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An Investigation of the Quantized Matrix Algebra $J_q^0(n)$ from a Computational Viewpoint

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Abstract: In the study of quantum groups, quantized matrix algebras have been widely investigated from the viewpoints of representation theory and noncommutative geometry. This paper addresses a computational approach to the investigation of quantized matrix algebra $J_q^0(n)$, namely, by employing the Shirshov algorithmic method, it is shown that the defining relations of $J_q^0(n)$ constitute a Gröbner-Shirshov basis; by constructing an appropriate monomial ordering on $J_q^0(n)$, it is shown that $J_q^0(n)$ is a solvable polynomial algebra. Consequently, it is shown that several further structural properties of $J_q^0(n)$, such as $J_q^0(n)$ being a Noetherian domain, having Hilbert series $\frac{1}{(1-t)^{n^2}}$, having GK dimension n^2, having global homological dimension n^2, and being a classical quadratic Koszul algebra, may be derived in a constructive-computational way. Moreover, applying the foregoing structural properties in turn to investigate several structural properties of modules over $J_q^0(n)$, such as constructing finite free resolutions of finitely generated modules, establishing the stability of finitely generated projective modules, establishing the K_0-groups of $J_q^0(n)$, computing minimal graded generating sets of finitely generated graded modules, and establishing the elimination property of one-sided ideals (finitely generated modules), it is shown that all of those properties may be obtained and realized in a computational way.

Keywords: Quantized Matrix Algebra, Gröbner-Shirshov Basis, PBW Basis, Solvable Polynomial Algebra

1. Introduction

Let $K = \mathbb{C}$ be the field of complex numbers. by modifying the standard quantized matrix algebra $M_q(n)$ in the sense of [4], a class of quadratic matrix algebras $M_q^{\mathcal{P}}(n)$ associated to the quantized enveloping algebra $U_q(A_{2n-1})$ was naturally introduced in [5], where algebras belonging to $M_q^{\mathcal{P}}(n)$ are called *modified algebra* for short. Except for the quantized matrix algebra $M_q(n)$, the so-called Dipper Donkin quantized matrix algebra $D_q(n)$ introduced in [3] is also a modified algebra, and recently some structural properties of both algebras $M_q(n)$ and $D_q(n)$, as well as their modules, have been established in a constructive-computational way in [15] and [16] respectively. This paper investigates another modified algebra $J_q^0(n)$ appeared in [5], which has its own justification. More precisely, in Section 2 it is shown explicitly that the defining relations of $J_q^0(n)$ form a Gröbner-Shirshov basis, and from which a PBW K-basis of $J_q^0(n)$ is obtained. As consequences, several global structural properties of $J_q^0(n)$ are derived. In Section 3, it is shown that $J_q^0(n)$ may be equipped with an appropriate monomial ordering such that $J_q^0(n)$ is turned into a solvable polynomial algebra in the sense of [6], thereby $J_q^0(n)$ has an algorithmic Gröbner basis theory for (two-sided, one-sided) ideals and modules. As consequences, more structural properties of $J_a^0(n)$ are derived, and several structural-computational properties of modules over $J_q^0(n)$ are obtained.

Throughout this paper, K denotes a field of characteristic 0, $K^* = K - \{0\}$, and all K-algebras considered are associative with multiplicative identity 1. If S is a nonempty subset of an algebra A, then $\langle S \rangle$ is written for the two-sided ideal of A generated by S.

2. The Gröbner-Shirshov Defining Relations of $J_q^0(n)$, Some Consequences

In this section it is shown that the defining relations of the quantized matrix algebra $J_q^0(n)$ constitute a Gröbner-Shirshov basis. As consequences, several global structural properties of $J_q^0(n)$ are derived. For classical Gröbner-Shirshov basis theory of noncommutative associative free algebras, one is referred to, for instance [2].

Let $K = \mathbb{C}$ be the field of complex numbers, $I(n) = \{(i,j) \mid i,j = 1,2,\dots,n\}$ with $n \geq 2$, and let $J_q^0(n)$ be the associative K-algebra generated by n^2 elements J_{ij} $(i,j = 1,....,n)$ subject to the relations:

$$\begin{array}{ll} J_{ij}J_{st} = q^{s+t-i-j}J_{st}J_{ij}, & \text{if } (s-i)(t-j) \leq 0, \\ J_{st}J_{ij} = q^{i+j-t-s+2}J_{ij}J_{st} - q^{i-s+1}(q-q^{-1})J_{it}J_{sj}, & \text{if } s > i\ ,\ t > j, \end{array}$$

Where $i,j,s,t = 1,2,...,n$ and $q \in K^*$ is the quantum parameter. Now, bearing in mind the defining relations of $J_q^0(n)$, the same set of symbols $J = \{J_{ij} \mid (i,j) \in I(n)\}$ to denote the generating set of the free associative K-algebra $K\langle J\rangle$, and let S be the subset of $K\langle J\rangle$ consisting of elements

$$\begin{array}{ll} f_{ijst} = J_{ij}J_{st} - q^{s+t-i-j}J_{st}J_{ij}, & \text{if } i > s,\ t \geq j, \\ h_{stij} = J_{st}J_{ij} - q^{i+j-t-s+2}J_{ij}J_{st} + q^{i-s+1}(q-q^{-1})J_{it}J_{sj}, & \text{if } s > i,\ t > j \end{array}$$

Then, $J_q^0(n) \cong K\langle J\rangle/\langle S\rangle$ as K-algebra, where $\langle S\rangle$ denotes the (two-sided) ideal of $K\langle J\rangle$ generated by S, i.e., $J_q^0(n)$ is presented as a quotient of $K\langle J\rangle$. Our aim below is to show that S forms a Gröbner-Shirshov basis with respect to a certain monomial ordering on $K\langle J\rangle$. To this end, the deg-lex ordering $\prec_{d\text{-}lex}$ (i.e., the degree-preserving lexicographic ordering) on J^* is employed below, where J^* is the set of all mono words in J, i.e., all words of finite length like $u = J_{ij}J_{kl}...J_{st}$. More precisely, first take the lexicographic ordering $\prec_{lex}$ on J^* which is the natural extension of the ordering on the set J of generators of $K\langle J\rangle$: for $J_{ij}, J_{kl} \in J$,

$$J_{ij} < J_{st} \Leftrightarrow \begin{cases} i < s, \\ \text{or } i = s \text{ and } j > t. \end{cases}$$

and for two words $u = J_{i_1j_1}J_{i_2j_2}...J_{i_sj_s}, v = J_{s_1t_1}J_{s_2t_2}...J_{s_kt_k} \in J^*$,

$$\begin{array}{lll} u \prec_{lex} v & \Leftrightarrow & \text{there exists an } m \geq 1, \text{ such that} \\ & & J_{i_1j_1} = J_{s_1t_1}, J_{i_2j_2} = J_{s_2t_2}, \dots, J_{i_{m-1}j_{m-1}} = J_{s_{m-1}t_{m-1}}, \\ & & \text{but } J_{i_mj_m} < J_{s_mt_m}. \end{array}$$

(note that conventionally the empty word $1 < J_{st}$ for all $J_{st} \in J$). For instance

$$J_{32}J_{21}J_{31} \prec_{lex} J_{43}J_{13}J_{43} \prec_{lex} J_{42}J_{23}J_{41}.$$

