic tools, projection method and fixed point theorems and then exploiting these to study the existence of solutions and convergence analysis of iterative algorithms developed for some classes of Variational inequality problem. The main objective of this paper is to study the existence of solutions of some classes of Variational inequalities using fixed point theorems for multivalued and using Banach contraction theorem we prove the existence of a unique solution of multi value Variational inequality problem.

Keywords: Fixed Points Theorems, Variational Inequality Problems, Strongly Lipschitz Operator

1. Introduction

Variational inequalities and complementarity problem play equally important roles in applied mathematics, physics, control theory and optimization, equilibrium theory of transportation and economics, mechanics, and engineering sciences. We study the existence and convergence of solutions of some classes of Variational inequalities using fixed point theorem for multivalued mappings we develop an iterative algorithm for prove the approximate solution converges to solution of multi value Variational inequality problem.

Definition (1-1): Let X be a metric space with metric d. A mapping $F: X \to X$ is called contraction mapping if:

$$d\big(F(a), F(b)\big) \leq \alpha\, d(a,b), for\ all\ a, b \in X$$

Definition (1-2): Let $a(.,.): H \times H \to R$ be a bilinear from, K a nonempty closed convex set of H.

Definition (1-2): If $H = \mathbb{R}^2$ and project any point $x = (x_1, x_2)$ on the x_1 axis then the projection $P_K(x) = (x_1, 0)$.

Theorem (1-3): Let H be a real Hilbert space, let $K \subset H$ be a nonempty closed convex set and let P_K due the projection mapping on K. Then P_K is non expansive, monotone, but not strictly monotone and strongly continuous.

Proof: Let us first note that the characterization of the projection $(y - x, y - \eta) \leq 0$,

$for\ all\ y \in K$ can be written

$$(P_K(x) - x, P_K(x) - y) for\ all\ y \in K \tag{1}$$

To show that P_K is monotone, let us fix $x_1 and\ x_2\ in\ H$ and write

$$(P_K(x_1) - x_1, P_K(x_1) - y) for\ all\ y \in K \tag{2}$$

$$(P_K(x_2) - x_2, P_K(x_2) - y) for\ all\ y \in K \tag{3}$$

Putting $y = P_K(x_2)$ in $(1.2)\ and\ y = P_K(x_1)\ in\ (1)$, we have

$$\big(P_K(x_1) - x_1 - P_K(x_2) + x_2, P_K(x_1) - P_K(x_2)\big) \leq 0 \tag{4}$$

And therefore

$$\|P_K(x_1) - P_K(x_2)\|^2 \leq \big(x_1 - x_2, P_K(x_1) - P_K(x_2)\big) for\ all\ x_1, x_2 \in H \tag{5}$$

Which in particular implies the monotonicity of $\mathrm{P_K}$, further if $\mathrm{K = H}$ then $\mathrm{P_K} = 1$, an identity mapping, and one has strict monotonicity, but in general for $\mathrm{K \neq H}$ then P_K is not injective and hence not strictly monotone, If $x \notin K$ then

$P_K(x) \neq 0$ but $P_K\big(P_K(x)\big) = P_K$.

To show that P_K is non expansive it is enough to apply the Schwartz inequality to (1.5) and obtain thus

$$\|P_K(x_1) - P_K(x_2)\|^2 \leq \|x_1 - x_2\| \, \|P_K(x_1) - P_K(x_2)\| \quad (6)$$

Or, dividing by $\|P_K(x_1) - P_K(x_2)\| = 0$ then $\|P_K(x_1) - P_K(x_2)\| \leq \|x_1 - x_2\|$. the strong continuity follows immediately form (5).

2. Preliminaries

Definition (2-1): Let H be a real Hilbert space and $\|.\|$ and $\langle .,. \rangle$ denote norm and inner product on H respectively. Given multivalued mappings $T, A: H \to 2^H$ is a power set of H, a nonlinear mapping $g: H \to H$ and a proper convex and lower.

