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The Optimal Harvesting of a Stochastic Gilpin-Ayala Model Under Regime Switching

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Abstract: In this paper, we consider a stochastic Gilpin-Ayala model under regime switching. Obtain the optimal harvesting effort and the maximum sustained yield by investigating the condition of average boundness of the system, and the ergodicity of the Markov chain. Also, through an example, we have proved our conclusion.

Keywords: Optimal Harvesting, Stochastic Differential Equation, Markov Chain

1. Introduction

In recently, many authors have discussed population systems subject to the white noise (see [1-6, 10]). Also, the optimal harvesting in managing natural resources has received much attention. Because the growth of species in the natural world is inevitably affected by environmental noise, many scholars have considered the optimal harvesting of stochastic population systems. By solving the corresponding Fokker-Planck equation, Beddington and May(1977) established the optimal harvesting policy for a stochastic logistic model. Using the same method, Li and Wang (2010) obtained the optimal harvesting policy for a stochastic Gilpin-Ayala model. The optimal harvesting of the stochastic population model was also examined in Alvarez and Shepp(1998), Braumann (2002), Lande et al.(1995), Liu and Bai(2014), Ludwig and Varah(1979), Song et al.(2011) and Zou and Wang (2014).

A famous Gilpin-Ayala population model with harvesting is described by the ordinary differential equation (ODE)

$$dN(t) = N(t)(a(t) - h - b(t)N^{\theta}(t))dt \tag{1}$$

Where h is the harvesting effort and $\theta > 1$ is a constant. If we still use $a(t)$ to denote the average growth rate, but incorporate white noise, and the intrinsic growth rate becomes

$$a(t) \to a(t) + \alpha(t)\dot{B}(t)$$

Where $\dot{B}(t)$ is white noise and $\alpha^2(t)$ represents the intensity of the noise. Then this environmentally perturbed system may be described by the Ito's equation

$$dN(t) = N(t)[(a(t) - h - b(t)N^{\theta}(t))dt + \alpha(t)dB(t)].$$

Where $B(t)$ is the 1-dimensional standard Brownian motion with $B(0) = 0$. As we known, there are various types of environmental noise. Let us now take a further step by considering another type of environmental noise, namely color noise, say telegraph noise (see e.g. [7, 8]). In this context, telegraph noise can be described as a random switching between two or more environmental regimes, which differ in terms of factors such as nutrition or rainfall. The switching is memoryless and the waiting time for the next switch has an exponential distribution. We can hence model the regime switching by a finite-state Markov chain. Assume that there are n regimes and the system obeys

$$dN(t) = N(t)[(a(1) - h - b(1)N^{\theta}(t))dt + \alpha(1)dB(t)], \tag{2}$$

When it is in regime 1, while it obeys another stochastic Gilpin-Ayala model

$$dN(t) = N(t)[(a(2) - h - b(2)N^{\theta}(t))dt + \alpha(2)dB(t)], \tag{3}$$

in regime 2 and so on. Therefore, the system obeys

$$dN(t) = N(t)[(a(i) - h - b(i)N^{\theta}(t))dt + \alpha(i)dB(t)], \quad (4)$$

In regime $i(1 \le i \le n)$. The switching between these n regimes is governed by a Markov chain $r(t)$ on the state space S={1,2,...,n}. The population system under regime switching can therefore be described by the following stochastic model

$$dN(t) = N(t)[(a(r(t)) - h - b(r(t))N^{\theta}(t))dt + \alpha(r(t))dB(t)]. \quad (5)$$

This system is operated as follows: If $r(0) = i_0$, the system obeys Eq.(4) with $i = i_0$ until time τ_1 when the Markov chain jumps to i_1 from i_0; the system will then obeys Eq.(4) with $i = i_1$ from τ_1 until τ_2 when the Markovian chain jumps to i_2 from i_1. The system will continue to switch as long as the Markovian chain jumps. In other words, Eq.(5) can be regarded as Eqs. (4) switching from one to another according to the law of the Markov chain. The different Eqs.(4) $(1 \le i \le n)$ are therefore referred to as the subsystem of Eq.(5).