And then, by assigning each J_{st} the degree 1, $1 \leq s,t \leq n$, and writing $|u|$ for the degree (i.e., length) of a word $u \in J^*$, the deg-lex ordering $\prec_{d\text{-}lex}$ is defined on the set J^*: for $u,v \in J^*$,

$$u \prec_{d\text{-}lex} v \Leftrightarrow \begin{cases} |u| < |v|, \\ \text{or } |u| = |v| \text{ and } u \prec_{lex} v. \end{cases}$$

For instance,

$$J_{24}J_{11} \prec_{d\text{-}lex} J_{32}J_{24} \prec_{d\text{-}lex} J_{32}J_{21} \prec_{d\text{-}lex} J_{11}J_{12}J_{13}.$$

It is straightforward to check that $\prec_{d\text{-}lex}$ is a monomial ordering on $K\langle J\rangle$, namely, $\prec_{d\text{-}lex}$ is a well-ordering and

$$u \prec_{d\text{-}lex} v \text{ implies } wur \prec_{d\text{-}lex} wvr \text{ for all } u,v,w,r \in J^*$$

With the monomial ordering $\prec_{d\text{-}lex}$ constructed above, the next goal is to prove the following result.

Theorem 2.1 With notation as fixed above, let $I = \langle S\rangle$ be the ideal of $J_q^0(n)$ generated by S. Then, with respect to the monomial ordering $\prec_{d\text{-}lex}$ on $K\langle J\rangle$, the set S is a Gröbner-Shirshov basis of the ideal I, i.e., the defining relations of $J_q^0(n)$ constitute a Gröbner-Shirshov basis.

Proof By [2], it is sufficient to check that all compositions determined by elements in S are trivial modulo S. In doing so, two more notations are fixed first. For an element $f \in \langle J\rangle$, $\overline{f}$ is written for the leading mono word of f with respect to $\prec_{d\text{-}lex}$, i.e., if $f = \sum_{i=0}^{s}\lambda_iu_i$ with $\lambda_i \in K^*$, $u_i \in J^*$, such that $u_1 \prec_{d\text{-}lex} u_2 \prec_{d\text{-}lex} \dots \prec_{d\text{-}lex} u_s$, then $\overline{f} = u_s$. Thus, the set S of defining relations of $J_q^0(n)$ has the set of leading mono words

$$\overline{S} = \left\{ \begin{array}{l} \overline{f}_{ijst} = J_{ij}J_{st}, \text{if } i > s, t \geq j, \\ \overline{g}_{stij} = J_{st}J_{ij}, \text{if } s > i, t > j. \end{array} \right\}$$

By means of $\overline{S}$ above, the compositions of intersections determined by elements in S, are presented as follows:

1. The case $(f_{ijst} \wedge f_{stkl})$ with $\omega = J_{ij}J_{st}J_{kl}$, where $k < s < i, t \geq j, t > l$. Since in this case $\omega_1 = J_{ij}J_{st}J_{kl} = \overline{f}_{ijst}J_{kl} = J_{ij}\overline{f}_{stkl}$ with $f_{ijst} = J_{ij}J_{st} - q^{s+t-i-j}J_{st}J_{ij}$, where $i \geq s$ and $j \leq t$, and $f_{stkl} = J_{st}J_{kl} - q^{k+l-s-t}J_{kl}J_{st}$, where $s \geq k$ and $l \geq t$, it follows that

$$\begin{aligned}
(f_{ijst}, f_{stkl})_w &= f_{ijst}J_{kl} - J_{ij}f_{stkl} \\
&= -q^{s+t-i-j}J_{st}J_{ij}J_{kl} + q^{k+l-s-t}J_{ij}J_{kl}J_{st} \\
&\equiv -q^{s+t+k+l-2i-2j}J_{st}J_{kl}J_{ij} + q^{2k+2l-s-t-i-j}J_{kl}J_{ij}J_{st} \\
&\equiv -q^{2k+2l-2i-2j}J_{kl}J_{st}J_{ij} + q^{2k+2l-2i-2j}J_{kl}J_{st}J_{ij} \\
&\equiv 0 \bmod(S, w)
\end{aligned}$$

2. The case $(f_{ijst} \wedge g_{stkl})$ with $\omega_1 = J_{ij}J_{st}J_{kl}$, where $k \leq s < i, j \leq t \leq l$.
Since in this case $\omega_1 = J_{ij}J_{st}J_{kl} = \overline{f}_{ijst}J_{kl} = J_{ij}\overline{g}_{stkl}$ with $f_{ijst} = J_{ij}J_{st} - q^{s+t-i-j}J_{st}J_{ij}$, where $i \geq s$ and $j \leq t$, and $g_{stkl} = J_{st}J_{kl} - q^{k+l-t-s+2}J_{kl}J_{st} + q^{k-s+1}(q - q^{-1})J_{kt}J_{sl}$, where $s > k$ and $t > l$, there are three cases to deal with.
Case 1. If $j \leq l$, then $k < s < i, j \leq l < t$, and it follows that

$$\begin{aligned}
(f_{ijst}, g_{stkl})_{w_1} &= f_{ijst}J_{kl} - J_{ij}g_{stkl} \\
&= -q^{s+t-i-j}J_{st}J_{ij}J_{kl} + q^{k+l-t-s+2}J_{ij}J_{kl}J_{st} - q^{k-s+1}(q - q^{-1})J_{ij}J_{kt}J_{sl} \\
&\equiv -q^{s+t+k+l-2i-2j}J_{st}J_{kl}J_{ij} + q^{2k+2l-t-s-i-j+2}J_{kl}J_{ij}J_{st} \\
&\quad -q^{2k+t-s-i-j+1}(q - q^{-1})J_{kt}J_{ij}J_{sl} \\
&\equiv -q^{2k+2l-2i-2j+2}J_{kl}J_{st}J_{ij} + q^{t+2k+l-2i-2j+1}(q - q^{-1})J_{kt}J_{sl}J_{ij} \\
&\quad +q^{2k+2l-2i-2j+2}J_{kl}J_{st}J_{ij} - q^{2k+t+l-2i-2j+1}(q - q^{-1})J_{kt}J_{sl}J_{ij} \\
&\equiv 0 \bmod(S_1, w_1)
\end{aligned}$$

Case 2. If $j > l$, then $k < s < i, l < j < t$, and it follows that

$$\begin{aligned}
(f_{ijst}, g_{stkl})_{w_1} &= f_{ijst}J_{kl} - J_{ij}g_{stkl} \\
&= -q^{s+t-i-j}J_{st}J_{ij}J_{kl} + q^{k+l-t-s+2}J_{ij}J_{kl}J_{st} - q^{k-s+1}(q - q^{-1})J_{ij}J_{kt}J_{sl} \\
&\equiv -q^{s+t+k+l-2i-2j+2}J_{st}J_{kl}J_{ij} + q^{s+t+k-2i-j+1}(q - q^{-1})J_{st}J_{kj}J_{il} \\
&\quad +q^{2k+2l-t-s-j-i+4}J_{kl}J_{ij}J_{st} \\
&\equiv -q^{2k+2l-2i-2j+4}J_{kl}J_{st}J_{ij} + q^{t+2k+l-2i-2j+3}(q - q^{-1})J_{kt}J_{sl}J_{ij} \\
&\quad +q^{2k-2i+3}(q - q^{-1})J_{kj}J_{st}J_{il} - q^{2k+t-2i-j+2}(q - q^{-1})^2J_{kt}J_{sj}J_{il} \\
&\quad +q^{2k+2l-2i-2j+4}J_{kl}J_{st}J_{ij} - q^{2k-2i+3}(q - q^{-1})J_{kj}J_{st}J_{il} \\
&\quad -q^{2k+t+l-2i-2j+3}(q - q^{-1})J_{kt}J_{sl}J_{ij} + q^{2k+t-2i-j+2}(q - q^{-1})^2J_{kt}J_{sj}J_{il} \\
&\equiv 0 \bmod(S_1, w_1)
\end{aligned}$$

