Science Journal of Applied Mathematics and Statistics
2018; 6(1): 16-27
http://www.sciencepublishinggroup.com/j/sjams
doi: 10.11648/j.sjams.20180601.13
ISSN: 2376-9491 (Print); ISSN: 2376-9513 (Online)

Pricing Options on Ghanaian Stocks Using Black-Scholes Model

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To cite this article:
Osei Antwi, Francis Tabi Oduro. Pricing Options on Ghanaian Stocks Using Black-Scholes Model. *Science Journal of Applied Mathematics and Statistics.* Vol. 6, No. 1, 2018, pp. 16-27. doi: 10.11648/j.sjams.20180601.13

Received: September 13, 2017; **Accepted**: September 30, 2017; **Published**: February 1, 2018

Abstract: We present a succinct new approach to derive the Black-Scholes partial differential equation and subsequently the Black-Scholes formula. We proceed to use the formula to price options using stocks listed on Ghana stock exchange as underlying assets. From one year historical stock prices we obtain volatilities of the listed stocks which are subsequently used to compute prices of three month European call option. The results indicate that it is possible to use the Black Scholes formula to price options on the stocks listed on exchange. However, it was realised that most call option prices tend to zero either due to very low volatilities or very low stock prices. On the other hand put options were found to give positive prices even for stocks with very low volatilities or low stock prices.

Keywords: Option Price, Volatility, Stochastic Process, Brownian Motion, Geometric Brownian Motion, Black−Scholes Formula

1. Introduction

An option is a contract between two parties in which the option buyer or holder purchases the right to buy or sell an underlying asset at a fixed time. Options are usually traded on stock exchanges and Over the Counter markets. The two types of options are *call* options and *put* options. An option gives the holder the right to buy or sell an underlying asset but he is in no way obliged to exercise this right. This is the main feature that distinguishes an option from other instruments such as futures and contracts. Currently, options are not traded on Ghana Stock Exchange (GSE) but it is a very lucrative business so much so that in 1972 the Chicago Stock Exchange was purposely opened in the United States of America to trade solely in options. The introduction of options trading on the GSE will greatly enhance the financial sector and attract hedgers with huge foreign investments. In addition, businesses, government institutions and other establishments can reduce the inherent market and credit risk in contracts and other market variables through hedging in options. Moreover, it will also create avenues for job opportunities as financial concerns set up hedge funds and brokerages to trade options and related derivatives.

One basic reason why options are currently not traded on the Ghanaian market is that the mathematical models required to price options have not been rigorously examined among researchers in Ghana. The aim of this paper is to begin the mathematical debate on the possibility of using various models to price options on Ghanaian assets. In view of this, we believe that there is no better starting point than the Black Scholes model. Specific objectives of this paper are to

- explain and simplify the mathematical models required to evaluate an option price
- use the model to price options with stocks listed on the GSE as the underlying assets
- test the model's behaviour when the underlying asset exhibit very low volatility or low stock price which is synonymous with the GSE.

In using the Black−Scholes model, we must emphasize here that although one may point to the inadequacies of the

formula as a pricing model we still wish to use it because we believe this paper will serve as a platform to ignite interest in researchers and financial engineers in the country to develop more robust models in pricing options. It is our hope that this paper would serve as a foundation upon which other option strategies and derivative pricing methods could develop. In addition, it is important to realise that despite its limitations the Black – Scholes model still remain the benchmark of option valuation and it is the standard to which all other pricing models are compared.

The theory of option pricing began in the 1900s when Louis Bachelier (1900) provided a valuation for stock options based on the assumption that stock prices follow a Brownian motion. Kendall (1953), Roberts (1959), Osborne (1959), Sprenkle (1961) all conducted studies into stock price behaviour and concluded that the stock price processes follow the lognormal distribution instead of a normal distribution assumed by Bachelier. The lognormal assumption ruled out the possibility of negative stock prices. Boness (1964) improved the option pricing model by accounting for the time value of money. However, it was Samuelson (1965) who proposed the Geometric Brownian Motion (GBM) as the model for the underlying stock in pricing options. Samuelson option pricing model was however not very popular as it required one to compute individual risk. The lack of certainty about a measure of an individual's risk characteristics meant that investors and sellers could not agree on a single option price. Despite these uncertainties, most financial engineers and economists have accepted the GBM as a model for the underlying stock price because it is everywhere positive as against Brownian motion which can give negative stock prices. Thus, GBM is now the most widely accepted formula for modelling stock price behaviour. Despite these early developments in option price modeling it was not until 1973 when Black and Scholes published a seminal paper in which they obtained a closed-form formula to calculate European calls that option trading took off in earnest. The introduction of the Black–Scholes formula is often regarded as the apogee of the option pricing theory and its introduction was so illuminating in structure and function that it created inflation in option trading and marked the beginning of a rapid expansion in derivatives trading in both European and American as well as Asian markets. The key idea underlying the Black–Scholes model was to set up a portfolio of one risky asset (stock) and one riskless asset (bond) and to buy and sell these assets by constantly adjusting the proportions of stocks and bonds in the portfolio so as to completely eliminate all the risk in the portfolio. Merton (1973), examined the Black and Scholes formula and provided an alternative derivation by relaxing some of the assumptions in the model. Merton's model was more functional and also provided several extensions of the Black-Scholes model including introducing dividend payments on the underlying asset. As a result, the Black-Scholes model is often referred to as the Black – Scholes – Merton model. The Black – Scholes – Merton formula was, and still popular because it provided a straightforward method that requires one to compute only the volatility of the stock in other to obtain the option price. In addition, the equation is independent of the investors risk appetites and as such individual risk measures cannot affect the solution as was the case in Samuelson's model. Cox, et al (1979) presented a discrete-time option pricing model known as the Binomial model whose limiting form is the Black–Scholes formula. Since the introduction of Black–Scholes–Merton and Cox–Ross–Rubinstein models several other models have emerged to price options but they are all either variants or improvements on these two fundamental models. Recent studies in option pricing have however focused primarily on novel computational applications and efficiency of the models. Monte Carlo simulation for instance has gained prominence and has widely been employed as an effective simulation technique. Bally, et al (2005), Egloff (2005), Moreno et. al. (2003), Dagpunar (2007) all examined the effectiveness of the Monte Carlo technique in options pricing. Mehrdoust et. al. (2017) examined the Monte Carlo option pricing under the constant elasticity of variance model.