Semi – continuous function $j: H \to R \cup \{+\infty\}$ with $Img \cap dom\, \partial j \neq \varphi$, where ∂j denote the subdifferential of j. We consider the following Variational inequality

problem (GVIP):

GVIP: Find $u \in H, x \in T(u), y \in A(u)$ such that $g(u) \cap dom\, \partial j \neq \varphi$ and

$$\langle x - y, v - g(u) \rangle \geq j\big(g(u)\big) - j(v), \forall v \in H.$$

Definition (2-2): let $g: H \to H$ is said to be:

(i) $v - stongly\, monotone$, if there exists a constant $v > 0$ such that

$$\langle g(u) - g(v), u - v \rangle \geq v\|u - v\|^2, \forall u, v \in H$$

(ii) $\rho - Lipschitz$ continuous, if there exists a constant $\rho > 0$ such that

$$\|g(u) - g(v)\| \leq \rho\, \|u - v\|, \forall u, v \in H$$

Definition (2-3): A multivalued mapping $T: H \to 2^H$ is said to be:

(i) $\alpha - stongly\, monotone$, if there exists a constant $\alpha > 0$ such that

$$\langle x - y, u - v \rangle \geq \alpha\|u - v\|^2, \forall u, v \in H, x \in T(u) and\ y \in T(v)$$

(ii) $\rho - \eta_1\ Lipschitz$ continuous, if there exists a constant $\rho > 0$ such that

$$\rho\,(T(u), T(v)) \leq \eta_1\, \|u - v\|, \forall u, v \in H$$

$$where\ \eta_1(A, B) = \sup\,\{\|a - b\|: a \in A, b \in B\}\ \ for\ any\ A, B \in 2^H$$

Definition (2-4): $Let\ (X, d)\ be\ a\ metric\ space\ and\ T: X \to X\ be\ amapping.$

A point x is said to be fixed point T if $x = x$.

Fixed Point Problem: Let T be a mapping defined on a metric space (X, d) into itself, find $x \in X$ such that $Tx = x$.

Definition (2-5): If F is multivalued mapping on X into itself. Then a point $x \in X$ is called a fixed point of F if $\in X$.

Lemma (2-6): A mutivalued mapping $F\ on\ X\ into\ Y$ is continuous at point x_0 if and only if $F(x_n) \to F(x_0)$ for all sequence $\{x_n\}\ in\ X\ with\ x_n \to x_0$

Proof: Suppose that $x_n \to x_0$. Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0 - x\| < \delta \implies \|u - v\| < \epsilon\ for\ all\ u \in F(x_0) and\ v \in F(x)$$

And there exists a positive integer N such that

$$\|x_0 - x_n\| < \delta\ for\ all\ n \geq N$$

Thus, if $n \geq N$ we have

$$\|u_n - u\| < \epsilon\ for\ all\ u_n \in F(x_n)\ and\ u \in F(x)$$

Therefore $F(x_n) \to F(x)$.

Conversely, suppose that $F(x_n) \to F(x), i.e\ if\ n \geq N$

$$\|u_n - u\| < \epsilon\ for\ all\ u_n \in F(x_n)\ and\ u \in F(x)$$

And $\|x_n - x\| < \delta$. Suppose that F is not continuous at x_0. Then there exists > 0, for each $\delta > 0$ there exists $x \in X$ such that

$$\|x_0 - x\| < \delta \implies \|u - v\| < \epsilon\ for\ all\ u \in F(x_0) and\ v \in F(x)$$

In particular, for each positive integer n there exists $x_n \in X$ such that

$$\|x_0 - x\| < \frac{1}{n}\ and\ \|u_n - u_0\| < \epsilon\ for\ all\ u_n \in F(x_n)\ and\ u_0 \in F(x_0)$$

Clearly, x_n converges to x but $F(x_n)$ does not converges to (x_0). which is a contradiction, this prove the lemma #

Theorem (2-7): $Let\ X\ be\ a\ Banach\ space$. If F is multivalued contraction mapping on X into itself Then F has a fixed point.