Case 3. If $k < s < i, l < j = t$, and it follows that

$$\begin{aligned}
(f_{ijst}, g_{stkl})_{w_1} &= f_{ijst}J_{kl} - J_{ij}g_{stkl} \\
&= -q^{s+t-i-j}J_{st}J_{ij}J_{kl} + q^{k+l-t-s+2}J_{ij}J_{kl}J_{st} - q^{k-s+1}(q - q^{-1})J_{ij}J_{kt}J_{sl} \\
&\equiv -q^{s+t+k+l-2i-2j+2}J_{st}J_{kl}J_{ij} + q^{s+t+k-2i-j+1}(q - q^{-1})J_{st}J_{kj}J_{il} \\
&\quad +q^{2k+2l-t-s-j-i+4}J_{kl}J_{ij}J_{st} - q^{2k+l-t-s-i+3}(q - q^{-1})J_{kj}J_{il}J_{st} \\
&\quad -q^{2k+t-s-i-j+1}(q - q^{-1})J_{kt}J_{ij}J_{sl} \\
&\equiv -q^{2k+2l-2i-2j+4}J_{kl}J_{st}J_{ij} + q^{t+2k+l-2i-2j+3}(q - q^{-1})J_{kt}J_{sl}J_{ij} \\
&\quad +q^{2k-2i+1}(q - q^{-1})J_{kj}J_{st}J_{il} + q^{2k+2l-2i-2j+4}J_{kl}J_{st}J_{ij} - q^{2k-2i+3}(q - q^{-1})J_{kj}J_{st}J_{il} \\
&\quad -q^{2k+t+l-2i-2j+3}(q - q^{-1})J_{kt}J_{sj}J_{il} + q^{2k+t-2i-j+2}(q - q^{-1})^2J_{kt}J_{sj}J_{il} \\
&\equiv 0 \bmod(S_1, w_1)
\end{aligned}$$

3. The case $(g_{stij} \wedge f_{ijkl})$ with $\omega_2 = J_{st}J_{ij}J_{kl}$, where $k \leq i < s, t > j, l \geq j$.
Since in this case $\omega_2 = J_{st}J_{ij}J_{kl} = \overline{g}_{stij}J_{kl} = J_{st}\overline{f}_{ijkl}$ with $g_{stij} = J_{st}J_{ij} - q^{i+j-t-s+2}J_{ij}J_{st} + q^{i-s+1}(q - q^{-1})J_{it}J_{sj}$, where $s > i$ and $t > j$, and $f_{ijkl} = J_{ij}J_{kl} - q^{k+l-i-j}J_{kl}J_{ij}$, where $i \geq k$ and $l \geq j$, there are three cases to deal with.
Case 1. If $t \leq l$, then $k \leq i < s, j < t \leq l$, and it follows that

$$\begin{aligned}
(g_{stij}, f_{ijkl})_{w_2} &= g_{stij}J_{kl} - J_{st}f_{ijkl}\\
&= -q^{i+j-t-s+2}J_{ij}J_{st}J_{kl} + q^{i-s+1}(q-q^{-1})J_{it}J_{sj}J_{kl} + q^{k+l-i-j}J_{st}J_{kl}J_{ij}\\
&\equiv -q^{i+j+k+l-2t-2s+4}J_{ij}J_{kl}J_{st} + q^{i+j+k-t-2s+3}(q-q^{-1})J_{ij}J_{kt}J_{sl}\\
&\quad +q^{i+k+l-2s-j+1}(q-q^{-1})J_{it}J_{kl}J_{sj} + q^{2k+2l-i-j-t-s+2}J_{kl}J_{st}J_{ij}\\
&\quad -q^{2k+l-i-j-s+1}(q-q^{-1})J_{kt}J_{sl}J_{ij}\\
&\equiv -q^{2k+2l-2t-2s+4}J_{kl}J_{ij}J_{st} + q^{2k-2s+3}(q-q^{-1})J_{kt}J_{ij}J_{sl}\\
&\quad +q^{2k+2l-2s-t-j+3}(q-q^{-1})J_{kl}J_{it}J_{sj} - q^{2k+l-2s-j+2}(q-q^{-1})J_{kt}J_{il}J_{sj}\\
&\quad +q^{2k+2l-2t-2s+4}J_{kl}J_{ij}J_{st} - q^{2k+2l-2s-t-j+3}(q-q^{-1})J_{kl}J_{it}J_{sj}\\
&\quad -q^{2k-2s+3}(q-q^{-1})J_{kt}J_{ij}J_{sl} + q^{2k+l-2s-j+2}(q-q^{-1})J_{kt}J_{il}J_{sj}\\
&\equiv 0 \text{ mod}(S_1, w_2)
\end{aligned}$$

Case 2. If $t > l$, then $k \le i < s, t > l > j$, and it follows that

$$\begin{aligned}
(g_{stij}, f_{ijkl})_{w_2} &= g_{stij}J_{kl} - J_{st}f_{ijkl}\\
&= -q^{i+j-t-s+2}J_{ij}J_{st}J_{kl} + q^{i-s+1}(q-q^{-1})J_{it}J_{sj}J_{kl} + q^{k+l-i-j}J_{st}J_{kl}J_{ij}\\
&\equiv -q^{i+j+k+l-2t-2s+2}J_{ij}J_{kl}J_{st} + q^{i+k+l-2s-j+1}(q-q^{-1})J_{it}J_{kl}J_{sj}\\
&\quad +q^{2k+2l-i-j-s-t}J_{kl}J_{st}J_{ij} - q^{2k+l-t-s-i+3}(q-q^{-1})J_{kj}J_{il}J_{st}\\
&\quad -q^{2k+t-s-i-j+1}(q-q^{-1})J_{kt}J_{ij}J_{sl}\\
&\equiv -q^{2k+2l-2t-2s+2}J_{kl}J_{ij}J_{st} + q^{2k+2l-2s-t-j+1}(q-q^{-1})J_{kl}J_{it}J_{sj}\\
&\quad +q^{2k+2l-2t-2s+2}J_{kl}J_{ij}J_{st} - q^{2k+2l-2s-t-j+1}(q-q^{-1})J_{kl}J_{it}J_{sj}\\
&\equiv 0 \text{ mod}(S_1, w_2)
\end{aligned}$$