2. Methodology

The basic idea of Black – Scholes equation was to construct a portfolio from stocks and bonds that yields the same return as a portfolio consisting only of an option. In this hedged portfolio the risky stock is modeled as a stochastic process and the riskless bond is modeled as deterministic process.

2.1. The Black–Scholes Partial Differential Equation

We begin by looking at the arguments leading to the developments of the Black – Scholes partial differential equation. The method of solution to the partial differential equation leads to the Black–Scholes formula. The Black-Scholes model prices options using stocks as the underlying assets. The diffusion of the stock price process is however captured as a Geometric Brownian Motion and as such we begin by examining GBM.

2.1.1. Geometric Brownian Motion

In a risky stock, the stock price $S(t)$ is assumed to follow the lognormal process and is modelled by the Geometric Brownian Motion (GBM) as

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1}$$

where μ is the return on the stock, σ is the standard deviation of μ or simply the stock's volatility and $W(t)$ is the standard Brownian motion or the Wiener process with mean 0 and standard deviation t. To determine the solution to the Geometric Brownian Motion, let

$$f(x) = InS(t)$$

$$f'(x) = \frac{1}{S(t)}$$

$$f''(x) = -\frac{1}{S^2(t)}$$

Since $S(t)$ is a stochastic process it follows that $f(x)$ is an Ito process and so by Ito formula

$$d\big(fS(t)\big) = f'dS(t) + \frac{1}{2}f''\sigma^2 dt$$

Hence

$$d\big(InS(t)\big) = \frac{1}{S(t)}dS(t) + \frac{1}{2}\Big(-\frac{1}{S^2(t)}\Big)\sigma^2 S^2(t)dt$$

$$\frac{1}{S(t)}\big(\mu S(t)d(t) + \sigma S(t)dW(t)\big) - \frac{1}{2}\sigma^2 dt$$

$$dInS(t) = \mu - \frac{1}{2}\sigma^2 dt + \sigma dW(t)$$

And dropping arguments

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)} \quad (2)$$

2.1.2. Derivation of the Black−Scholes Partial Differential Equation

The key mathematical theory underlying the Black-Scholes equation is the Itô's lemma.

Itô's Lemma

Assume that $X(t)$ is a stochastic process with the stochastic differential

$$dX(t) = \mu dt + \sigma dW(t)$$

where μ and σ are adapted processes. Let $f(X(t), t)$ be an ↑ → Ito process and a twice differentiable function, then $f(X(t), t)$ has the stochastic differential

$$df(X(t), t) = \Big(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X(t)}\partial f + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X(t)^2}\Big)\,dt + \sigma\frac{\partial f}{\partial X(t)}\,dW(t) \quad (3)$$

The derivation of the Black Scholes model can be summarized in three key arguments:

- Ito formula application to the value of a replicating portfolio of the option
- The hedging argument to create a riskless portfolio
- The no arbitrage arguments of a risk free return of the portfolio

Ito Formula Application to the Value of a Replicating Portfolio

Consider an asset whose price process $S(t)$ follows the Geometric Brownian Motion such that

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad (4)$$

Equation (4) is an Ito process with mean $\mu S(t)$ and $Var(S(t)) = [\sigma S(t)]^2$.

Consider a contingent claim on $S(t)$ whose value $V(S(t), t)$ depends on the $S(t)$ and t. By Ito lemma of Equation (3), the change in $V(S(t), t)$ is given by Equation (5). →

$$dV(S(t), t) = \Big(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t}\Big)dt + \Big(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\Big)dW(t) \quad (5)$$

Thus the stochastic process followed by V$(S(t), t)$ in Equation 5 is also an Ito process with drift

$$\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t}\right)$$

and variance

$$\left(\sigma S\frac{\partial V(S(t),t)}{\partial S(t)}\right)^2$$

The Hedging Argument to Create a Riskless Portfolio

Now construct a portfolio in which we buy 1 option ↑ →with value $V(S(t), t)$ and an unknown amount of stocks. The question here is how much of the stocks should be purchased in order to create a riskless or hedged portfolio. Let this amount be Δ stocks. The portfolio now consist of an option and stocks and has a value given by $\pi = V(S(t), t) - \Delta S(t)$. In a small time step the change in the portfolio's value is given by

$$d\pi = dV(S(t), t) - \Delta dS(t) \quad (6)$$

Substitute Equations (4) and Equation (5) into Equation (6) we obtain

$$d\pi = \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t}\right)dt + \left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)dW(t) + \Delta[\mu S(t)dt + \sigma S(t)dW(t)]$$

Grouping terms with dt and $dB(t)$ we have

$$d\pi = \Big(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t} - \mu\Delta S(t)\Big)dt + \Big(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)} - \sigma\Delta S(t)\Big)\,dW(t) \quad (7)$$

We also realise that the stochastic process for the hedged portfolio is an Ito process with drift parameter

$$\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t} - \mu\Delta S(t)\right)$$

and variance

$$\left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)} - \sigma\Delta S(t)\right)^2$$