$proof$: $Let\ \mu < 1$ be contraction constant for F and let $x_0 \in X$. Choose $x_1 \in F(x_0)$. Since $F(x_0)\ and\ F(x_1)$ are subsets of X and $x_1 \in F(x_0)$ there is a $x_2 \in F(x_1)$ such that

$$\|x_1 - x_2\| \leq \mu \|x_0 - x_1\|.$$

Now, since $F(x_1)\ and\ F(x_2)$ are subsets of X and $x_2 \in F(x_1)$, there is a point $x_3 \in F(x_2)$ such that $\|x_2 - x_3\| \leq \mu \|x_1 - x_2\| \leq \mu^2 \|x_0 - x_1\|$

We product a sequence $\{x_n\}$ of points of X such that $x_{n+1} \in F(x_n)\ and$

$$\|x_n - x_{n+1}\| \leq \mu \|x_{n-1} - x_n\| \leq \mu^2 \|x_0 - x_1\|, for\ all\ n \geq 1$$

Now

$$\begin{aligned} \|x_n - x_{n+m}\| &\leq \|x_n - x_{n+1}\| \leq \|x_{n+1} - x_{n+2}\| + \cdots + \|x_{n+m-1} - x_{n+m}\| \\ &\leq \mu^n \|x_0 - x_1\| + \mu^{n+1} \|x_0 - x_1\| + .. + \mu^{n+m-1} \|x_0 - x_1\| \\ &= (\mu^n + \mu^{n+1} + \cdots + \mu^{n+m-1}) \|x_0 - x_1\| \\ &\leq \mu^n \left(\sum_{i=0}^{\infty} \mu^i\right) \|x_0 - x_1\|\ for\ all\ n, m \geq 1 \\ &= \frac{\mu^n}{1-\mu} \|x_0 - x_1\|\ for\ all\ n, m \geq 1 \end{aligned}$$

$If\ n, m \rightarrow \infty$, then the sequence $\{x_n\}$ is a Cauchy sequence. Since X is Banach space

Lemma (2-8) : $u \in H, x \in T(u), y \in A(u)$ is a solution of GVIP if and only if for some given $\alpha > 0$, the mapping $F: H \rightarrow 2^H$ defined by

$$F(u) = \bigcup_{x \in T(u)} \bigcup_{y \in A(u)} \{u - g(u) + p_\alpha^j(g(u) - \alpha(x - y))\}$$

Has a fixed point u, where $p_\alpha^j = (I + \alpha\partial j)^{-1}$ is called proximal mapping. I stands for the identity mapping on H.

Proof: Let $u \in H, x \in T(u), y \in A(u)$ be a solution of GVIP,$i.e. u \in H, x \in T(u), y \in A(u)$ satisfy $\langle x - y, v - g(u) \rangle \geq j\langle(g(u)\rangle - j(v), \forall v \in H$. By definition of ∂j we have $x - y, \in \langle g(u)\rangle$

$$\begin{aligned} &\Rightarrow g(u) - \alpha(x - y) \in g(u) + \alpha\partial j\langle(g(u)\rangle \\ &\Rightarrow g(u) - \alpha(x - y) \in (I + \alpha\partial j)\langle(g(u)\rangle \\ &\Rightarrow p_\alpha^j(g(u) - \alpha(x - y) = g(u) \\ &\Rightarrow u = u - (g(u) + p_\alpha^j((g(u) - \alpha(x - y)) \\ &\in \bigcup_{x \in T(u)} \bigcup_{y \in A(u)} \{u - g(u) + p_\alpha^j(g(u) - \alpha(x - y))\} \\ &\Rightarrow u \in F(u) \end{aligned}$$