Takeuchi et al. [7] investigated a 2-dimensional autonomous predator-prey Lotka-Volterra system with regime switching and revealed a very interesting and surprising result: If two equilibrium states of the subsystems are different, all positive trajectories of this system always exit from any compact set of R^{2+} with probability 1; on the other hand, if the two equilibrium states coincide, then the trajectory either leaves any compact set of R^{2+} or converges to the equilibrium state. In practice, two equilibrium states are usually different, in which case Takeuchi et al. [7] showed that the stochastic population system is neither permanent nor dissipative. This is an important result as it reveals the significant effect of environmental noise on the population system: both its subsystems develop periodically but switching between them makes them become neither permanent nor dissipative. Therefore, these factors motivate us to consider the Gilpin-Ayala population system subject to both white noise and color noise, described by (SDE)

$$dN(t) = N(t)[(a(r(t)) - h - b(r(t))N^{\theta}(t))dt + \alpha(r(t))dB(t)], \quad (6)$$

where for each $i \in S, a(i), b(i)$ and $\alpha(i)$ are all nonnegative constants and $\theta > 1$. Our aim is to reveal the optimal harvesting of the system (6) with the environmental noise affects.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, F, \{F_t\}t \ge 0, P)$ be a complete probability space with a filtration $\{F_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is right continuous and F_0 contains all P-null sets). Let $B(t), t \ge 0,$ be a scalar standard Brownian motion defined on this probability space. We also denote by R^+ the open interval $(0, +\infty)$, and denote by $\overline{R}^+$ the interval $[0, +\infty)$. Let r(t) be a right-continuous Markov chain on the probability space, taking values in a finite state space S={1,2,...,n}, with the generator $\Gamma = (\gamma_{uv})n \times n$ given by

$$P\{r(t+\delta) = v \mid r(t) = u\} = \begin{cases} \gamma_{uv}\delta + o(\delta) & if\, u \ne v, \\ 1 + \gamma_{uv}\delta + o(\delta) \ge 0 & if \;\; u = v, \end{cases} \quad (7)$$

where $\delta > 0$. Here γ_{uv} is the transition rate from u to v and $\gamma_{uv} \ge 0$ if $u \ne v$, while

$$\gamma_{uu} = -\sum_{u \ne v} \gamma_{uv},$$

we assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is right continuous step function with a finite number of jumps in any finite subinterval of $\overline{R}^+$. As a standing hypothesis we assume in this paper that the Markov chain r(t) is irreducible. This is a very reasonable assumption, as it means that the system can switch from any regime to any other regime. This is equivalent to the condition that for any $u, v \in S$, one can find finite numbers $i_1, i_2, ...i_k \in S$ such that $\gamma_{u,i_1}, \gamma_{i_1,i_2}, ..., \gamma_{i_k,v} > 0$ Note that Γ always has an eigen value 0. The algebraic interpretation of irreducibility is that rank $(\Gamma) = n-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, ..., \pi_n) \in \mathcal{R}^{1\times n}$ which can be determined by solving the following linear equation

$$\pi\Gamma = 0 \quad (8)$$

subject to

$$\sum_{i=1}^{n} \pi_i = 1 \quad and \quad \pi_i > 0 \quad for \quad any \quad i \in S.$$

For convenience and simplicity in the following discussion, define

$$\hat{f} = \min_{i \in S} f(i), \; \check{f} = \max_{i \in S} f(i),$$

where $\{f(i)\}_{i\in S}$ is a constant vector.

Let $N(t)$ be a solution of Eq.(6),by Itˆo's formula

$$dN^{\theta}(t) = \theta N^{\theta}(t)[(a(r(t)) - h + \frac{\alpha^2(r(t))(\theta - 1)}{2} - b(r(t))N^{\theta}(t))dt + \alpha(r(t))dB(t)]. \quad (9)$$

Let $Y(t)=N^{\theta}(t)$, then the system (9) can be written as

$$dY(t)=\theta Y(t)[(a(r(t))-h+\frac{\alpha^2(r(t))(\theta-1)}{2}-b(r(t))Y(t))dt+\alpha(r(t))dB(t)]. \tag{10}$$

Similarly to the Theorem 2.1 in [9], we have the following Lemma.