Equation (7) consist of two parts: a deterministic part given by

$$\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu\Delta S(t)\right)dt$$

and a stochastic part given by

$$\left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}-\sigma\Delta S(t)\right)dW(t) \quad (8)$$

To make the portfolio completely riskless the stochastic part i.e. Equation (8) must vanish. In otherwords we should have

$$\left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}-\sigma\Delta S(t)\right)\,dW(t)=0 \quad (9)$$

Equation (9) yields

$$\Delta=\left(\frac{\partial V(S(t),t)}{\partial S(t)}\right)$$

If Equation (9) holds, then Equation (7) becomes

$$d\pi=\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu\Delta S(t)\right)dt \quad (10)$$

Replace Δ in Equation (10) by $\frac{\partial V(S(t),t)}{\partial S(t)}$ then we have

$$d\pi=\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)dt \quad (11)$$

It follows that to completely hedge the portfolio we must purchase $\frac{\partial V(S(t),t)}{\partial S(t)}$ of the underlying asset. It means that to have a riskless portfolio we must purchase an amount of stocks that is equal to the ratio of how much the option value changes relative to the change in value of the stock. However, this situation is only valid in a small time interval and so we must continuously change the amount of stocks purchased to rebalance $\left(\frac{\partial V(S(t),t)}{\partial S(t)}\right)$.

The no Arbitrage Arguments and a Risk Free Return of the Portfolio

The change in the portfolio's value is $d\pi$. So what is the return of this riskless portfolio in a small time step dt. Black and Scholes suggested that the return must be the risk free rate r otherwise there will be arbitrage opportunities. If this is the case then owning π amount of the portfolio would provide a return of $r\pi dt$ in a small time interval dt. Consequently,

$$d\pi=r\pi dt$$

Replacing $d\pi$ by $r\pi dt$ in Equation 11 we have

$$r\pi dt=\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)dt$$

$$r\pi=\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)$$

But $\pi=(V(S(t),t)-\Delta S(t))=\left(V(S(t),t)-S(t)\frac{\partial V}{\partial S(t)}\right)$ and so

$$r\left(V(S(t),t)-S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)=\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)$$

$$rV(S(t),t)-rS(t)\frac{\partial V(S(t),t)}{\partial S(t)}=\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}$$

$$\frac{\partial V(S(t),t)}{\partial t}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+rS(t)\frac{\partial V(S(t),t)}{\partial S(t)}-rV(S(t),t)=0 \quad (12)$$

At maturity, the price of the option $C(S(t),t)$ is equal to the value of the hedged portfolio and so $C(S(t),t)=V(S(t),t)$. Hence Equation (12) is rewritten as

$$\frac{\partial C(S(t),t)}{\partial t}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 C(S(t),t)}{\partial S(t)^2}+rS(t)\frac{\partial C(S(t),t)}{\partial S(t)}-rC(S(t),t)=0 \quad (13)$$

Equation is the Black Scholes partial differential equation. For a European call option the boundary conditions are

$$C(0,t)=0,\ C(S(T),T)=max(S(T)-K,0),\ t\geq 0$$

2.2. The Black−Scholes Formula

Theorem 1

Let $f(S(t),t)$ be a differentiable function satisfying the partial differential equation

$$\frac{\partial f(S(t),t)}{\partial t} + \mu(X(t),t)\frac{\partial f(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma(X(t),t)^2\frac{\partial^2 f(S(t),t)}{\partial S(t)^2} - r(X(t),t)f(S(t),t) = 0$$

with boundary condition $f(X(T),T) = g(x)$. Then $C(S(t),t)$ is the solution and

$$C(S(t),t) = \mathbb{E}_{\mathbb{Q}}\left(\left(e^{-\int_t^T r(X(u),u)du}\right)g(S(T))|\mathcal{F}_t\right)$$

Theorem 1 asserts that if $f(S(t),t)$ satisfies the Black-Scholes partial differential equation then $C(S(t),t)$ can be represented as an expectation. It follows that the option price can be considered as the discounted value of the expected option payoff under the martingale measure $\mathbb{Q}$ such that

$$C(S(t),T) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[max(S(T)-K,0)] \quad (14)$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation taken under the equivalent martingale measure $\mathbb{Q}$, $S(T)$ is the terminal stock price given by

$$S(T) = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W(T)} \quad (15)$$

The price of the expected payoff of the option is now given by

$$C(S(t),t) = e^{-r(T-t)}\int_0^{+\infty} f(x)max(x-K,0)\,dx \quad (16)$$

Where $f(x)$ is the density of the lognormal random variable X given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}exp\left(-\frac{(\mu-Inx)^2}{2\sigma^2}\right)$$

Where $\mu = \mathbb{E}(InS(T))$ = mean stock price and $\sigma^2 = Var(InS(T))$ the variance of the return. Now if $S(T) < K$, the option will not be exercised and so $max(S(T)-K,0)$ will be 0. We are therefore interested in the price distribution when $S(T) > K$. So we can write

$$C(S(t),t) = e^{-r(T-t)}\int_k^{+\infty} f(x)(x-K)\,dx$$

$$C(S(t),t) = e^{-r(T-t)}\int_k^{+\infty} xf(x)dx - Ke^{-r(T-t)})\int_k^{+\infty} f(x)dx)\,dx \quad (17)$$

Now the last integral $\int_k^{+\infty} f(x)dx)$ is the probability of the event that $S(T) > K$. So the last integral is equivalent to the statement $P(S(T)) > K)$.