Conversely, let u be a fixed point of $F, i.e. \exists x \in T(u), y \in A(u)$ such that

$$\begin{aligned} u &= u - g(u) + p_\alpha^j(g(u) - \alpha(x - y)) \\ &\Rightarrow g(u) = p_\alpha^j(g(u) - \alpha(x - y) \\ &\Rightarrow g(u) - \alpha(x - y) \in (I + \alpha\partial j)\langle(g(u)\rangle \\ &\Rightarrow g(u) - \alpha(x - y) \in g(u) + \alpha\partial j\langle(g(u)\rangle \\ &\Rightarrow -\alpha(x - y) \in \alpha\partial j\langle(g(u)\rangle \end{aligned}$$

$$\Rightarrow y - x \in \alpha\partial j\left(g(u)\right), since\ \alpha > 0$$

Hence, by definition of ∂j
$\langle x - y, v - g(u) \rangle \geq j\ g(u)) - j(v), \forall v \in H$. This complete the proof #

3. Main Result

Let $T, A: H \to 2^H$ be multivalued mappings then multivalued Variational inequality problem is to find $u \in K$ such that

$$(p, v - u) \geq (q, v - u), for\ all\ v \in K\ and\ p \in T(u), q \in A(u) \tag{7}$$

Let $T\ and\ A$ are nonlinear mappings on H into H, the single value Variational inequality problem is to find $u \in K$ such that :

$$\langle T(u), v - u\rangle \geq A(u), v - u, for\ all\ v \in K \tag{8}$$

Lemma (3-1): $(u, p, q)\ with\ u \in H, p \in T(u), q \in A(u)$, is a solution of multivalued Variational inequality problem
(3-1) if and only if $u \in H$ is a fixed point of mapping $F: H \to 2^H$ defined as

$$F(u) = \bigcup_{\substack{p\in T(u)\\ q\in A(u)}} (P_K(v - \xi(p - q)))$$

For some positive ξ.
Proof :
Suppose (u, p, q) satisfies if it satisfies $(p - q, v - u) \geq 0, for\ all\ v \in K$
Or $(u - (u - \xi(p - q, v - u)v - u) \geq 0, for\ all\ v \in K$
If and only, the Theorem (2-3) (u, p, q) satisfies

$$u = P_K(v - \xi(p - q))$$

Or $u \in F(u) = \bigcup_{\substack{p\in T(u)\\ q\in A(u)}}(P_K(v - \xi(p - q)))$
We prove the existence of a unique solution of multivalued Variational inequality problem (3-1)
Theorem (3-2): Let $T: H \to 2^H$ be a $\eta_1 -$ Lipschitz continuous and λ_1-strongly monotone multivalued mapping and let $A: H \to 2^H$ be ρ_1 −Lipschitz continuous multivalued mapping. then multivalued Variational inequality problem (3-1) has a solution.
Proof: By Lemma (3-1), it is enough to prove that multivalued mapping F is contraction mapping.
Let $w_1 \in F(u_1)\ and\ w_2 \in F(u_2)$, we have

$$w_1 = P_K\left(u_1 - \xi(p_1 - q_1)\right) for\ p_1 \in T(u_1)\ and\ q_1 \in A(v_1)$$

$$w_2 = P_K\left(u_2 - \xi(p_2 - q_{21})\right) for\ p_2 \in T(u_2)\ and\ q_2 \in A(v_2)$$

Now

$$\begin{aligned}\|w_1 - w_2\| &= \left\| P_K\left(u_1 - \xi(p_1 - q_1)\right) - P_K\left(u_2 - \xi(p_2 - q_2)\right) \right\| \\ &\leq \left\|\left(u_1 - \xi(p_1 - q_1)\right) - \left(u_2 + \xi(p_2 - q_2)\right)\right\| \\ &\leq \|u_1 - u_2 - \xi(p_1 - p_2)\| + \xi\|q_1 - q_2\| \\ &\leq \|u_1 - u_2 - \xi(p_1 - p_2)\| + \xi\ \rho_1\|q_1 - q_2\|\end{aligned}$$