Lemma 1. There exists a unique continuous solution N(t) to SDE (6) for any initial value N(0)=N0>0, which is global and represented by

$$N^{\theta}(t)=\frac{\exp\{(\theta\int_0^t[a(r(s))-h+\frac{\alpha^2(r(s))(\theta-1)}{2}]ds+\alpha(r(s))dB(s))\}}{\frac{1}{N_0{}^{\theta}}+\theta\int_0^t b(r(s))\exp\{(\theta\int_0^t[a(r(u))-h+\frac{\alpha^2(r(u))(\theta-1)}{2}]du+\alpha(r(u))dB(u))\}ds}$$

. Since $Y(t)$ and $N(t)$ have the same monotone and extreme points in $\overline{R}^{+}$, then we can investigate the optimal harvesting of the system (10) instead of (6).

The solution of system (10) with initial value $Y(0)=Y_0=N^{\theta}(0)=N^{\theta}{}_0$ is

$$Y(t)=\frac{\exp\{(\theta\int_0^t[a(r(s))-h+\frac{\alpha^2(r(s))(\theta-1)}{2}]ds+\alpha(r(s))dB(s))\}}{\frac{1}{Y_0}+\theta\int_0^t b(r(s))\exp\{(\theta\int_0^t[a(r(u))-h+\frac{\alpha^2(r(u))(\theta-1)}{2}]ds+\alpha(r(u))dB(u))\}ds}$$

which is positive and global.

We first give some definitions about the optimal harvesting of the system (10) with the environmental noise affects.

Definition. The harvesting effort h^* is said to be optimal, if

$$h^*\overline{x}(h^*)=\sup_{h>0}\{h\overline{x}(h)\}$$

where $\overline{x}(h)=\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds$.

For system (10), we introduce the following basic assumptions:

(H1) For each $i\in S, b(i)>0$;

(H2) $\sum_{i=1}^{n}\pi_i[a(i)-h+\frac{\theta-1}{2}\alpha^2(i)]>0$;

(H3) For each $i\in S$, $a(i)-h+\frac{\theta-1}{2}\alpha^2(i)>0.$

For the system(10), we have the following results.

3. The Main Results

We firstly have the following lemma.

Lemma 2. If assumption (H1) holds, for an arbitrary given positive constant p, the solution Y(t) of SDE (10) with any given positive initial value has the property that

$$\limsup_{t\to\infty}E(Y(t)^p)\le K(p) \tag{11}$$

where

$$K(p)=\begin{cases}(\dfrac{\check{a}-h+\dfrac{\check{\alpha}^2(\theta-1)}{2}}{\hat{b}})^p, & 0<p<1;\\[2ex] (\dfrac{\check{a}-h+\dfrac{\check{\alpha}^2(\theta p-1)}{2}}{\hat{b}})^p, & p\ge 1.\end{cases} \tag{12}$$

Proof By the generalized It^o formula, we have

$$\begin{aligned}dY^p(t)&=pY^{p-1}(t)dY(t)+\frac{1}{2}p(p-1)Y^{p-2}(t)(dY(t))^2\\&=\theta pY^p(t)[(a(r(t))-h+\frac{(\theta-1)\alpha^2(r(t))}{2}-b(r(t))Y(t))dt\\&+\alpha(r(t))dB(t)]+\frac{\theta^2}{2}p(p-1)Y(t)\alpha^2(r(t))dt.\end{aligned} \tag{13}$$

Integrating it from 0 to t and taking expectations of both sides, we obtain that

$$\begin{aligned}E(Y^p(t))-E(Y^p(0))&=\theta p\int_0^t E[Y^p(s)(a(r(s))-h+\frac{(\theta-1)}{2}\alpha^2(r(s))\\&-b(r(s)))Y(s)]ds+\frac{\theta^2}{2}p(p-1)\int_0^t E[\alpha^2(r(s))Y^p(s)]ds.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{dE(Y^p(t))}{dt}&=\theta pE[Y^p(t)(a(r(t))-h+\frac{(\theta-1)}{2}\alpha^2(r(t))\\&-b(r(t)))Y(t)]+\frac{\theta^2}{2}p(p-1)E[\alpha^2(r(t))Y^p(t)].\end{aligned} \tag{14}$$