Now $S(t) = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W(T)}$ and so

$$P(S(T)) > K) = P\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W(T)} > K\right)$$

$$P(S(T)) > K) = P\left(e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W(T)} > \frac{K}{S_0}\right)$$

$$= P\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma W(T) > In\left(\frac{K}{S_0}\right)\right)$$

$$= P\left(\sigma W(T) > In\left(\frac{K}{S_0}\right) - \left(r-\frac{\sigma^2}{2}\right)T\right)$$

$$= P\left(W(T) > \frac{In\left(\frac{K}{S_0}\right) - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma}\right)$$

Dividing by $\sqrt{T}$ we have

$$P(S(T)) > K) = P\left(\frac{W(T)}{\sqrt{T}} > \frac{In\left(\frac{K}{S_0}\right) - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= 1 - N\left(\frac{In\left(\frac{K}{S_0}\right) - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

But $1 - N(x) = N(-x)$, hence

$$1 - N\left(\frac{In\left(\frac{K}{S_0}\right) - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = N\left((-)\frac{In\left(\frac{K}{S_0}\right) - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= N\left(\frac{ln\left(\frac{S_0}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Let $d_2 = \left(\frac{In\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ then we have

$$P(S(T)) > K) = N(d_2) \quad (18)$$

We now compute the first integral

$$\int_k^{+\infty} xf(x)dx$$

Let I $= \int_k^{+\infty} xf(x)dx$

But $f(x) = \frac{1}{x\sigma\sqrt{2\pi}}exp\left(-\frac{(\mu-Inx)^2}{2\sigma^2}\right)$

Hence

$$\text{I} = \int_k^{+\infty}\left(x\frac{1}{x\sigma\sqrt{2\pi}}exp\left(-\frac{(\mu-Inx)^2}{2\sigma^2}\right)\right)dx$$

$$\text{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_k^{+\infty} exp\left(-\frac{(\mu-Inx)^2}{2\sigma^2}\right)dx$$

Now the first natural change of variables is $Inx = s$, $x = e^s$, and $dx = e^s ds$ and this gives

$$\text{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_k^{+\infty} exp\left(-\frac{(\mu-s)^2}{2\sigma^2}\right)e^s ds$$

$$\mathrm{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_{k}^{+\infty} exp\left(-\frac{(\mu - s)^2}{2\sigma^2} + s\right) ds$$

Now completing the square we have

$$-\frac{(\mu - s)^2}{2\sigma^2} + s = \frac{(s - (\mu + \sigma^2)^2}{2\sigma^2} + \frac{\sigma^2}{2} + \mu$$

Hence

$$\mathrm{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_{Ink}^{+\infty} exp\left[-\frac{(s - (\mu + \sigma^2)^2}{2\sigma^2} + \frac{\sigma^2}{2} + \mu\right] ds$$

$$\mathrm{I} = exp\left(\frac{\sigma^2}{2} + \mu\right)\frac{1}{\sigma\sqrt{2\pi}}\int_{Ink}^{+\infty} exp\left[-\frac{(s - (\mu + \sigma^2)^2}{2\sigma^2} + \right] ds$$

The expression under the integrand is the density function of a normal variable with mean $\mu = InS_0 + \left(r - \frac{\sigma^2}{2}\right)T$ and variance $\sigma^2 T$.

Now $\mathrm{I} = exp\left(\frac{\sigma^2}{2} + \mu\right) \cdot 1 - N(InK;\ \mu + \sigma^2,\ \sigma^2)$

and $S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W(T)}$

Hence we replace σ^2 in equation (15) by $\sigma^2 T$ and μ by $InS_0 + \left(r - \frac{\sigma^2}{2}\right)T$. We obtain

$$\mathrm{I} = exp\left(\frac{\sigma^2 T}{2} + InS_0 + \left(r - \frac{\sigma^2}{2}\right)T\right) \cdot 1 - N(InK;\ InS_0 + \left(r + \frac{\sigma^2}{2}\right)T, \sigma^2 T)$$

Now $exp\left(\frac{\sigma^2 T}{2} + InS_0 + rT - \frac{\sigma^2 T}{2}\right) = exp(rT + InS_0) = S_0 exp(rT)$

$$\mathrm{I} = S_0 e^{rT} \cdot 1 - N(x)$$

$$1 - N(x) = 1 - N(InK;\ InS_0 + \left(r + \frac{\sigma^2}{2}\right)T,\ \sigma^2 T)$$

$$= 1 - N\left(\frac{InK - InS_0 - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

and so

$$\mathrm{I} = S_0 e^{rT} \cdot \left[1 - N\left(\frac{InK - InS_0 - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right]$$

$$= S_0 e^{rT} N(-)\left(\frac{InK - InS_0 - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\mathrm{I} = S_0 e^{rT} N\left(\frac{-InK + InS_0 + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= S_0 e^{rT} N\left(\frac{In\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Let $d_1\left(\frac{In\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

Then

$$\mathrm{I} = S_0 e^{rT} N(d_1)$$

Now from Equation (13) we can write

$$C(S(t), t) = e^{-r(T-t)}\mathrm{I} - Ke^{-r(T-t)}(P(S(T)) > K)) \quad (19)$$

Replace T by $T - t$ in Equation (17) and substitute Equations (14) and (17) into Equation (18) we have

$$C(S(t), t) = e^{-r(T-t)}\mathrm{I} - Ke^{-r(T-t)}P(S(T)) > K)$$
$$= e^{-r(T-t)}S_0 e^{r(T-t)}N(d_1)$$
$$- Ke^{-r(T-t)}N(d_2)$$

$$C(S(t), t) = S_0 N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Replacing S_0 by $S(t)$ we obtain the Black − Scholes formula for the price of an European call option as

$$C(S(t), t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \quad (20)$$

where

$$d_1 = \frac{In\left(S_0/K\right) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \quad (21)$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{In\left(S_0/K\right) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \quad (22)$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

$N(x)$ is the cumulative distribution function of a standard normal variable.

