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# Symmetry analysis to $f''' + \beta f f'' - \alpha f'^2$=0 arising in boundary layer theory

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**Abstract:** In this paper we analyze the boundary layer equation $f''' + \beta f f'' - \alpha f'^2$=0 using a group theoretical method known as symmetry method. We obtain the symmetry group admitted by the boundary layer equation. We then construct exact invariant solutions and outline a symmetry reduction. The invariant solution is examined under common boundary conditions.

**Keywords:** Lie Symmetries, Group-Invariant Solutions, Analytic Solution, Boundary Layer Equation

## 1. Introduction

The third order autonomous nonlinear differential equation

$$f'''(z) + \beta f(z) f''(z) - \alpha f'^2(z)=0, \quad (1.1)$$

subject to suitable boundary conditions arises in industrial processes and physical applications such as boundary layer flow near a stretching surface [1-5], Nano boundary layer fluid flows [6], free convection boundary layer flow near vertical flat plate embedded in a porous medium [7-9] and high frequency excitation of liquid metal in an antisymmetric magnetic field within a boundary layer approximation [10, 11].

Eq. (1.1) has received considerable attention in the literature and has been investigated under certain boundary conditions by many researchers. For$\beta = \alpha = 1$, Weidman and Magyari [2] outlined an analytical solution known as Crane's solution and obtained solutions for generalized Crane flows. Wang [3] found an analytic solution in a general form that may recover Crane's solution. For $\beta = 1, \alpha = 2m/(m+1)$,Magyari and Keller [1] illustrated an exact analytic solution, Bognar [4] proved the existence of exponential series solution and Liao and Pop [8] applied homotopy analysis method to derive explicit analytic solutions. Kudenatti et al. [5] and Awati et al. [9] solved Eq. (1.1) when $\beta = \beta, \alpha = -\alpha$ using Dirichlet series, method of stretching of variables, and asymptotic function method. Aly and Ebaid [6] investigated theoretically Eq. (1.1) using $G'/G$ −expansion method when $\beta = m, \alpha = 1$. Belhachmi et al. [7], Brighi and Hoernel [10] and Tsai [11] examined the existence, uniqueness and nonexistence of concave convex solutions when $\beta = (\alpha + 1)/2$ with different values of $\alpha$.

So far, to the best of our knowledge, there has been no study in the literature concerning the symmetry analysis of Eq. (1.1). Therefore the aim of the present work is to analyze Eq. (1.1) using Lie symmetry group method [12,13] in order to search for possible exact solutions for all $\alpha$ and $\beta$. The significant of Lie group method lies in its ability to obtain exact invariant solutions just by determining the symmetry group admitted by the equation.

The paper is organized as follows: In Section 2, we obtain the symmetry group of the boundary layer equation (1.1). In Section 3, we construct exact invariant solutions to Eq. (1.1) using symmetry group. In Section 4, we derive a symmetry reduction for Eq. (1.1). In Section 5, we impose boundary conditions to an invariant solution. Finally, we give a brief conclusion in Section 6.

## 2. Symmetry Group

In this section, we use Lie symmetry group method [12, 13] to obtain the symmetry group of the boundary layer equation (1.1). Consider a one-parameter Lie group of infinitesimal transformation of the form

$$z \to z + \varepsilon \xi(z, f), \quad (2.1)$$

$$f \to f + \varepsilon \eta(z, f), \quad (2.2)$$

with infinitesimal generator

$$X = \xi(z,f)\frac{\partial}{\partial z} + \eta(z,f)\frac{\partial}{\partial f}.$$

Requiring that Eq. (1.1) is invariant under (2.1)-(2.2), then eliminating $f'''(z)$ using Eq. (1.1), and setting to zero all coefficients of the derivatives of $f(z)$ we obtain the following determining equations for the infinitesimals $\xi(z,f), \eta(z,f)$:

$$\eta_{zzz} + \beta\eta_{zz}f = 0,$$

$$3\eta_{zzf} - \xi_{zzz} + \beta(2\eta_{zf} - \xi_{zz})f - 2\alpha\eta_z = 0,$$

$$3(\eta_{zff} - \xi_{zzf}) + \alpha(\eta_f - 3\xi_z) + \beta(\eta_{ff} - 2\xi_{zf})f - 2\alpha(\eta_f - \xi_z) = 0,$$

$$\eta_{fff} - 3\xi_{zff} - 4\alpha\xi_f - \beta\xi_{ff}f + 2\alpha\xi_f = 0, \quad (2.3)$$

$$3(\eta_{zf} - \xi_{zz}) + \beta\xi_z f + \beta\eta = 0,$$

$$3(\eta_{ff} - 3\xi_{zf}) + \beta\xi_z f = 0,$$

$$\xi_{fff} = \xi_{ff} = \xi_f = 0.$$

Solving Eqs. (2.3) for the infinitesimals, we find

$$\xi(z,f) = c_2 - c_1 z,$$

$$\eta(z,f) = c_1 f.$$

Where $c_1$ and $c_2$ are arbitrary constants. Thus, the infinitesimal generators of Eq. (1.1) are:

$$X_1 = -z\frac{\partial}{\partial z} + f\frac{\partial}{\partial f}, \quad X_2 = \frac{\partial}{\partial z}.$$

Hence the corresponding one parameter symmetry groups of Eq. (1.1) are:

$$G_1: (z,f) \to (e^{-\varepsilon}z, e^{\varepsilon}f),$$

$$G_2: (z,f) \to (z+\varepsilon, f).$$

We can see that $G_1$ is a scaling transformation and $G_2$ is a translation transformation in $z$.

# 3. Exact Group-Invariant Solutions

Here we utilize the infinitesimal generators of Eq. (1.1) obtained in Section 2 to construct exact group-invariant solutions for Eq. (1.1).