If 0<p<1. we obtain

$$\frac{dE(Y^p(t))}{dt} \le \theta p(\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2)E(Y^p(t)) - \hat{b}\theta p E(Y^{p+1}(t))$$

$$\le \theta p(\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2)E(Y^p(t)) - \hat{b}\theta p(E(Y^p(t))^{\frac{p+1}{p}} \tag{15}$$

$$= \theta p E(Y^p(t))[(\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2) - \hat{b}(E(Y^p(t))^{\frac{1}{p}}],$$

while if $p \ge 1$, we obtain

$$\frac{dE(Y^p(t))}{dt} \le \theta p[(\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2)E(Y^p(t)) - \hat{b}\theta p E(Y^{p+1}(t))] + \frac{\theta^2}{2}p(p-1)\check{\alpha}^2 E[Y^p(s)]$$

$$\le \theta p[(\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2)E(Y^p(t)) - \hat{b}\theta p E(Y^p(t))^{\frac{p+1}{p}}] + \frac{\theta^2}{2}p(p-1)\check{\alpha}^2 E[Y^p(s)] \tag{16}$$

$$= \theta p E(Y^p(t))[(\check{a}-h+\frac{(\theta p-1)}{2}\check{\alpha}^2) - \hat{b}E(Y^p(t))^{\frac{1}{p}}].$$

Therefore, letting $z(t) = E(Y^p(t))$. we have

$$z(t) \le \begin{cases} \theta p z(t)[(\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2) - \hat{b}(z(t))^{\frac{1}{p}}], & 0<p<1; \\ \theta p z(t)[(\check{a}-h+\frac{(\theta p-1)}{2}\check{\alpha}^2) - \hat{b}(z(t))^{\frac{1}{p}}], & p \ge 1. \end{cases} \tag{17}$$

Notice that if 0<p<1, the solution of equation

$$\tilde{z}(t) = \theta p \tilde{z}(t)[(\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2) - \hat{b}(\tilde{z}(t))^{\frac{1}{p}}],$$

obeys

$$\tilde{z}(t) \to \left(\frac{\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2}{\hat{b}}\right)^p \quad as \quad t\to\infty,$$

similarly, if p≥1 the solution of equation

$$\bar{z}(t) = \theta p \bar{z}(t)[(\check{a}-h+\frac{(\theta p-1)}{2}\check{\alpha}^2) - \hat{b}(\bar{z}(t))^{\frac{1}{p}}]$$

as $t\to\infty$ is such that $\bar{z}(t) \to \left(\frac{\check{a}-h+\frac{(\theta p-1)}{2}\check{\alpha}^2}{\hat{b}}\right)^p$.

Thus by the comparison argument we get

$$\limsup_{t\to\infty} z(t) \le \begin{cases} \left(\frac{\check{a}-h+\frac{(\theta-1)}{2}\check{\alpha}^2}{\hat{b}}\right)^p & 0<p<1; \\ \left(\frac{\check{a}-h+\frac{(\theta p-1)}{2}\check{\alpha}^2}{\hat{b}}\right)^p, & p\ge 1. \end{cases} \tag{18}$$

By the definitions of z(t), we obtain the assertion (11).