In order to arrive at the formula Black and Scholes made the following assumptions on the stock price and the market.

- There are no dividends payment on the stock during the option's life
- The model assumes European-style
- Markets are assumed to be efficient
- There are no transaction costs in buying or selling the asset or the option, no barriers to trading and no taxes.
- Interest rates remain constant and equal to the risk-free rate
- Returns on the underlying stock are lognormally distributed
- The price of the underlying asset is divisible so that we can trade any fractional share of assets

2.3. Interest Rate and Volatility Models of a Stock Price

Return processes are called interest rates and they can either be a constant or following a stochastic model. Examples of stochastic interest rate models include the Vasicek model with a return process

$$dR(t) = \big(\alpha - \beta R(t)\big)dt + \sigma dW(t) \quad (23)$$

α, β and σ are positive constants. The solution to the Vasicek model is the process

$$R(t) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW(s)$$

The other most generally used stochastic interest rate model is the Cox−Ross−Ingesoll model whose return rate process is given by

$$dR(t) = \left(\alpha - \beta R(t)\right)dt + \sigma\sqrt{R(t)}dW(t) \qquad (24)$$

where $\alpha > 0$, $\beta > 0$ and $\sigma > 0$ are all constants.

Other commonly used stochastic interest rate models include The Hull–White model, also called the extended Vasicek model, the Ho-Lee model and the Black–Karasinski model. The most popular stochastic volatility model is the Heston's model where the stock prices process is modeled as

$$dS(t) = \mu S(t)dt + \sqrt{\mathcal{V}(t)}S(t)dW_1(t) \qquad (25)$$

with $d\mathcal{V}(t) = \xi\mathcal{V}dt + \sigma\sqrt{\mathcal{V}}dW_2(t)$

- $\mathcal{V}$ is the variance, which follows a square−$r$oot process.
- dW_1 and dW_2 are two Wiener processes having correlation ρ.
- μ is constant riskfree ratea and both μ ξ may depend on $\mathcal{V}$ and t.

Other stochastic volatility models include the constant elasticity of variance model and the Generalized Autoregressive conditional Heteroskedasticity (GARCH) model.

For the purpose of this study we will assume a constant interest rate and constant volatility for the option. We compute a one year historical volatility and use it to compute a three month call and put option prices for the stocks listed on GSE. Data of stock prices for (2015) on GSE are used to compute the volatilities.

The Ghana government 90-day Treasury bill rate is used as the constant risk free interest rate. A time interval of three months is selected as the lifetime of the option. The closing stock price of January 4, 2016 is assumed as the initial stock price (S_0) of the option. The volatility (σ) of a stock is calculated as

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\mu^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}\mu\right)^2} \qquad (26)$$

where n is the number of trading days in a particular year and

$$\mu = \frac{1}{n}\sum_{i=1}^{n} ln\left(\frac{S(t_i)}{S(t_{i-1})}\right) \qquad (27)$$

The summary of volatility values for all listed stocks on GSE can be found in Table 3.

3. Results

We now evaluate the price of option sold on the stocks using the Black Scholes formula, i.e. Equation (23). Let's first consider a theoretical example.

Example 1

The stock price 6 months from expiry of an option is ¢42. The strike price of the option is ¢40. The risk free interest rate is 10% per annum and the volatility is 20% per annum. The price of an option written on this stock is given by

$$C(S(t),t) = S_0N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where $d_1 = \frac{In\left(S_0/K\right) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{In\left(S_0/K\right) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Now $S_0 = 42$, $K = 40$, $r = 0.1$, $T - t = \Delta t = 0.5$, $\sigma = 0.2$

$$d_1 = \frac{In\left(42/40\right) + \left(0.1 + \frac{0.2^2}{2}\right)(0.5)}{0.2\sqrt{0.5}} = 0.7693$$

$$N(d_1) = 0.7791$$

$$d_2 = \frac{In\left(42/40\right) + \left(0.1 - \frac{0.2^2}{2}\right)(0.5)}{0.2\sqrt{0.5}} = 0.6278$$

$$N(d_2) = 0.7349$$

$$C(S(t),t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$= 42(0.7791) - 40e^{-0.1\times0.5}(0.7349)$$
$$= 42(0.7791) - 38.0492(0.7349)$$
$$= 32.7222 - 27.9622$$
$$C(S(t),t) = 4.76$$

Thus, a call option sold on this stock would cost ¢4.76p

The put price is given by

$$P(S(t),t) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1)$$

$$N(-d_1) = N(-0.7693) = 0.2209$$

$$N(-d_2) = N(-0.6278) = 0.2651$$

$$P(S(t),t) = 40e^{-0.1\times0.5}(0.2651) - 42\times(0.2209)$$
$$= 38.049\times0.2651 - 42\times(0.2209)$$
$$= 0.8089$$
$$P(S(t),t) = 0.81$$

Thus, a put option sold on this stock would cost 81p.

We use Microsoft Excel to evaluate the option price and the results are presented in Figure 1. In Excel, we add form/active control to all the parameters; Stock Price, Strike Price, Volatility, Risk Free Rate and Time. This allows us to evaluate the option price at different parameter levels for any stock.