(i) $X_2$

The group-invariant solution corresponding to $X_2$ is $f(z) = c$ (constant).

(ii) $X_1$

The generator $X_1$ gives the group-invariant solution $f(z) = 6/(2\beta - \alpha)z$, for all$\alpha$ and $\beta$provided that $2\beta \neq \alpha$.

(iii) $X_1 + cX_2$

The linear combination between generators $X_1$ and $X_2$, where $c$ is constant, gives the group-invariant solution

$$f(z) = -6/[(2\beta - \alpha)(c - z)], \quad (3.1)$$

for all $\alpha$ and $\beta$ provided that $2\beta \neq \alpha$.

# 4. Symmetry Reduction

The boundary layer equation (1.1) admits a two parameter Lie group of transformations. Therefore the order of the differential equation may be reduced by two. To determine which infinitesimal generator to use first, we compute the commutator: $[X_1, X_2] = X_1X_2 - X_2X_1 = X_2$.

We begin with the infinitesimal generator $X_2$, with first extension $X_2^{(1)} = \frac{\partial}{\partial z}$ [12]. The invariants of $X_2$ satisfying $X_2 q = 0$, and $X_2^{(1)}p = 0$ are:

$$q(z,f) = f, \; p(z,f,f') = f', \quad (4.1)$$

Where $p = p(q)$. Substitution of Eq. (4.1) into Eq. (1.1) gives the reduced second order ordinary differential equation

$$pp'' + (p')^2 + \beta qp' - \alpha p = 0. \quad (4.2)$$

Infinitesimal generator $X_1$ is admitted by Eq. (4.2) since it is admitted by Eq. (1.1). The invariants of $X_1$ satisfying $X_1^{(1)}\omega = 0$, and $X_1^{(2)}g = 0$ are:

$$\omega(q,p) = q^{-2}p, \; g(q,p,p') = q^{-1}p', \quad (4.3)$$

Where $g = g(\omega)$. Substitution of Eq. (4.3) into Eq. (4.2) gives the reduced first order ordinary differential equation

$$(g - 2\omega)\omega g' + (\omega + \beta)g + g^2 - \alpha\omega = 0. \quad (4.4)$$

Let $g = N(\omega, c)$ be a general solution to Eq. (4.4), where $c$ is constant, then the first order ODE: $g = q^{-1}p' = N(q^{-2}p, c)$ may be solved by the use of canonical coordinates $r(q,p)$ and $s(q,p)$ [12]. By satisfying that $X_2^{(1)}r(q,p) = 0$ and $X_2^{(1)}s(q,p) = 1$ we obtain $r(q,p) = q^{-2}p$ and $s(q,p) = -0.5\ln p$. Thus

$$s'(r) = -N(r,c)/2(rN(r,c) - 2r^2). \quad (4.5)$$

Integration of Eq. (4.5) and substitution for $(r,s)$ in terms of $(q,p)$ gives

$$p = c_2\exp\left[\int^{q^{-2}p} N(\rho,c_1)/(\rho N(\rho,c_1) - 2\rho^2)d\rho\right], \quad (4.6)$$

Where $c_1$ and $c_2$ are constants. By using Eq. (4.1) in Eq. (4.6) we obtain a general solution of Eq. (1.1) in solved form

$\int \frac{df}{M(f,c_1,c_2)} = z + c_3$, where $c_3$ is constant.

***Special Case***

When $\beta = 2\alpha$ Eq. (4.4) has a special solution $g(\omega) = \omega/2$. Hence, using Eq. (4.1) and Eq. (4.3) we obtain a solution for Eq. (1.1) in the form

$$f(z) = (c_1 z + c_2)^2, \quad (4.7)$$

Where $c_1$ and $c_2$ are constants.

# 5. Boundary Conditions

Usually Eq. (1.1) is associated with the boundary conditions:

$$f(0) = a, f'(0) = b, f'(\infty) = 0, \quad (5.1)$$

Where $a$ and $b$ are parameters describing the mass transfer at the surface and the surface movement respectively. For $a > 0$ mass suction occurs, $a < 0$ mass injection and $a = 0$ the surface is impermeable. When $b > 0$ the surface moves in the same direction to the main stream, $b < 0$ opposite direction and $b = 0$ the surface is fixed.

Imposing the boundary conditions (5.1) on the invariant solution (3.1), we obtain the solution

$$f(z) = 6a/(6 + a(2\beta - \alpha)z), \quad (5.2)$$

Provided that $a$ and $b$ are related by the equation $a^2(2\beta - \alpha) + 6b = 0$. When $\alpha = 1$ and $b = 1$ the solution (5.2) is similar to the one in [6].

The skin friction at the surface is given by $f''(0) = a^3(2\beta - \alpha)^2/18$. Solution (5.2) is defined for two cases: when $2\beta > \alpha, a > 0, b < 0$ and $2\beta < \alpha, a < 0, b > 0$.

# 6. Conclusions

In this paper, we applied Lie group method to derive the symmetry group admitted by the boundary layer equation (1.1). Under the symmetry group we obtained exact invariant solutions to Eq. (1.1) for all $\alpha$ and $\beta$provided that $2\beta \neq \alpha$. Furthermore, we reduced the order of Eq. (1.1) to a first order and found an analytic solution for a special case $\beta = 2\alpha$. Finally, we forced the invariant solution to satisfy boundary conditions (5.1) and constructed an exact solution to Eq. (1.1) with conditions (5.1). Solution (5.2) is not applicable for fixed impermeable surfaces. However, it is appropriate for two cases: when $2\beta > \alpha, a > 0, b < 0$ and $2\beta < \alpha, a < 0, b > 0$.

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