Case 3. If $k < i < s, t > j = l$, and it follows that

$$\begin{aligned}
(g_{stij}, f_{ijkl})_{w_2} &= g_{stij}J_{kl} - J_{st}f_{ijkl}\\
&= -q^{i+j-t-s+2}J_{ij}J_{st}J_{kl} + q^{i-s+1}(q-q^{-1})J_{it}J_{sj}J_{kl} + q^{k+l-i-j}J_{st}J_{kl}J_{ij}\\
&\equiv -q^{i+j+k+l-2t-2s+4}J_{ij}J_{kl}J_{st} + q^{i+j+k-t-2s+3}(q-q^{-1})J_{ij}J_{kt}J_{sl}\\
&\quad +q^{i+k+l-2s-j+1}(q-q^{-1})J_{it}J_{kl}J_{sj} + q^{2k+2l-i-j-s-t+2}J_{kl}J_{st}J_{ij}\\
&\quad -q^{2k+l-i-j-s+1}(q-q^{-1})J_{kt}J_{sl}J_{ij}\\
&\equiv -q^{2k+2l-2t-2s+4}J_{kl}J_{ij}J_{st} + q^{2k-2s+3}(q-q^{-1})J_{kt}J_{ij}J_{sl}\\
&\quad +q^{2k+2l-2s-j-t+3}(q-q^{-1})J_{kl}J_{it}J_{sj} - q^{2k+l-2s-j+2}(q-q^{-1})^2J_{kt}J_{il}J_{sj}\\
&\quad +q^{2k+2l-2s-2t+4}J_{kl}J_{ij}J_{st} - q^{2k+2l-j-2s-t+3}(q-q^{-1})J_{kl}J_{it}J_{sj}\\
&\quad -q^{2k-2s+1}(q-q^{-1})J_{kt}J_{ij}J_{sl}\\
&\equiv 0 \text{ mod}(S_1, w_2)
\end{aligned}$$

4. The case $(g_{stij} \wedge g_{ijkl})$ with $\omega = J_{st}J_{ij}J_{kl}$, where $k < i < s, t > j, l < j < t$. Since in this case $\omega = J_{st}J_{ij}J_{kl} = \overline{g}_{stij}J_{kl} = J_{st}\overline{g}_{ijkl}$ with $g_{stij} = J_{st}J_{ij} - q^{i+j-t-s+2}J_{ij}J_{st} + q^{i-s+1}(q-q^{-1})J_{it}J_{sj}$, where $s > i$ and $t > j$, and $g_{ijkl} = J_{ij}J_{kl} - q^{k+l-i-j+2}J_{kl}J_{ij} + q^{k-i+1}(q-q^{-1})J_{kj}J_{il}$, where $i > k$ and $j > l$, it follows that

$$\begin{aligned}
(g_{stij}, g_{ijkl})_w &= g_{stij}J_{kl} - J_{st}g_{ijkl}\\
&= -q^{i+j-t-s+2}J_{ij}J_{st}J_{kl} + q^{i-s+1}(q-q^{-1})J_{it}J_{sj}J_{kl} + q^{k+l-i-j+2}J_{st}J_{kl}J_{ij}\\
&\quad -q^{k-i+1}(q-q^{-1})J_{st}J_{kj}J_{il}\\
&\equiv -q^{i+j+k+l-2t-2s+4}J_{ij}J_{kl}J_{st} + q^{i+j+k-t-2s+3}(q-q^{-1})J_{ij}J_{kt}J_{sl}\\
&\quad +q^{i+k+l-2s-j+3}(q-q^{-1})J_{it}J_{kl}J_{sj} - q^{i+k-2s+2}(q-q^{-1})^2J_{it}J_{kj}J_{sl}\\
&\quad +q^{2k+2l-i-j-t-s+4}J_{kl}J_{st}J_{ij} - q^{2k+l-j-i-s+3}(q-q^{-1})J_{kt}J_{sl}J_{ij}\\
&\quad -q^{2k+j-i-t-s+3}(q-q^{-1})J_{kj}J_{st}J_{il} + q^{2k-i-s+2}(q-q^{-1})^2J_{kt}J_{sj}J_{il}\\
&\equiv -q^{2k+2l-2t-2s+6}J_{kl}J_{ij}J_{st} + q^{j+2k+l-2t-2s+5}(q-q^{-1})J_{kj}J_{il}J_{st}\\
&\quad +q^{2t-2s+3}(q-q^{-1})J_{kt}J_{ij}J_{sl} + q^{2k+2l-2s-j-t+5}(q-q^{-1})J_{kl}J_{it}J_{sj}\\
&\quad -q^{2k+l-2s-j+4}(q-q^{-1})^2J_{kt}J_{il}J_{sj} - q^{2k+j-2s-t+4}(q-q^{-1})^2J_{kj}J_{it}J_{sl}\\
&\quad +q^{2k-2s+3}(q-q^{-1})^3J_{kt}J_{ij}J_{sl} + q^{2k+2l-2t-2s+6}J_{kl}J_{ij}J_{st}\\
&\quad -q^{2k+2l-j-t-2s+5}(q-q^{-1})J_{kl}J_{it}J_{sj} - q^{2k-2s+3}(q-q^{-1})J_{kt}J_{ij}J_{sl}\\
&\quad -q^{2k+j+l-2t-2s+5}(q-q^{-1})J_{kj}J_{il}J_{st} + q^{2k+j-t-2s+4}(q-q^{-1})^2J_{kj}J_{it}J_{sl}\\
&\quad +q^{2k+l-2s-j+4}(q-q^{-1})^2J_{kt}J_{il}J_{sj} - q^{2k-2s+3}(q-q^{-1})^3J_{kt}J_{ij}J_{sl}\\
&\equiv 0 \text{ mod}(S_1, w).
\end{aligned}$$

This finishes the proof of the theorem. □

Immediately, Theorem 2.1 has the following

Corollary 2.2 The quantized matrix algebra $J_q^0(n) \cong K\langle J\rangle/I$ has the linear basis, or more precisely, the PBW basis

$$\mathcal{B} = \left\{ J_{1n}^{k_{1n}} J_{1(n-1)}^{k_{1(n-1)}} \cdots J_{11}^{k_{11}} J_{2n}^{k_{2n}} \cdots J_{21}^{k_{21}} \cdots J_{nn}^{k_{nn}} \cdots J_{n1}^{k_{n1}} \;\middle|\; k_{ij} \in \mathbb{N}, (i,j) \in I(n) \right\}.$$

Proof With respect to the monomial ordering $\prec_{d\text{-}lex}$ on the set J^* of mono words of $K\langle J\rangle$, note that

$$\begin{aligned} &J_{1n} \prec_{d\text{-}lex} J_{1(n-1)} \prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{11} \prec_{d\text{-}lex} J_{2n} \prec_{d\text{-}lex} J_{2(n-1)} \prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{21} \\ &\prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{nn} \prec_{d\text{-}lex} J_{n(n-1)} \prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{n1}, \end{aligned}$$

and the Gröbner-Shirshov basis S of the ideal $J = \langle S\rangle$ has the set of leading mono words consisting of

$$\begin{aligned} \overline{f}_{ijst} &= J_{ij}J_{st}, \text{ with } J_{st} \prec_{d\text{-}lex} J_{ij} \text{ where } i > s, j \le t, \\ \overline{g}_{stij} &= J_{st}J_{ij}, \text{ with } J_{st} \prec_{d\text{-}lex} J_{ij} \text{ where } s > i, t > j. \end{aligned}$$

It follows from classical Gröbner-Shirshov basis theory that the set of normal forms of J^* (mod S) is given as follows:

$$\left\{ J_{1n}^{k_{1n}} J_{1(n-1)}^{k_{1(n-1)}} \cdots J_{11}^{k_{11}} J_{2n}^{k_{2n}} \cdots J_{21}^{k_{21}} \cdots J_{nn}^{k_{nn}} \cdots J_{n1}^{k_{n1}}, \;\middle|\; k_{ij} \in \mathbb{N},\ (i,j) \in I(n) \right\}.$$

Therefore, $J_q^0(n)$ has the desired PBW basis. □

To give some applications of Theorem 2.1 and Corollary 2.2, first recall three results of [11] in one proposition below, for the reader's convenience.