Table 1. *Option Price Results for Example 1.*

Parameter		d_1	$N(d_{1)}$	d_2	$N(d_2)$	Ke^{-rt}	Call Price	$N(-d_1)$	$N(d_2)$	Put Price
Stock Price	42	0.7693	0.7791	0.6278	0.7349	38.0492	4.7594	0.2209	0.2651	0.8086
Strike Price	40									
Volatility	20									
Interest Rate	10									
Time to Expiry	6									

We present here the call and put option price for one of the listed stocks-Tullow Oil. The closing stock price for Tullow Oil for January 4, 2016 was ¢28 and the annual volatility of its stock price is 10%. The Ghana government risk free rate is 23% and we choose a strike price of ¢30. The price of a three month European call option sold on Tullow stocks is computed as follows:

$$C(S(t),t) = S(t)N(d_1) + Ke^{-r(T-t)}N(d_2)$$

where $d_1 = \frac{In\left(S_0/K\right)+\left(r+\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{In\left(S_0/K\right)+\left(r-\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Now $S_0 = 28,\ K = 30,\ r = 0.23,$
$T - t = \Delta t = 3\,months = 0.25, \sigma = 0.1$

$$d_1 = \frac{In\left(28/30\right) + \left(0.23 + \frac{0.1^2}{2}\right) \times (0.25)}{0.1 \times \sqrt{0.25}} = -0.2049$$

$$N(d_1) = 0.4188$$

$$d_2 = \frac{In\left(28/30\right) + \left(0.1 - \frac{0.1^2}{2}\right) \times (0.25)}{0.1 \times \sqrt{0.25}} = -0.2549$$

$$N(d_2) = 0.3994$$

$$C(S(t),t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$= 28(0.4188) - 30e^{-0.23\times0.25}(0.3994)$$
$$= 28(0.4188) - 28.3237(0.3994)$$
$$C(S(t),t) = 0.4139$$

The price of a 3-month European call option sold on Tullow stocks would cost 41p.

The put price is given by

$$P(S(t),t) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1)$$

$$N(-d_1) = N(-(-0.2049)) = 0.5812$$

$$N(-d_2) = N(-(-0.2549)) = 0.6006$$

$$P(S(t),t) = 30e^{-0.23\times0.25}(0.6006) - 28 \times (0.5812)$$

$$P(S(t),t) = 0.7384$$

The put option price sold on Tullow stocks is 74p. The results in are presented in Table 2.

Table 2. *Results of Option Pricing on Tallow Stock.*

Parameter		d_1	$N(d_1)$	d_2	$N(d_2)$	Ke^{-rt}	Call Price	$N(-d_1)$	$N(-d_2)$	Put Price
Stock Price	28	0.2049	0.4188	0.2549	0.3994	28.3237	0.4146	0.5812	0.6006	0.7383
Strike Price	30									

Parameter		d_1	$N(d_1)$	d_2	$N(d_2)$	Ke^{-rt}	Call Price	$N(-d_1)$	$N(-d_2)$	Put Price
Volatility	10									
Risk-free Rate	23									
Time to Expiry	3									

We repeat this procedure for stocks listed on GSE and the summary of the call and put prices are tabulated in Table 3.

Table 3. *Call and Put Prices for Stocks on GSE.*

Stock	African Champion Ltd.	AngloGold Ashanti Ltd.	Aluworks Limited	Ayrton Drug Manufacturing Ltd.	Benso Oil Palm Plantation	Cal Bank Ltd.	Clydestone (Ghana) Ltd.	Camelot Ghana Ltd.	Cocoa Processing Co. Ltd.
Stock Price (S_0)	0.02	37	0.14	0.16	2.5	1.01	0.03	0.12	0.02
Strike Price (K)	0.03	40	0.16	0.18	2.55	1.03	0.05	0.14	0.04
Volatility (σ)	0.0625	1	0.0681	0.0205	0.0205	0.0173	0	0	0.1531
Risk-free Rate (r)	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
Call Price $C(S(t),t)$	0	0.2	0	0	0.09	0	0	0	0
Put Price $P(S(t),t)$	0.01	0.02	0.01	0.01	0	0	0.02	0.01	0.02

Stock	Ecobank Ghana Ltd.	Enterprise Group Ltd.	Ecobank Transnational Inc.	Fan Milk Ltd.	Ghana Commercial Bank Ltd.	Guinness Ghana Breweries Ltd.	Ghana Oil Company Ltd.	Golden Star Resources Ltd.	Golden Web Ltd.
Stock Price (S_0)	7	2.4	0.27	7.35	3.79	1.99	1.4	1.99	0.01
Strike Price (K)	9	3	0.3	9.35	4.5	3	2	3	0.03
Volatility (σ)	0.0131	0.0133	0.0223	0.0894	0.0842	0.0189	0.0124	0.0103	0.051
Risk-free Rate (r)	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
Call Price $C(S(t),t)$	0	0	0	0	0	0	0	0	0
Put Price $P(S(t),t)$	1.5	0.43	0.01	1.48	0.93	0.84	0.49	0.84	0.02

Stock	HFC Bank (Ghana) Ltd.	Mechanical Lloyd Co. Ltd.	Pioneer Kitchenware Ltd.	Produce Buying Company Ltd.	PZ Cussons Ghana Ltd.	Standard Chartered Bank Ghana. Ltd.	SIC Insurance Company Ltd.	Starwin Products Ltd.	Societe Generale Ghana Ltd.
Stock Price (S_0)	0.9	0.19	0.05	0.1	0.34	16.3	0.14	0.04	0.8
Strike Price (K)	2	1.21	0.07	0.2	0.36	18	0.16	0.06	2
Volatility (σ)	0.0252	0.0245	0.0116	0.0196	0.0173	0.0063	0.0376	0.066	0.0124
Risk-free Rate (r)	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
Call Price $C(S(t),t)$	0	0	0	0	0	0.02	0	0	0
Put Price $P(S(t),t)$	0.99	0.75	0.02	0.09	0.6	1.01	0.8	0.9	1.09