By λ_1-strongly monotonicity and η_1 −Lipschitz continuous of T, we have

$$\begin{aligned}\|u_1 - u_2 - \xi(p_1 - p_2)\|^2 &\leq \|u_1 - u_2\|^2 - 2\xi(p_1 - p_2, u_1 - u_2) + \xi^2\|p_1 - p_2\|^2 \\ &\leq \|u_1 - u_2\|^2 - 2\xi\lambda_1\|u_1 - u_2\|^2 + \xi^2\eta_1^2\|u_1 - u_2\|^2 \\ &\leq (1 - 2\xi\lambda_1 + \xi^2\eta_1^2)\ \|u_1 - u_2\|^2\end{aligned}$$

Therefore:

$$\begin{aligned}\|w_1 - w_2\| &\leq \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2}\ \|u_1 - u_2\| + \xi\ \rho_1\|u_1 - u_2\| \\ &= \xi\ \rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2}\ \|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|\end{aligned}$$

Where $\theta = \xi\ \rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2} < 1$

$$\xi < \frac{2(\lambda_1 - \rho_1)}{\eta_1^2 - \rho_1^2},\ \rho_1 < \lambda_1\ and\ \rho_1 < \lambda_1$$

Hence F is contraction multivalued mapping. By Theorem (3-2), F has a fixed point, say $u, i.e, u \in F(u)$ then

$$u = P_K\big(u - \xi(p - q)\big)\ and\ p \in T(u), q \in A(u)$$

This completes the proof #

Iterative Algorithm (3-3):

For any given $u_0 \in K$, compute u_{n+1} defined as

$$u_{n+1} = P_K\big(u_n - \xi(p_n - q_n)\big)\ (9)$$

$p_{n+1} \in T(u_{n+1})\ and\ q_{n+1} \in A(u_{n+1})\ for\ some\ constant\ \xi$

Theorem (3-4): Let $T: H \to 2^H$ be a $\eta_1 -$ Lipschitz continuous and λ_1 -strongly monotonicity multivalued mapping and let $A: H \to 2^H$ be $\rho_1 -$ Lipschitz continuous multivalued mapping. If $(u_{n+1}, p_{n+1}, q_{n+1})\ and\ (u, p, q)$ are solution of (7) and (8) respectively then u_{n+1} converges strongly to $u\ in\ H, p_{n+1}$ converges strongly to $p\ in\ H\ and\ q_{n+1}$ converges strongly to $q\ in\ H\ for$

$$\xi < \frac{2(\lambda_1 - \rho_1)}{\rho_1^2 - \eta_1^2},\ \rho_1 > \lambda_1\ and\ \rho_1 > \lambda_1$$

Proof: By Lemma (3-1) and Iterative Algorithm (3-4), we have

$$u_{n+1} - u = P_K\big(u_n - \xi(p_n - q_n)\big) - P_K\big(u - \xi(p - q)\big)$$

Therefore,

$$\|u_{n+1} - u\| = \big\|P_K\big(u_n - \xi(p_n - q_n)\big) - P_K\big(u - \xi(p - q)\big)\big\|$$

$$\leq \left(\xi\, \rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2}\ \|u_1 - u_2\|\right)$$

$$= \theta\|u_1 - u_2\|$$

By theorem

Where $\theta = \left(\xi\,\rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2}\right) < 1$

For $\xi < \frac{2(\lambda_1 - \rho_1)}{\eta_1^2 - \rho_1^2},\ \rho_1 < \lambda_1\ and\ \rho_1 < \lambda_1$

Then by iteration, we have

$$\|u_{n+1} - u\| \leq \theta^n\|u_1 - u_2\|$$

Since $\theta < 1$. we have u_{n+1} converges to u strongly in, we have $p_{n+1} \to p$ strongly in H and $q_{n+1} \to q$ strongly in H. This completes the proof #

4. Conclusions

We study the existence of solutions of some classes of Variational inequalities using fixed point theorems for multivalued and using Banach contraction theorem we prove the existence of a unique solution of multi value Variational inequality problem discussed in the article research.

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Statement of the Problem

Find solutions of some classes of Variational inequalities.

Research Objectives

Study the existence of solutions of some classes of Variational inequalities using fixed point theorems for multivalued and using Banach contraction theorem

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