

The Least-Energy Sign-Changing Solutions for Planar Schrödinger-Newton System with an Exponential Critical Growth

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Abstract: In 1990, the notion of critical growth in \mathbb{R}^2 was introduced by Adimurthi and Yadava. Soon afterwards, de Figueiredo, Miyagaki and Ruf also studied the solvability of elliptic equations in dimension two. They treated the problems in the subcritical and the critical case. In 2019, Alves and Figueiredo proved the existence of a positive solution for a planar Schrödinger-Poisson system, where the nonlinearity is a continuous function with exponent critical growth. Also, in 2020, Chen and Tang investigated the planar Schrödinger-Poisson system with the critical growth nonlinearity. Under the axially symmetric assumptions, they obtained infinitely many pairs solutions and ground states. In this work, motivated by the works mentioned above and Z. *Angew. Math. Phys.*, 66 (2015) 3267-3282, Z. *Angew. Math. Phys.*, 67 (2016) 102, 18, we study the planar Schrödinger-Newton system with a Coulomb potential where the nonlinearity f is autonomous nonlinearity which belongs to C^1 and satisfies super-linear at zero and exponential critical at infinity. Moreover, we need that f satisfies the Nehari type monotonic condition. We obtain a least-energy sign-changing solution via the variational method. To be more precise, we define the sign-changing Nehari manifold. And the least-energy sign-changing solution is obtained by minimizing the energy functional on the sign-changing Nehari manifold.

Keywords: Schrödinger-Newton System, The Exponential Critical Growth, Sign-Changing Solutions, Sign-Changing Nehari Manifold

1. Introduction and Main Results

In the present paper, we are concerned with the following Schrödinger-Newton system with a Coulomb potential

$$\begin{cases} -\Delta u + u + 2\pi\phi u + \int_{\mathbb{R}^2} \frac{[u(y)]^2}{|x-y|} dy = f(u), & \text{in } \mathbb{R}^d, \\ \Delta\phi = u^2, & \text{in } \mathbb{R}^d. \end{cases} \quad (1)$$

Problems of the type (1) arise in many problems from physics. And we refer the readers to [17], where (1) appears in a quantum mechanical context in the case $d \leq 3$. For the case $d = 3$, (1) is called the Schrödinger-Poisson system and it has been well studied, see for example [3, 5, 16, 20, 22] and the references therein. However, much less is known about the

case $d = 2$.

Motivated by the papers [2, 10, 12, 15, 19], the purpose of this paper is to study the existence of least-energy sign-changing solutions of the planar problem (1) with an exponential critical growth (see [2]). We mention that this notion of criticality was introduced by Adimurthi and Yadava [1], see also de Figueiredo, Miyagaki and Ruf [13]. We refer the readers to [6, 7, 8, 9, 11] for related problems and for recent advances on planar Schrödinger-Newton system. In order to state our main result, we assume that

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(u) = o(|u|)$ as $u \rightarrow 0$;

(f₂) There exists an $\alpha_0 > 0$ such that

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{\exp(\alpha u^2)} = 0 \text{ for } \alpha > \alpha_0,$$

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{\exp(\alpha u^2)} = +\infty \text{ for } \alpha < \alpha_0;$$

(f₃) $\frac{f(u)}{|u|^3}$ is a strictly increasing function of $u \in \mathbb{R} \setminus \{0\}$;

(f₄) There is a $q > 4$ such that $F(u) \geq \frac{1}{q}|u|^q$, where $F(u) = \int_0^u f(t)dt$.

Remark 1.1. $f(u) = |u|^5 \exp(\frac{\pi-1}{m}u^2)$ satisfies the assumptions (f₁) – (f₄).

Now we state our first result as follows.

Theorem 1.1. For $d = 2$, suppose that (f₁) – (f₄) are

satisfied. Then for any $\alpha \in (0, \frac{\pi}{m})$, where $m > 0$ will be defined in (20), (1) possesses a least-energy sign-changing solution.