Stock	Sam Woode Ltd.	Trust Bank Ghana Ltd.	Total Petroleum Ghana Ltd.	Transaction Solutions (Ghana) Ltd.	Tullow Oil Plc	Unilever Ghana Ltd.	UT Bank Ltd.	Mega African Capital Ltd.
Stock Price (S_0)	0.04	0.37	5.1	0.03	28	8.5	0.1	6
Strike Price (K)	0.06	0.5	7	0.05	30	10	0.2	8
Volatility (σ)	1	0.0156	0.0143	0	0.1044	0.0142	0.0302	0.009
Risk-free Rate (r)	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
Call Price $C(S(t),t)$	0	0	0	0	0.41	0.01	0	0
Put Price $P(S(t),t)$	0.02	0.57	1.51	0.91	0.74	0.95	0.84	1.55

4. Discussion

Table 3 shows the values of initial stock price, the strike price, the stock volatility as well as the call and put price obtained for listed stocks on GSE using the Black-Scholes formula. The risk free interest rate used is 23% per annum. We realise that stocks on the exchange are characterized by low volatilities and low stock prices. For these reasons we examine the effect of low volatilities and low stock prices on an option price.

Effect of low volatility on an option price

When volatility approaches zero the stock is almost virtually riskless and so at maturity time T its price will grows at a rate r to S_0e^{-rT}. The payoff from the option is $max(S_0e^{-rT} - K, 0)$.

Discounting at a rate of r, the value of the call today is

$$e^{-rT}max\ (S_0e^{-rT} - K, 0) = max\ (S_0 - Ke^{-rT}, 0)$$

Now consider the Black-Scholes formula

$$C(S(t),t) = S(t)N(d_1) + Ke^{-r(T-t)}N(d_2)$$

with $d_1 = \frac{In\left(S_0/K\right)+\left(r+\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{In\left(S_0/K\right)+\left(r-\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

As $\sigma \to 0$, $d_1 and\ d_2 \to +\infty$ so that $N(d_1)$ and $N(d_2) \to 1$ and

$C(S(t),t) = S(t)N(d_1) - Ke^{-r(T-t)}$ approaches $S(t) - Ke^{-r(T-t)}$

Thus $lim_{\sigma\to 0}\, C(S(t),t) = S(t) - Ke^{-r(T-t)}$

The call price in the limit as volatility approaches zero is

$lim_{\sigma\to 0}\, C(S(t),t) = max(S_0e^{-rT} - K, 0)$.

A graph of stock volatility and call option price shows that call option prices are close to zero for most of the stocks. See Figure 1.

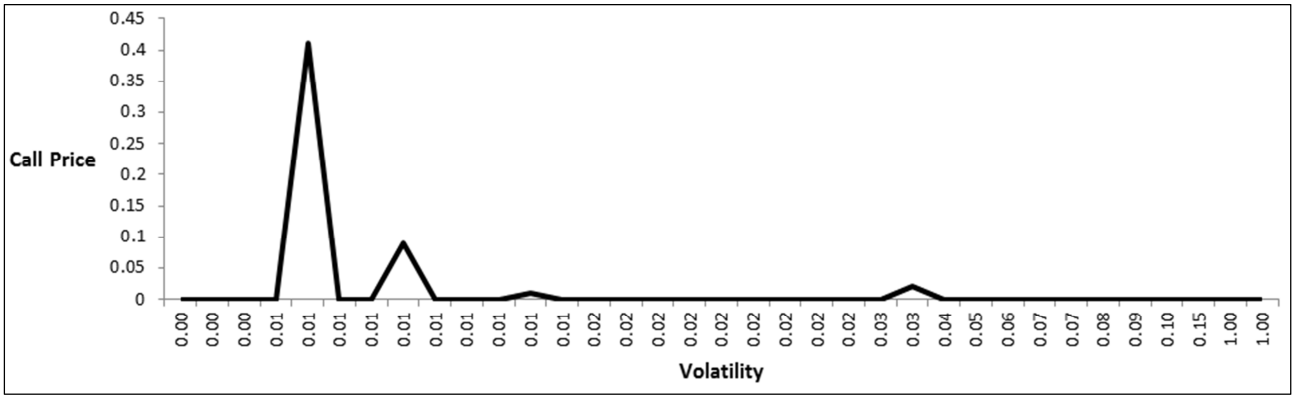

Figure 1. *Call option price against volatility.*

The put option price as stock volatility approaches zero does not tend to zero but rather to $max(Ke^{-rT} - S_0, 0)$. That is