Lemma 3. Under (H1), the solution Y(t) of SDE (10) with any positive initial value has the property

$$\limsup_{t\to\infty} \frac{\ln(Y(t))}{\ln t} \le 1 \quad a.s. \tag{19}$$

Proof By Lemma 1, the solution Y(t) with positive initial value will remain in R^+. We have

$$dY(t) = \theta Y(t)[(a(r(t)) - h + \frac{\alpha^2(r(t))(\theta-1)}{2} - b((r(t))Y(t))dt + \alpha((r(t))dB(t)]$$
$$\le \theta(\check{a}-h+\frac{\check{\alpha}^2(\theta-1)}{2})Y(t)dt + \theta\alpha(r(t))Y(t)dB(t).$$

We can also derive from this that

$$E(\sup_{t\le u\le t+1} Y(u)) \le \theta E(Y(t)) + \theta(\check{a}-h+\frac{\check{\alpha}^2(\theta-1)}{2})\int_t^{t+1} E(Y(s))ds$$
$$+\theta E(\sup_{t\le u\le t+1}\int_t^u \alpha(r(s))Y(s)dB(s)).$$

From Lemma 2, we know that

$$\limsup_{t\to\infty} E(Y(t)) \le K(1). \tag{20}$$

But, by the well-known Burkholder-Davis-Gundy inequality and the H"older inequality, we derive that

$$E(\sup_{t\le u\le t+1}\int_t^u \alpha(r(s))Y(s)dB(s)) \le 3E[\int_t^{t+1}(\alpha(r(s))Y(s))^2 ds]^{\frac{1}{2}}$$
$$\le E[9\check{\alpha}^2\int_t^{t+1}(Y(s))^2 ds]^{\frac{1}{2}}$$
$$\le E[\sup_{t\le u\le t+1} Y(u)\bullet 9\check{\alpha}^2\int_t^{t+1}(Y(s))^2 ds]^{\frac{1}{2}}$$
$$\le E[(\frac{1}{2}\sup_{t\le u\le t+1} Y(u))^2 + (9\check{\alpha}^2\int_t^{t+1}(Y(s))ds)^2]^{\frac{1}{2}} \tag{21}$$
$$\le E[\frac{1}{2}\sup_{t\le u\le t+1} Y(u) + 9\check{\alpha}^2\int_t^{t+1}(Y(s))ds]$$
$$\le \frac{1}{2}E(\sup_{t\le u\le t+1} Y(u) + 9\check{\alpha}^2\int_t^{t+1}E(Y(s))ds].$$

Therefore

$$E(\sup_{t\le u\le t+1} Y(u)) \le 2\theta E(Y(t))$$
$$+2\theta(\check{a}-h+\frac{\check{\alpha}^2(\theta-1)}{2})\int_t^{t+1}E(Y(s))ds + 18\theta\check{\alpha}^2\int_t^{t+1}EY(s)ds.$$

This, together with (20), yields

$$\limsup_{t\to\infty} E(\sup_{t\le u\le t+1} Y(u)) \le 2\theta(1+\check{a}-h+\frac{\check{\alpha}^2(\theta-1)}{2}+9\check{\alpha}^2)K(1). \quad (22)$$

To prove assertion (19), we observe from (22) that there is a positive constant $\overline{K}$ such that

$$E(\sup_{k\le t\le k+1} Y(t)) \le \overline{K}, \quad k=1,2,3\ldots$$

Let $\varepsilon>0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$P\{\sup_{k\le t\le k+1} Y(t) > k^{1+\varepsilon}\} \le \frac{\overline{K}}{k^{1+\varepsilon}}, \quad k=1,2,3\ldots$$

Applying the well-known Borel-Cantelli lemma, we obtain that for almost all $\omega\in\Omega$

$$\sup_{k\le t\le k+1} Y(t) \le k^{1+\varepsilon} \quad (23)$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega\in\Omega$, for which (23) holds whenever $k>k_0$. Consequently, for almost all $\omega\in\Omega$ and $k\le t\le k+1$,

$$\frac{\ln(Y(t))}{\ln t} \le \frac{(1+\varepsilon)\ln k}{\ln k} = 1+\varepsilon.$$

Therefore

$$\limsup_{t\to\infty}\frac{\ln(Y(t))}{\ln t} \le 1+\varepsilon \quad a.s.$$

Letting $\varepsilon\to 0$ we obtain the desired assertion (19). The proof is therefore complete.