Remark 1.2. We point out that we can define m without using the condition $\alpha \in (0, \frac{\pi}{m})$. It seems that the condition $\alpha \in (0, \frac{\pi}{m})$ is fussy but it is used to prove the minimizing sequence of m is bounded, and please see Lemma 3.1. Up to now, we have not been able to remove it.

The paper is organized as follows. In Section 2, let (f₁) – (f₄) be satisfied, we are going to establish the variational setting and give some preliminaries. In Section 3, with the additional condition $\alpha \in (0, \frac{\pi}{m})$, we are devoted to show that m is achieved and the minimizer is a critical point. The section 4 is the conclusion, we summarize our main results and the main idea of the proof.

2. Variational Setting and Preliminaries

We formally formulate problem (1) in a variational way as

$$I(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{4} \int_{\mathbb{R}^2} (\ln(|x|) * u^2)u^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} (|x|^{-1} * u^2)u^2 dx - \int_{\mathbb{R}^2} F(u)dx, \quad u \in X, \quad (2)$$

where

$$\ln(|\cdot|) * u^2(x) = \int_{\mathbb{R}^2} \ln(|x-y|)u^2(y)dy, \quad |\cdot|^{-1} * u^2(x) = \int_{\mathbb{R}^2} \frac{u^2(y)}{|x-y|}dy, \quad (3)$$

and X is defined as tha in [2].

Using (f₁) – (f₂), for each $\varepsilon > 0$ and $\alpha > \alpha_0$, we have

$$|F(u)| \leq \varepsilon|u|^2 + C(\varepsilon)|u|^p[\exp(\alpha|u|^2) - 1], \quad (4)$$

where $p > 2$. At first, according to [2, 4, 14], we give the following lemma which is used to estimate the nonlinearity.

Lemma 2.1. If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then $\exp(\alpha u^2) - 1 \in L^1(\mathbb{R}^2)$. Moreover, if $\|\nabla u\|_{L^2} \leq 1$, $\|u\|_{L^2} \leq M$ and $\alpha < 4\pi$, then there exists $C > 0$ independent of u such that

$$\int_{\mathbb{R}^2} [\exp(\alpha u^2) - 1]dx \leq C. \quad (5)$$

Similar to [2, 12], in view of Lemma 2.1, it is easy to check that the functional I belongs to $C^1(X, \mathbb{R})$. Define constraint:

$$M = \{u \in X : u^\pm \neq 0, \langle I'(u), u^+ \rangle = \langle I'(u), u^- \rangle = 0\}. \quad (6)$$

Lemma 2.2. For $u \in E$ with $u^\pm \neq 0$, then there exists a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in M$. Moreover, if $s, t \geq 0$ and $(s, t) \neq (s_u, t_u)$, we have

$$I(su^+ + tu^-) < I(s_u u^+ + t_u u^-). \quad (7)$$

Proof. For $u \in E$ with $u^\pm \neq 0$, similar to [18], we define

$$g(\mu, s, t) := s^2 \|u^+\|_{H^1}^2 + s^4 \int_{\mathbb{R}^2} \{[(\ln|x| * |u^+|^2) + (|x|^{-1} * |u^+|^2)]|u^+|^2\}dx$$

$$+ \mu s^2 t^2 \int_{\mathbb{R}^2} \{[(\ln|x| * |u^-|^2) + |x|^{-1} * |u^-|^2)]|u^+|^2\}dx - \int_{\mathbb{R}^3} f(su^+)su^+ dx, \quad (8)$$

and

$$h(\mu, s, t) := t^2 \|u^-\|_{H^1}^2 + t^4 \int_{\mathbb{R}^2} \{[(\ln|x| * |u^-|^2) + (|x|^{-1} * |u^-|^2)]|u^-|^2\}dx$$

$$+ \mu s^2 t^2 \int_{\mathbb{R}^2} \{[(\ln|x| * |u^+|^2) + |x|^{-1} * |u^+|^2)]|u^-|^2\}dx - \int_{\mathbb{R}^3} f(tu^-)tu^- dx. \quad (9)$$