Proposition 2.3 Adopting notations used in [11], let $K\langle X\rangle = K\langle X_1, X_2, ..., X_{ni}\rangle$ be the free $K-$algebra with the set of generators $X = X_1, X_2, ..., X_n$, and let $\prec$ be a monomial ordering on $K\langle X\rangle$. Suppose that G is a Gröbner-Shirshov basis of the ideal $I = \langle \mathcal{G}\rangle$ with respect to $\prec$, such that the set of leading monomials $LM(\mathcal{G}) = \{X_jX_i | 1 \le i < j \le n\}$. Considering the algebra $A = K\langle X\rangle/I$, the following statements hold.

(i) [11, P.167, Example 3] The Gelfand-Kirillov dimension GK.dim$A = n$.

(ii) [11, P.185, Corollary 7.6] The global homological dimension gl.dim$A = n$, provided $\mathcal{G}$ consists of homogeneous elements with respect to a certain N-gradation of $K\langle X\rangle$. (Note that in this case $G^{\mathbb{N}}(A) = A$, with the notation used in loc. cit.)

(iii) [11, P.201, Corollary 3.2] A is a classical quadratic Koszul algebra, provided $\mathcal{G}$ consists of quadratic homogeneous elements with respect to the N-gradation of $K\langle X\rangle$ such that each X_i is assigned the degree 1, $1 \le i \le n$. (Note that in this case $G^{\mathbb{N}}(A) = A$, with the notation used in loc. cit.)

Remark Let $j_1j_2\cdots j_n$ be a permutation of $1, 2, \ldots, n$. One may notice from the respectively quoted references in Proposition 4.1 that if, in the case of Proposition 4.1, the monomial ordering $\prec$ employed there is such that

$$\begin{aligned} &X_{j_1} \prec X_{j_2} \prec \cdots \prec X_{j_n}, \text{ and} \\ &LM(\mathcal{G}) = \{X_{j_k}X_{j_t} \mid X_{j_t} \prec X_{j_k},\ 1 \le j_k, j_t \le n\}, \\ &\text{or} \\ &LM(\mathcal{G}) = \{X_{j_k}X_{j_t} \mid X_{j_k} \prec X_{j_t},\ 1 \le j_k, j_t \le n\}, \end{aligned}$$

then all results still hold true.

Applying Proposition 2.3 and the above remark to $J_q^0(n) \cong K\langle J\rangle/I$, the result below is obtained.

Theorem 2.4 The quantized matrix algebra $J_q^0(n)$ has the following structural properties.

(i) The Hilbert series of $J_q^0(n)$ is $\frac{1}{(1-t)^{n^2}}$.

(ii) The Gelfand-Kirillov dimension GK.dim$J_q^0(n) = n^2$.

(iii) The global homological dimension gl.dim$J_q^0(n) = n^2$.

(iv) $J_q^0(n)$ is a classical quadratic Koszul algebra.

Proof Recalling that with respect to the monomial ordering $\prec_{d\text{-}lex}$ on the set J^* of mono words of $K\langle J\rangle$,

$$J_{st} \prec_{d\text{-}lex} J_{ij} \Leftrightarrow \begin{cases} s < i,\ t < j, \\ s < i,\ j < t, \\ s < i,\ j = t, \\ s = i,\ t > j. \end{cases} \quad (i,j) \in I(n),$$

$$\begin{aligned} &J_{1n} \prec_{d\text{-}lex} J_{1(n-1)} \prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{11} \prec_{d\text{-}lex} J_{2n} \prec_{d\text{-}lex} J_{2(n-1)} \prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{21} \\ &\prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{nn} \prec_{d\text{-}lex} J_{n(n-1)} \prec_{d\text{-}lex} \cdots \prec_{d\text{-}lex} J_{n1}, \end{aligned}$$

it follows that the Gröbner-Shirshov basis S of the ideal $I = \langle S\rangle$ has the set of leading mono words consisting of

$$\begin{aligned} \overline{f}_{ijst} &= J_{ij}J_{st}, \text{ with } J_{st} \prec_{d\text{-}lex} J_{ij} \text{ where } i \ge s, j \le t, \\ \overline{g}_{stij} &= J_{st}J_{ij}, \text{ with } J_{st} \prec_{d\text{-}lex} J_{ij} \text{ where } s > i, t > j. \end{aligned}$$

This means that $J_q^0(n)$ satisfies the conditions of Proposition 2.3. Therefore, the assertions (i)-(iv)are established as follows.

(i) Since $J_q^0(n)$ has the PBW K-basis as described in Corollary 2.2, it follows that the Hilbert series of $J_q^0(n)$ is $\frac{1}{(1-t)^{n^2}}$. (ii) This follows from Theorem 2.1, Proposition 2.3(i).

Note that $J_q^0(n)$ is an N-graded algebra defined by a quadratic homogeneous Gröbner-Shirshov basis (Theorem

2.1), where each generator J_{ij} is assigned the degree 1, $(i,j) \in I(n)$. The assertions (iii) and (iv) follow from Proposition 2.3(ii) and Proposition 2.3(iii), respectively.

3. Equipping $J_q^0(n)$ with the Structure of a Solvable Polynomial Algebra, Some Consequences

In this section it is shown, by constructing an appropriate monomial ordering on the PBW basis of the quantized matrix algebra $J_q^0(n)$, that $J_q^0(n)$ is a solvable polynomial algebra in the sense of [6]. As consequences, some further more structural properties of $J_q^0(n)$ are derived, and several constructive-computational results for modules over $J_q^0(n)$ are gained.

For convenience, it is necessary to start by recalling the following definitions and notations. Suppose that a finitely generated K-algebra $A = K[a_1, \ldots, a_n]$ has the PBW K-basis $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$, and that $\prec$ is a total ordering on $\mathcal{B}$. Then every nonzero element $f \in A$ has a unique expression

$$\begin{aligned} f &= \lambda_1 a^{\alpha(1)} + \lambda_2 a^{\alpha(2)} + \cdots + \lambda_m a^{\alpha(m)}, \\ &\quad \text{such that } a^{\alpha(1)} \prec a^{\alpha(2)} \prec \cdots \prec a^{\alpha(m)}, \\ &\quad \text{where } \lambda_j \in K^*,\ a^{\alpha(j)} = a_1^{\alpha_{1j}} a_2^{\alpha_{2j}} \cdots a_n^{\alpha_{nj}} \in \mathcal{B},\ 1 \le j \le m. \end{aligned}$$

Since elements of $\mathcal{B}$ are conventionally called *monomials*, the *leading monomial of* f is defined as $LM(f) = a^{\alpha(m)}$, the *leading coefficient of* f is defined as $LC(f) = \lambda_m$, and the *leading term of* f is defined as $LT(f) = \lambda_m a^{\alpha(m)}$.