Corollary 1. Under (H1), the solution of SDE (10) with any positive initial value has the property

$$\limsup_{t\to\infty}\frac{\ln(Y(t))}{t} \le 0 \quad a.s. \quad (24)$$

Proof From Lemma 3, we have

$$\limsup_{t\to\infty}\frac{\ln(Y(t))}{t} = \limsup_{t\to\infty}\frac{\ln(Y(t))}{\ln t}\limsup_{t\to\infty}\frac{\ln t}{t} \le \limsup_{t\to\infty}\frac{\ln t}{t} = 0.$$

Consequently, we will get some results about SDE(10).

Theorem 1. The solution of SDE (10) with any positive initial value has the property

$$\frac{\ln(Y(t))}{t} = 0 \qquad a.s. \quad (25)$$

Proof From Corollary 1, we have

$$\limsup_{t\to\infty}\frac{\ln(Y(t))}{t} \le 0 \quad a.s.$$

Hence, we only need to prove the following conclusion

$$\liminf_{t\to\infty}\frac{\ln(Y(t))}{t} \ge 0 \quad a.s. \quad (26)$$

Let

$$M(t) = \int_0^t \alpha(r(s))dB(s).$$

The quadratic variation of this martingale is

$$\langle M, M\rangle = \int_0^t \alpha^2(r(s))ds ? \le \check{\alpha}^2 t$$

By the strong law of large numbers for martingales, we therefore have

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0 \quad a.s.$$

For any positive constant $\varepsilon>0$ and small enough, there is a positive constant $T<\infty$, such that

$$|\int_0^t \alpha(r(s))dB(s)| \le \varepsilon t \quad a.s \quad t\ge T$$

Therefore, for any t>s>T, there have

$$|\int_s^t \alpha(r(\tau))dB(\tau)| \le |\int_0^t \alpha(r(\tau))dB(\tau)| + |\int_o^s \alpha(r(\tau))dB(\tau)| \le \varepsilon(t+s) \quad a.s. \quad (27)$$

By (27), for any t>T, we have

$$\begin{aligned}
Y^{-1}(t) &= Y^{-1}(T)\exp\{-\theta(\int_0^t [a(r(s))-h+\frac{\alpha^2(r(s))(\theta-1)}{2}]ds+\alpha(r(s))dB(s))\}\\
&+\theta\int_0^t b(r(s))\exp\{-\theta(\int_0^s [a(r(u))-h+\frac{\alpha^2(r(u))(\theta-1)}{2}]du+\alpha(r(u))dB(u))\}ds\\
&\le Y^{-1}(T)\exp\{-\theta(\int_0^t [a(r(s))-h+\frac{\alpha^2(r(s))(\theta-1)}{2}]ds+\varepsilon(t+T))\}\\
&+\theta\int_0^t b(r(s))\exp\{-\theta(\int_0^s [a(r(u))-h+\frac{\alpha^2(r(u))(\theta-1)}{2}]du+\varepsilon(t+s))\}ds.
\end{aligned}$$

There has a constant $K>0$ such that

$$e^{-2\varepsilon(t+T)}Y^{-1}(t) \le Y^{-1}(T)\exp\{-\theta(\int_0^t [a(r(s))-h+\frac{\alpha^2(r(s))(\theta-1)}{2}]ds-\varepsilon(t+T))\}$$
$$+\theta\int_0^t b(r(s))\exp\{-\theta(\int_0^t [a(r(u))-h+\frac{\alpha^2(r(u))(\theta-1)}{2}]du-\varepsilon(t+s)-2\varepsilon T)\}ds$$
$$\le K<\infty \quad a.s.$$

That is $Y^{-1}(t) \le Ke^{2\varepsilon(t+T)}$ $a.s.$ There have

$$\frac{\ln(Y^{-1}(t))}{t} \le \frac{1}{t}(\ln K + 2\varepsilon(t+T)) \quad a.s.$$

and

$$\limsup_{t\to\infty}\frac{\ln(Y^{-1}(t))}{t} \le 2\varepsilon \quad a.s.$$

or

$$\liminf_{t\to\infty}\frac{\ln(Y^{-1}(t))}{t} \ge -2\varepsilon \quad a.s.$$

Therefore we obtain the assertion (26) and complete the proof.