Consider the solvability of the following system

$$\begin{cases} g(\mu, s, t) = 0, \\ h(\mu, s, t) = 0. \end{cases} \quad (10)$$

Define $\mathcal{Z} := \{\zeta : 0 \leq \zeta \leq 1 \text{ such that (10) is uniquely solvable in } (\mathbb{R}_+, \mathbb{R}_+)\}$, where $\mathbb{R}_+ = (0, \infty)$. Obviously, $g(0, s, t)$ is independent of t and $h(0, s, t)$ is independent of s . By (f_1) and (f_2) , noting that

$$\begin{aligned} \ln(r) &= \ln(1+r) - \ln(1+\frac{1}{r}), \\ 0 \leq \ln(1+\frac{1}{r}) &\leq \frac{1}{r}, \text{ for } r > 0, \end{aligned} \quad (11)$$

Thus,

$$\begin{aligned} \frac{\partial g(\mu, s, t)}{\partial s} \Big|_{(\mu_0, \bar{s}, \bar{t})} &= \bar{s} \|u^+\|_{H^1}^2 + 3\bar{s}^3 \int_{\mathbb{R}^2} \{[(\ln|x| * |u^+|^2) + (|x|^{-1} * |u^+|^2)]|u^+|^2\} dx \\ &\quad + \mu_0 \bar{s} \bar{t}^2 \int_{\mathbb{R}^2} \{[(\ln|x| * |u^-|^2) + |x|^{-1} * |u^-|^2]|u^+|^2\} dx - \int_{\mathbb{R}^3} f'(\bar{s}u^+) \bar{s} |u^+|^2 dx \\ &< -2\bar{s} \|u^+\|_{H^1}^2 - 2\mu_0 \bar{s} \bar{t}^2 \int_{\mathbb{R}^2} \{[(\ln|x| * |u^-|^2) + |x|^{-1} * |u^-|^2]|u^+|^2\} dx. \end{aligned} \quad (14)$$

In view of (11), we have

$$\frac{\partial g(\mu, s, t)}{\partial s} \Big|_{(\mu_0, \bar{s}, \bar{t})} < -2\bar{s} \|u^+\|_{H^1}^2 - 2\mu_0 \bar{s} \bar{t}^2 \int_{\mathbb{R}^2} [(\ln(1+|x|) * |u^-|^2)] |u^+|^2 dx < 0. \quad (15)$$

and

$$\frac{\partial g(\mu, s, t)}{\partial t} \Big|_{(\mu_0, \bar{s}, \bar{t})} = 2\mu_0 \bar{s}^2 \bar{t} \int_{\mathbb{R}^2} \{[(\ln|x| * |u^-|^2) + |x|^{-1} * |u^-|^2]|u^+|^2\} dx. \quad (16)$$

And for $h(\mu, s, t)$, we have similar results. Therefore, it yields that

$$\det \begin{pmatrix} \frac{\partial g(\mu, s, t)}{\partial s} \Big|_{(\mu_0, \bar{s}, \bar{t})} & \frac{\partial g(\mu, s, t)}{\partial t} \Big|_{(\mu_0, \bar{s}, \bar{t})} \\ \frac{\partial h(\mu, s, t)}{\partial s} \Big|_{(\mu_0, \bar{s}, \bar{t})} & \frac{\partial h(\mu, s, t)}{\partial t} \Big|_{(\mu_0, \bar{s}, \bar{t})} \end{pmatrix} > 0. \quad (17)$$

Jointly with the implicit function theorem, we can obtain the desired conclusion. We deduce that $I_\mu(su^+ + tu^-) \rightarrow -\infty$ uniformly as $|(s, t)| \rightarrow \infty$. Up to this stage, it is sufficient to check that a maximum point cannot be achieved on $(0 \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times 0)$. Otherwise, if $(0, \bar{t})$ is a maximum point. In view of (11), for $s > 0$ small enough, with [2, Lemma 2.5] in hand, it holds that

$$\frac{\partial I(su^+ + \bar{t}u^-)}{\partial s} \geq 2sC_1 + s^3C_2 + sC_3 - \varepsilon sC_4 - C(\varepsilon)s^pC_5 > 0, \quad (18)$$

which is a contradiction.