Definition 3.1 Suppose that the K-algebra $A = K[a_1, \ldots, a_n]$ has the PBW basis $\mathcal{B}$. If $\prec$ is a total ordering on $\mathcal{B}$ that satisfies the following three conditions:

1. $\prec$ is a well-ordering (i.e., every nonempty subset of $\mathcal{B}$ has a minimal element);
2. For $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$, if $a^\gamma \ne 1$, $a^\beta \ne a^\gamma$, and $a^\gamma = LM(a^\alpha a^\beta a^\eta)$, then $a^\beta \prec a^\gamma$ (thereby $1 \prec a^\gamma$ for all $a^\gamma \ne 1$);
3. For $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathcal{B}$, if $a^\alpha \prec a^\beta$, $LM(a^\gamma a^\alpha a^\eta) \ne 0$, and $LM(a^\gamma a^\beta a^\eta) \notin \{0, 1\}$, then $LM(a^\gamma a^\alpha a^\eta) \prec LM(a^\gamma a^\beta a^\eta)$, then $\prec$ is called a *monomial ordering* on $\mathcal{B}$ (or a monomial ordering on A).

Definition 3.2 A finitely generated K-algebra $A = K[a_1, \ldots, a_n]$ is called a *solvable polynomial algebra* if A has the PBW K-basis $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ and a monomial ordering $\prec$ on $\mathcal{B}$, such that for some $\lambda_{ji} \in K^*$ and $f_{ji} \in A$,

$$\begin{aligned} &a_j a_i = \lambda_{ji} a_i a_j + f_{ji},\ 1 \le i < j \le n, \\ &LM(f_{ji}) \prec a_i a_j \text{ whenever } f_{ji} \ne 0. \end{aligned}$$

Following the definitions above, it is ready now to give and prove the main result of this section.

Theorem 3.3 Let $J_q^0(n)$ be the quantized matrix algebra over the field $K = \mathbb{C}$. Then $J_q^0(n)$ is a solvable polynomial algebra in the sense of Definition 3.2.

Proof Let $I(n) = \{(i,j) \mid i,j = 1, 2, \cdots, n\}$ with $n \ge 2$. Recall that $J_q^0(n)$ is the associative K-algebra generated by the set of n^2 generators $J = \{J_{ij} \mid (i,j) \in I(n)\}$ subject to the relations:

$$\begin{array}{lll} \mathrm{F}_1: & J_{ij}J_{st} = q^{s+t-i-j} J_{st} J_{ij}, & \text{if } (s-i)(t-j) \le 0, \\ \mathrm{F}_2: & J_{st}J_{ij} = q^{i+j-t-s+2} J_{ij} J_{st} - q^{i-s+1}(q-q^{-1}) J_{it} J_{sj}, & \text{if } s > i\ ,\ t > j, \end{array}$$

where $i, j, s, t = 1, 2, ..., n$ and $q \in K^*$ is the quantum parameter, and by Corollary 2.2, $J_q^0(n)$ has the PBW K-basis

$$\mathcal{B} = \left\{ J_{1n}^{k_{1n}} J_{1(n-1)}^{k_{1(n-1)}} \cdots J_{11}^{k_{11}} J_{2n}^{k_{2n}} \cdots J_{21}^{k_{21}} \cdots J_{nn}^{k_{nn}} \cdots J_{n1}^{k_{n1}} \;\middle|\; k_{ij} \in \mathbb{N}, (i,j) \in I(n) \right\}.$$

In order to show that $\mathcal{B}$ may be equipped with a monomial ordering $\prec$ such that the condition of Definition 3.2 is satisfied, first observe that if $\mathcal{B}$ is rewritten as

$$\mathcal{B} = \{1,\ J_{i_1 j_1} J_{i_2 j_2} \cdots J_{i_k j_k} \mid (i_t, j_t) \in I(n),\ k \ge 1\},$$

then $\mathcal{B}$ may obviously have an ordering $\prec$ inherited from the ordering $\prec_{d\text{-}lex}$ employed on J^* in Section 2, namely, for $u = J_{i_1 j_1} J_{i_2 j_2} \cdots J_{i_k j_k}$, $v = J_{i'_1 j'_1} J_{i'_2 j'_2} \cdots J_{i'_t j'_t} \in \mathcal{B}$,

$$v \prec u \Leftrightarrow \begin{cases} |v| < |u|, \\ \text{or } |v| = |u| \text{ and } v \prec_{lex} u, \end{cases}$$

where each generators J_{ij} of $J_q^0(n)$ is assigned the degree 1. Clearly $\prec$ is a well-ordering on $\mathcal{B}$. It remains to show that $\prec$ satisfies the conditions (2) and (3) of Definition 3.1, and that with respect to $\prec$ on $\mathcal{B}$, the relations F_1 and F_2 satisfied by generators of $J_q^0(n)$ have the property required by Definition 3.2. To this end, note that Definition 3.2 requires that the product $a_j a_i$ of two generators must be a linear combination of monomials in $\mathcal{B}$, so that $LM(f_{ji})$ is well defined and

$LM(f_{ji}) \prec a_i a_j$ with respect to the ordering $\prec$ defined on $\mathcal{B}$. Bearing in mind this basic requirement, it is seen immediately that the definition of $\prec$ entails that the relations F_1 and F_2 satisfied by generators of $J_q^0(n)$ have the property required by Definition 3.2. Next, let $J_{ij}, J_{st}, J_{kl} \in J$, and suppose that $J_{st} \prec J_{kl}$. If (i,j) and (s,t) are such that $i > s$ and $j > t$, then the relation F_2 gives rise to

$$\begin{aligned} J_{ij}J_{st} \;&=\; q^{2-i-j+s+t}J_{st}J_{ij} + (q-q^{-1})q^{1-t+j}J_{sj}J_{it} \\ &\quad \text{with } J_{st}J_{ij}, J_{sj}J_{it} \in \mathcal{B} \\ &\quad \text{and } J_{sj}J_{it} \prec J_{st}J_{ij} = LM(J_{ij}J_{st}). \end{aligned}$$

On the other hand, if (i,j) and (k,l) are such that $i > k$ and $j > l$, then the relation F_2 gives rise to

$$\begin{aligned} J_{ij}J_{kl} \;&=\; q^{2-i-j+k+l}J_{kl}J_{ij} + (q-q^{-1})q^{1-l+j}J_{kj}J_{il} \\ &\quad \text{with } J_{kl}J_{ij}, J_{kj}J_{il} \in \mathcal{B} \\ &\quad \text{and } J_{kj}J_{il} \prec J_{kl}J_{ij} = LM(J_{ij}J_{kl}). \end{aligned}$$