Theorem 2. Suppose (H1), (H2) hold and the Markov chain r(t) is irreducible, then the solution Y(t) of SDE (10) with any positive initial value has the property

$$\lim_{t\to\infty}\frac{\int_0^t b(r(s))Y(s)ds}{t} = \sum_{i=1}^n \pi_i[a(i)-h+\frac{\theta-1}{2}\alpha^2(i)] \quad a.s.$$

Proof From Itˆo equation,

$$\ln Y(t) = \ln Y_0 + \theta\int_0^t [(a(r(s))-h+\frac{\alpha^2\,(r(s))(\theta-1)}{2}-b(r(s))Y(s))ds+\alpha(r(s))dB(s)]. \tag{28}$$

Hence, we have

$$\theta\frac{\int_0^t b(r(s))Y(t)ds}{t} = \frac{\ln Y_0}{t} - \frac{\ln Y(t)}{t} + \theta\frac{\int_0^t (a(r(s))-h+\frac{\alpha^2\,(r(s))(\theta-1)}{2})Y(s))ds}{t}$$
$$+\theta\frac{\int_0^t \alpha(r(s))dB(s)}{t}.$$

Obviously, $\frac{\ln Y_0}{t} \to 0(t\to\infty)$, from Theorem 1, $\frac{\ln Y(t)}{t} \to 0$ a.s for $t\to\infty$.

By the ergodicity of the Markov chain r(t), as $t\to\infty$,

$$\frac{\int_0^t (a(r(s))-h+\frac{\alpha^2\,(r(s))(\theta-1)}{2})ds}{t} \to \sum_{i=1}^n \pi_i[a(i)-h+\frac{\theta-1}{2}\alpha^2(i)] \quad a.s.$$

that is $\lim_{t\to\infty}\frac{\int_0^t b(r(s))Y(t)ds}{t}$ exists and be equal to $\sum_{i=1}^n \pi_i[a(i)-h+\frac{\theta-1}{2}\alpha^2(i)]$.

On the optimal harvesting effort and the maximum sustainable yield, we have the following results.

Theorem 3. Under (H1), (H2) and (H3), the optimal harvesting effort of (10) is

$$h^* = \frac{1}{2}\sum_{i=1}^n \pi_i[a(i)+\frac{\theta-1}{2}\alpha^2(i)] \tag{29}$$

and the maximum sustainable yield satisfied

$$\frac{1}{4\check{b}}(\sum_{i=1}^{n}\pi_i[a(i)+\frac{\theta-1}{2}\alpha^2(i)])^2 \le F(h^*) \le \frac{1}{4\hat{b}}(\sum_{i=1}^{n}\pi_i[a(i)+\frac{\theta-1}{2}\alpha^2(i)])^2 \tag{30}$$

Proof By the theorem 2, we have

$$\limsup_{t\to\infty}\frac{\int_0^t Y(s)ds}{t} = \frac{1}{4\hat{b}}(\sum_{i=1}^{n}\pi_i[a(i)+\frac{\theta-1}{2}\alpha^2(i)])^2; \tag{31}$$

and

$$\liminf_{t\to\infty}\frac{\int_0^t Y(s)ds}{t} = \frac{1}{4\check{b}}(\sum_{i=1}^{n}\pi_i[a(i)+\frac{\theta-1}{2}\alpha^2(i)])^2. \tag{32}$$

It is easily to know that

$$h\liminf_{t\to\infty}\frac{\int_0^t Y(s)ds}{t} \le h\lim_{t\to\infty}\frac{\int_0^t Y(s)ds}{t} \le h\limsup_{t\to\infty}\frac{\int_0^t Y(s)ds}{t}.$$

Let

$$F(h) = h\lim_{t\to\infty}\frac{\int_0^t Y(s)ds}{t}$$

be the sustainable yield function, while F'(h)=0, we can get the unique extreme point. Noticed the h such that F'(h)=0 is independent on b(r(t)), therefore

$$\sum_{i=1}^{n}\pi_i[a(i)-2h+\frac{\theta-1}{2}\alpha^2(i)] = 0$$

and

$$h^* = \frac{1}{2}\sum_{i=1}^{n}\pi_i[a(i)+\frac{\theta-1}{2}\alpha^2(i)],$$

this is the optimal harvesting effort, taking it into (31)(32) and we can get the (30) easily. The proof is complete.