Since $u \in M$, by (f_3) , $f(u)u - 4F(u) \geq 0$, one has

$$I(u) = I(u) - \frac{1}{4} \langle I'(u), u \rangle \geq 0. \quad (19)$$

and similar to the proof of [2, Lemma 3.1], we have

$$g(0, s, t) \geq C_1 s^2 - C_2 s^4 - C_3 s^p. \quad (12)$$

It follows from (f_4) that

$$g(0, s, t) \leq C_1 s^2 + C_2 s^4 - C_3 s^q. \quad (13)$$

So $g(0, s, t) > 0$ for $s > 0$ small enough and $g(0, s, t) < 0$ for large s . Thus, there exists $s_0 > 0$ such that $g(0, s_0, t) = 0$ and (f_3) implies the uniqueness. Similarly, for $h(0, s, t)$. So we can obtain that $0 \in \mathcal{Z}$. Next, we want to prove that \mathcal{Z} is open and closed in $[0, 1]$. Suppose that $\mu_0 \in \mathcal{Z}$ and (\bar{s}, \bar{t}) is the unique solution of (4) with $\mu = \mu_0$. It follows from (f_3) that $f'(t)t^2 - 3f(t)t > 0$.

We can define the minimization problem

$$m := \inf_{u \in M} I(u). \quad (20)$$

3. The Proof of Theorem 1.1

Lemma 3.1. $m > 0$ can be achieved.

Proof. Let $\{u_n\} \subset M$ be a minimizing sequence. It follows that

$$m + o_n(1) = I(u_n) - \frac{1}{4} \langle I'(u_n), u \rangle \geq \frac{1}{4} \|u_n\|_{H^1}^2, \quad (21)$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. We can assume that $u_n^\pm \rightharpoonup u_0^\pm$ in $H^1(\mathbb{R}^2)$. Since $u_n \in M$, and \square

$$\int_{\mathbb{R}^2} (\ln|x| * u_n^2 + |x|^{-1} * u_n^2) |u_n^\pm|^2 dx \geq 0, \quad (22)$$

one has

$$\|u_n^\pm\|_{H^1}^2 \leq \int_{\mathbb{R}^2} f(u_n^\pm) u_n^\pm dx \leq \varepsilon \int_{\mathbb{R}^2} |u_n^\pm|^2 dx + C(\varepsilon) \int_{\mathbb{R}^2} |u_n^\pm|^p [\exp(\alpha |u_n^\pm|^2) - 1] dx. \quad (23)$$

If $\|u_n^\pm\|_{H^1} \rightarrow 0$, using Hölder inequality with $s > 1$ and $s \approx 1$, it holds that

$$\int_{\mathbb{R}^2} |u_n^\pm|^p [\exp(\alpha |u_n^\pm|^2) - 1] dx \leq \left(\int_{\mathbb{R}^2} |u_n^\pm|^{s'p} dx \right)^{\frac{1}{s'}} \left(\int_{\mathbb{R}^2} [\exp(\alpha s \|u_n^\pm\|_{H^1}^2 \left(\frac{|u_n^\pm|}{\|u_n^\pm\|_{H^1}} \right)^2) - 1] dx \right)^{\frac{1}{s}}. \quad (24)$$

By $\|u_n^\pm\|_{H^1} \rightarrow 0$, for n large enough, it holds that

$$\alpha s \|u_n^\pm\|_{H^1}^2 < 4\pi. \quad (25)$$

By Lemma 2.1 and Sobolev embedding, we get

$$(1 - C\varepsilon) \|u_n^\pm\|_{H^1}^2 \leq CC(\varepsilon) \|u_n^\pm\|_{H^1}^p. \quad (26)$$

It is absurd. So $m > 0$. And we claim that there are $R, \eta > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} |u_n^\pm|^2 dx \geq \eta. \quad (27)$$