Thus, it has been shown that if

$$\begin{aligned} &(i,j),(s,t) \in I(n) \text{ such that } i > s, j > t, \\ &(i,j),(k,l) \in I(n) \text{ such that } i > k, j > l, \end{aligned}$$

then

$$\begin{aligned} &J_{st} \prec d_{kl} \text{ implies } LM(J_{ij}J_{st}) = J_{st}J_{ij} \prec J_{kl}J_{ij} = LM(J_{ij}J_{kl}) \\ &\text{and the generating relations of } J_q^0(n) \text{ determined by } \mathrm{F}_2 \\ &\text{have the property required by Definition 3.2.} \end{aligned} \tag{1}$$

Similarly, in the case that

$$\begin{aligned} &(s,t),(i,j) \in I(n) \text{ such that } s > i, t > j, \\ &(k,l),(i,j) \in I(n) \text{ such that } k > i, l > j, \end{aligned}$$

with the aid of F_2 it is seen that

$$\begin{aligned} &J_{st} \prec J_{kl} \text{ implies } LM(J_{st}J_{ij}) = J_{ij}J_{st} \prec J_{ij}J_{kl} = LM(J_{kl}d_{ij}) \\ &\text{and the generating relations of } J_q^0(n) \text{ determined by } \mathrm{F}_2 \\ &\text{have the property required by Definition 3.2.} \end{aligned} \tag{2}$$

At this stage, bearing in mind the relations F_1, F_2, and the assertions (1) and (2) derived above, it is ready to conclude that

$$\begin{aligned} &\text{for any } j_{ij}, j_{st}, j_{kl} \in J, \text{ if } J_{st} \prec J_{kl}, \text{ then} \\ &LM(J_{ij}J_{st}) \prec LM(J_{ij}J_{kl}),\ LM(J_{st}J_{ij}) \prec LM(J_{kl}K_{ij}), and \\ &\text{the generating relations of } J_q^0(n) \text{ determined by } \mathrm{F}_1, \mathrm{F}_2, \\ &\text{have the property required by Definition 3.2.} \end{aligned} \tag{3}$$

Finally, by means of (1), (2), and (3) presented above, it is straightforward to check that the conditions (2) and (3) of Definition 3.1 are satisfied by $\prec$, thereby $\prec$ is a monomial ordering on $\mathcal{B}$, and consequently $J_q^0(n)$ is a solvable polynomial algebra in the sense of Definition 3.2, as desired. □

Now that $J_q^0(n) \cong K\langle J\rangle/\langle S\rangle$ is a solvable polynomial algebra by Theorem 3.3, it follows from [6] that every (two-sided, respectively one-sided) ideal of $J_q^0(n)$ and every submodule of a free (left) $J_q^0(n)$-module has a finite Gröbner basis with respect to a given monomial ordering, in particular, for one-sided ideals and submodules of free (left) modules there is a noncommutative Buchberger Algorithm which, nowadays, has been successfully implemented in the computer algebra system Plural [8]. At this point, it is possible to give several applications of Theorem 3.3 to $J_q^0(n)$ and their modules. In what follows, modules over $J_q^0(n)$ are meant *left $J_q^0(n)$-modules*.

Theorem 3.4 Let $J_q^0(n)$ be the quantized matrix algebra in the sense of [5]. Then the following statements hold.

(i) $J_q^0(n)$ is a Noetherian domain.

(ii) Let L be a nonzero left ideal of $J_q^0(n)$, and $J_q^0(n)/L$ the left $J_q^0(n)$-module. Considering Gelfand-Kirillov dimension, GK.dim$J_q^0(n)/L <$ GK.dim$J_q^0(n) = n^2$ holds true, and there is an algorithm for computing GK.dim$J_q^0(n)/L$.

(iii) Let M be a finitely generated $J_q^0(n)$-module. Then a finite free resolution of M can be algorithmically constructed, and the projective dimension of M can be algorithmically computed. Moreover, every finitely generated projective $J_q^0(n)$-module P is stably free, thereby the K_0-group of $J_q^0(n)$ is isomorphic to the additive group of integers $\mathbb{Z}$.

(iv) Let M be a finitely generated graded $J_q^0(n)$-module (note that $J_q^0(n)$ is an $\mathbb{N}$-graded algebra in which each generator has degree 1). Then a minimal homogeneous generating set of M can be algorithmically computed, and a minimal finite graded free resolution of M can be algorithmically constructed.

Proof (i) Though the property that $J_q^0(n)$ is a Noetherian domain may be (or may have been) established in some other ways, it is necessary to emphasize here that this property may also follow immediately from Theorem 3.3. More precisely, the property that $J_q^0(n)$ has no divisors of zero follows from the fact that $LM(fg) = LM(LM(f)LM(g)) \neq 0$ for all nonzero $f, g \in J_q^0(n)$, and that the Noetherianess of $J_q^0(n)$ follows from the fact that every nonzero one-sided ideal of a solvable polynomial algebra has a finite Gröbner basis (see [6]).

(ii) By Theorem 2.4(ii), GK.dim$J_q^0(n) = n^2$. Since $J_q^0(n)$ is a (quadratic) solvable polynomial algebra by Theorem 3.3, it follows from [10, CH.V] that GK.dim$J_q^0(n)/L < n^2$ (this may also follow from classical Gelfand-Kirillov dimension theory [7], for $J_q^0(n)$ is now a Noetherian domain), and that there is an algorithm for computing GK.dim$J_q^0(n)/L$.

(iii) This follows from [11, Ch.3].

(iv) This follows from [11, Ch.4]. □

This section is ended by concluding that the algebra $J_q^0(n)$ also has the elimination property for (one-sided) ideals in the sense of ([12], [13, A3]). To see this, let us first recall the Elimination Lemma given in [12]. Let $A = K[a_1, \dots, a_n]$ be a finitely generated K-algebra with the PBW basis $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ and, for a subset $U = \{a_{i_1}, ..., a_{i_r}\} \subset \{a_1, ..., a_n\}$ with $i_1 < i_2 < \cdots < i_r$, let

$$S = \left\{ a_{i_1}^{\alpha_1} \cdots a_{i_r}^{\alpha_r} \;\middle|\; (\alpha_1, ..., \alpha_r) \in \mathbb{N}^r \right\}, \quad V(S) = K\text{-span}S.$$

Lemma 3.5 [11, Lemma 3.1] With notation as fixed above, let L be a nonzero left ideal of A and A/L the left A-module defined by L. If there is a subset $U = \{a_{i_1}, \dots, a_{i_r}\} \subset \{a_1, \dots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$, such that $V(S) \cap L = \{0\}$, then

$$\text{GK.dim}(A/L) \geq r.$$

Consequently, if A/L has finite GK dimension GK.dim$(A/L) = d < n$ (= the number of generators of A), then

$$V(S) \cap L \neq \{0\}$$

holds true for every subset $U = \{a_{i_1}, ..., a_{i_{d+1}}\} \subset \{a_1, ..., a_n\}$ with $i_1 < i_2 < \cdots < i_{d+1}$, in particular, for every $U = \{a_1, \dots a_s\}$ with $d + 1 \leq s \leq n - 1$, $V(S) \cap L \neq \{0\}$ holds true. □

For stating the next theorem, it is convenient to write the set of generators of $J_q^0(n)$ as $J = \{J_1, J_2, \dots J_{n^2}\}$, i.e., $J_q^0(n) = K[J_1, J_2, \dots, J_{n^2}]$.