Corollary 2. Assume for some $i\in S, b(i)>0$, the subsystem of SDE(10) with Markov switching is

$$dY(t) = Y(t)[(a(i)-h+\frac{\theta\text{-}1}{2}\alpha^2(i)-b(i)Y(t))dt+\alpha(i)dB(t)]?$$

It has the optimal harvesting effort

$$h^* = \frac{1}{2}(a(i)+\frac{\theta-1}{2}\alpha^2(i))$$

and the maximum sustainable yield satisfied

$$F(h^*) = \frac{1}{4b(i)}(\sum_{i=1}^{n}\pi_i[a(i)+\frac{\theta-1}{2}\alpha^2(i)])^2.$$

4. Conclusions and example

In this paper, we investigate the optimal harvesting effort and the maximum sustainable yield of a stochastic Gilpin-Ayala model under regime switching, we get the optimal harvesting effort of the SDE (10) and estimate the value of the maximum sustainable yield. we get the value of the maximum sustainable yield of the subsystem of (10) without the SDE (10).

Making use of the results, we shall illustrate these conclusions through the following example.

Example. Consider a 3-dimensional stochastic differential equation with Markovian switching of the form

$$dN(t) = N(t)[(a(r(t))-h-b(r(t))N^{\theta}(t))dt+\alpha(r(t))dB(t)] \quad on\ t\ge 0. \tag{33}$$

Where r(t) is a right-continuous Markov chain taking values in S = {1,2,3}, and θ=2, r(t) and B(t) are independent. Here

$$a(1)=2,\ b(1)=3,\ \alpha(1)=1;$$
$$a(2)=1,\ b(2)=2,\ \alpha(2)=2;$$
$$a(3)=4,\ b(3)=1,\ \alpha(3)=3.$$

We rewrite the (33) as (34)

$$dY(t) = Y(t)[(a(r(t))-h+\frac{\theta\text{-}1}{2}\alpha^2(r(t))-b(r(t))Y(t))dt+\alpha(r(t))dB(t)] \tag{34}$$

We compute

$$a(1)+\frac{\theta-1}{2}\alpha^2(1)=\frac{5}{2};\ a(2)+\frac{\theta-1}{2}\alpha^2(2)=3;$$
$$a(3)+\frac{\theta-1}{2}\alpha^2(3)=\ 172.$$

Let the generator of the Markov chain r(t) be

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

By solving the linear equation (8) we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = (\frac{7}{15}, \frac{1}{5}, \frac{1}{3}).$$

Then the optimal harvesting effort of (34) is

$$h^* = \frac{1}{2}\sum_{i=1}^{3}\pi_i[a(i) + \frac{\theta-1}{2}\alpha^2(i)]$$
$$= \frac{1}{2}[\frac{7}{15}(2+\frac{1}{2}) + \frac{1}{5}(1+2) + \frac{1}{3}(4+\frac{9}{2})] = \frac{23}{10}$$

and the maximum sustainable yield separately, for i=1,2,3

$$F(h^*) = \frac{1}{4b(1)}(a(1) + \frac{\theta-1}{2}\alpha^2(1))^2 = \frac{25}{48};$$

$$F(h^*) = \frac{1}{4b(2)}(a(2) + \frac{\theta-1}{2}\alpha^2(2))^2 = \frac{9}{8};$$

$$F(h^*) = \frac{1}{4b(3)}(a(3) + \frac{\theta-1}{2}\alpha^2(3))^2 = \frac{289}{16}.$$

We get the optimal harvesting effort of $Y(t) = N^2(t)$ is $\frac{23}{10}$, then the optimal harvesting effort of $N^2(t)$ is $\frac{23}{10}$.

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