$$\lim_{\sigma\to 0} P(S(t),t) = max(Ke^{-rT} - S_0, 0)$$

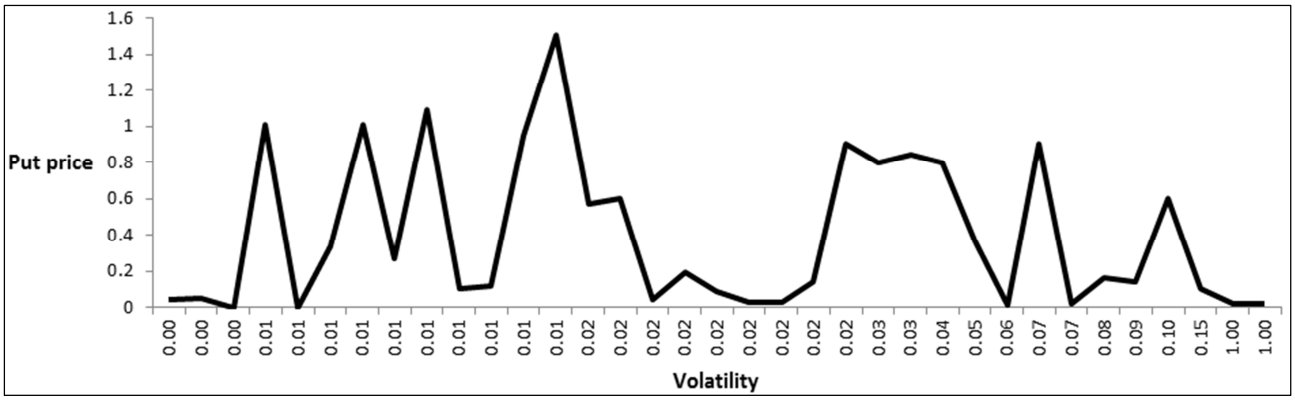

Figure 2. *Put Option Price against volatility.*

Figure 2 shows the graph of put option price against the stock volatility. We realise that the put price do not tend to zero as volatility decreases to zero.

Effect of low stock price on an option price

When stock price S_0 become very large both $d_1 \to +\infty$ and $d_2 \to +\infty$ and both $N(d_1)$ and $N(d_2) \to 1$. The price of European call option is given by

$$C(S(t),t) = S(t) - Ke^{-rt}$$

The call option is then almost certain to be exercised as $S(t) \gg Ke^{-rt}$. Note that a call option is likely to be exercised if and only if $S(t) > K$.

On the other-hand, if stock price S_0 become very small both d_1 and $d_2 \to -\infty$ and $N(d_1)$ and $N(d_2) \to 0$. The price of a call option therefore approaches zero. i.e.

$$lim_{S_0\to 0}\, C(S(t),t) = 0.$$

If $S(t) \ll Ke^{-rt}$ then it is almost certain the call option will not to be exercised. Note that a call option is likely to be exercised if and only if $S(t) > K$. This is consistent with our results. Figure 1 shows a graph of stock price against the call option price. We realise that as stock price decreases the call price also decreases to zero.

Figure 3. *Call option price against stock price.*

We realise from Figure 3 that the call option price for most of the low priced stocks approaches zero. This makes practical sense, of course, if the probability of exercising the option in the future is very small then its current price will certainly approach zero.

For a put option, if the stock price S_0 approaches zero the parameters $d_1 \to -\infty$ and $d_2 \to -\infty$ and consequently $N(-d_1)$ and $N(-d_2) \to 0$. The price of a put option then approaches zero. That is

$$\lim_{S_0 \to 0} P(S(t), t) = 0$$

In the Black-Scholes formula to calculate put option price, as the stock price S_0 approaches zero the parameters d_1 and d_2 approaches $+\infty$ and consequently $N(-d_1)$ and $N(-d_2)$ approaches 1. In this case the price of a put option

$$P(S(t), t) = Ke^{-rt}N(-d_2) - S_0N(-d_1) \to Ke^{-rt} - S_0$$

i.e. $lim_{S_0 \to 0} P(S(t), t) = Ke^{-rt} - S_0$

If $S_0 \ll K$ then the put price can be approximated by Ke^{-rt}. The behaviour of put prices in the presence of low stock prices is shown in figure 3. We clearly observe that put prices do not necessary tend to zero for low stock prices.

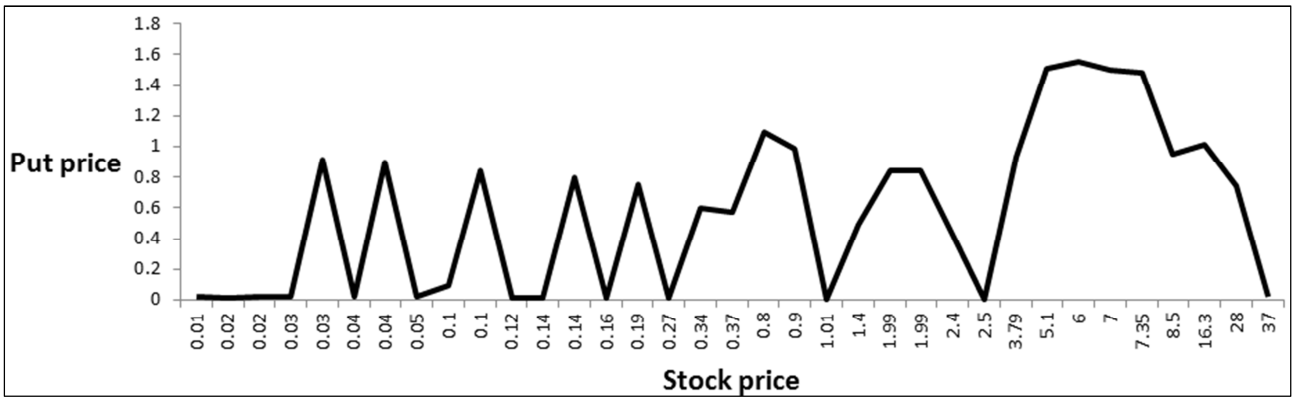

Figure 4. *Put option price against stock price.*

5. Conclusion

We have examined the possibility of pricing European options on Ghanaian stocks using the Black-Scholes formula. The results presented here shows that we can conveniently price options on Ghanaian stocks using the Black-Scholes model. However, it is realised that for stocks with very low volatilities and low initial stock price the call option price is zero in most cases. On the otherhand, we determined that put options do not tend to zero for low stock volatilities and low stock prices. The formula gives positive put prices for most of the listed stocks. This study suggest that if a broker wish to open trading in options stocks on GSE as underlying asset then he must concentrate e on selling put options until the situation on the exchange improve in regards to stock volatility and stock prices.

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