Theorem 3.6 With notation as fixed above, Let L be a nonzero left ideal of $J_q^0(n)$. Then the following two statements hold.

(i) GK.dim$J_q^0(n)/L < n^2 =$ GK.dim$J_q^0(n)$. If GK.dim$J_q^0(n)/L = t$, then

$$V(S) \cap L \neq \{0\}$$

holds true for every subset $U = \{J_{i_1}, J_{i_2}, ..., J_{t+1}\} \subset J$ with $i_1 < i_2 < \cdots < i_{t+1}$, in particular, for every $U = \{x_1, x_2 \dots x_s\}$ with $t + 1 \leq s \leq n^2 - 1$, $V(T) \cap L \neq \{0\}$ holds true.

(ii) Without exactly knowing the numerical value GK.dim$J_q^0(n)/L$, the elimination property for a left ideal $L = \sum_{i=1}^m J_q^0(n)\xi_i$ of $J_q^0(n)$ can be realized in a computational way, as follows:

Let $\prec$ be the monomial ordering on the PBW basis $\mathcal{B}$ of $J_q^0(n)$ as constructed in the proof of Theorem 3.3, and let $V(S)$ be as in (i). Then, employing an elimination ordering $<$ with respect to $V(S)$ (which can always be constructed if the existing monomial ordering on $\mathcal{B}$ is not an elimination ordering, see [13, Proposition 1.6.3]), a Gröbner basis $\mathcal{G}$ of L can be produced by running the noncommutative Buchberger algorithm for solvable polynomial algebras, such that

$$L \cap V(S) \neq \{0\} \Leftrightarrow \mathcal{G} \cap V(S) \neq \emptyset.$$

Proof (i) By Corollary 2.2, $J_q^0(n)$ has the PBW basis $\mathcal{B}$, Also by Theorem 2.4(ii), GK.dim$J_q^0(n) = n^2$, thereby GK.dim$J_q^0(n)/L < n^2$ by Theorem 3.4(ii), the desired elimination property follows from Lemma 3.5 mentioned above.

(ii) This follows from [13, Corollary 1.6.5]. □

Finally, it is necessary to point out that since $J_q^0(n)$ is now a solvable polynomial algebra, if $F = \oplus_{i=1}^s J_q^0(n)e_i$ is a free (left) $J_q^0(n)$-module of finite rank, then a similar (even much stronger) result of Theorem 3.6 holds true for any finitely generated submodule $N = \sum_{i=1}^m J_q^0(n)\xi_i$ of F. The detailed statements and proof is omitted here, but the interested reader may referred to [13, Section 2.4] for a comprehensive discussion.

4. Conclusion and Prospect of Further Exploration

In conclusion, by means of Gröbner-Shirshov basis theory for free associative algebras and Gröbner basis theory for (noncommutative) solvable polynomial algebras, several important structural properties of the quantized matrix algebra $J_q^0(n)$ and its modules have been established and realized in previous sections. As pointed out in the introduction part of this paper, the algebra $J_q^0(n)$ belongs to the class of quadratic matrix algebras $M_q^{\mathcal{P}}(n)$ associated to the quantized enveloping algebra $U_q(A_{2n-1})$, which was introduced by Jakobsen and Zhang in [5]. Combining the work of [15], [16], and the present investigation of $J_q^0(n)$, it is clear that the research results of loc. cit. may bring some prospects on effectively establishing and realizing structural properties of quantized matrix algebras (such as $M_q(n)$, $D_q(n)$, and $J_q^0(n)$) and their modules in a constructive-computational way. Thereby, in light of this viewpoint, in the forthcoming work, the whole class of quadratic matrix algebras $M_q^{\mathcal{P}}(n)$ associated to the quantized enveloping algebra $U_q(A_{2n-1})$ will be investigated, especially, except those structural properties as established before, the Auslander regularity, the Cohen-Macaulay property, and the Artin-Schelter regularity (see [1], [9], [14]), will be explored.

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References

[1] M. Artin, W. F. Schelter, Graded algebras of global dimension 3. *Adv. Math.*, 66 (2) (1987), 171-216.

[2] L. Bokut et al., *Gröbner-Shirshov Bases: Normal Forms, Combinatorial and Decision Problems in Algebra*. World Scientific Publishing, 2020. https://doi.org/10.1142/9287

[3] R. Dipper and S. Donkin, Quantum GL_n. *Proc, London Math. Soc.*, 63 (1991), 156-211.

[4] L. D. Faddeev, N. Yu. Reshetikhin, and L. A. Takhtajan, Quantization of Lie groups and Lie algebras. *Algebraic Analysis*, Academic Press (1988), 129-140.

[5] H. P. Jakobsen and H. Zhang , A class of quadratic matrix algebras arising from the quantized enveloping algebra $U_q(A_{2n-1})$. *J. Math. Phys.*, (41) (2000), 2310-2336.

[6] A. Kandri-Rody and V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type. *J. Symbolic Comput.*, 9 (1990), 1-26. Also available as: Technical Report University of Passau, MIP-8807, March 1988.

[7] G.R. Krause and T.H. Lenagan, *Growth of Algebras and Gelfand-Kirillov Dimension*. Graduate Studies in Mathematics. American Mathematical Society, 1991.

[8] V. Levandovskyy and H. Schönemann, Plural: a computer algebra system for noncommutative polynomial algebras. In: *Proc. Symbolic and Algebraic Computation*, International Symposium ISSAC 2003, Philadelphia, USA, 2003, 176-183.

[9] T. Levasseur, Some properties of noncommutative regular graded rings. *Glasgow Math. J.*, 34 (1992), 277-300.

[10] H. Li, *Noncommutative Gröbner Bases and Filtered-graded Transfer*. Lecture Notes in Mathematics, Vol. 1795, Springer, 2002. https://doi.org/10.1007/b84211

[11] H. Li, *Gröbner Bases in Ring Theory*. World Scientific Publishing Co., 2011. https://doi.org/10.1142/8223

[12] H. Li, An elimination lemma for algebras with PBW bases. *Communications in Algebra*, 46 (8) (2018), 3520-3532. https://doi.org/10.1080/00927872.2018.1424863

[13] H. Li, *Noncommutative Polynomial Algebras of Solvable Type and Their Modules: Basic Constructive-Computational Theory and Methods*. Chapman and Hall/CRC Press, 2021.

[14] H. Li, F. Van Oystaeyen, (1996, 2003). *Zariskian Filtrations*. K-Monograph in Mathematics, Vol. 2. Kluwer Academic Publishers, Berlin Heidelberg: Springer-Verlag.

[15] Lina Niu, Rabigul Tuniyaz, Structural properties of the quantized matrix algebra $D_q(n)$ established by means of Gröbner-Shirshov basis theory. arXiv: 2201.00631v2 [math. RA]. https://doi.org/10.48550/arXiv.2201.00631

[16] Rabigul Tuniyaz, On the standard quantized matrix algebra $M_q(n)$: from a constructive-computational viewpoint. *Journal of Algebra and Its Applications*. Published online: 18 January 2023. https://doi.org/10.1142/S0219498824500920