If it is false, using Lion's lemma, we get $u_n^\pm \rightharpoonup 0$ in $L^t(\mathbb{R}^2)$ for all $t \in [2, \infty)$. Noting (21) and taking into account $\alpha \in (0, \frac{\pi}{m})$, it yields that

$$\alpha s \|u_n^\pm\|_{H^1}^2 < 4\pi. \quad (28)$$

So we have

$$\|u_n^\pm\|_{H^1}^2 \leq \int_{\mathbb{R}^2} f(u_n^\pm) u_n^\pm dx \leq \varepsilon \int_{\mathbb{R}^2} |u_n^\pm|^2 dx + C(\varepsilon) C \left(\int_{\mathbb{R}^2} |u_n^\pm|^{s'p} dx \right)^{\frac{1}{s'}} \rightarrow 0, \quad (29)$$

which is a contradiction. Thus, $u_0^\pm \neq 0$. As our problem is invariant by translations, according to [2, Corollary 4.4], up to a translation, we have that $\{u_n\}$ is bounded in X . Furthermore, it gives that

$$\int_{\mathbb{R}^2} f(u_n^\pm) u_n^\pm dx = \int_{\mathbb{R}^2} f(u_0^\pm) u_0^\pm dx + o_n(1) \quad (30)$$

and

$$\int_{\mathbb{R}^2} F(u_n^\pm) dx = \int_{\mathbb{R}^2} F(u_0^\pm) dx + o_n(1). \quad (31)$$

By Lemma 2.2, there exists $(s, t) \in (0, \infty) \times (0, \infty)$ such that $su_0^+ + tu_0^- \in M$. Moreover, by Fatou's Lemma, combining with (11), we get

$$\int_{\mathbb{R}^2} \{[(\ln |x| * |u_0|^2) + (|x|^{-1} * |u_0|^2)] |u_0^\pm|^2\} dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} \{[(\ln |x| * |u_n|^2) + (|x|^{-1} * |u_n|^2)] |u_n^\pm|^2\} dx. \quad (32)$$

Thus, it follows that

$$\|u_0^\pm\|_{H^1}^2 \leq \int_{\mathbb{R}^2} f(u_0^\pm) u_0^\pm dx \text{ and } s^2 \|u_0^\pm\|_{H^1}^2 \leq \int_{\mathbb{R}^2} f(su_0^\pm) su_0^\pm dx. \quad (33)$$

Therefore, we get $0 < s \leq 1$. Similarly, $0 < t \leq 1$. By (f_3) , we have

$$uf(u) - 4F(u) \quad (34)$$

is a nonnegative function, increasing in $|u|$. So we have

$$\begin{aligned} m &\leq I(su_0^+ + tu_0^-) - \frac{1}{4} \langle I'(su_0^+ + tu_0^-), su_0^+ + tu_0^- \rangle \\ &\leq \frac{1}{4} \|u_0\|_{H^1}^2 + \frac{1}{4} \int_{\mathbb{R}^2} [u_0 f(u) - 4F(u_0)] dx \\ &\leq \liminf_{n \rightarrow \infty} [I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle] = m. \end{aligned} \quad (35)$$

The claim is done. □

Lemma 3.2. Suppose that $u \in M$ and $I(u) = m$, then u is a sign-changing critical point of I .

Proof. Assuming the contrary that u is not a critical point of I . We can find $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfying $\psi^\pm \neq 0$ such that

$\langle I'(u), \psi \rangle \leq -1$. Choose $\varepsilon > 0$ very small such that

$$\langle I'(su^+ + tu^- + \sigma\psi), \psi \rangle \leq -\frac{1}{2}, \quad \forall (s, t, \sigma) \in B_\varepsilon(1, 1, 0), \quad (36)$$

where $B_\varepsilon(1, 1, 0)$ denotes the ball of radius ε centered at $(1, 1, 0)$. And introduce a cut-off function $0 \leq \eta \leq 1$ such that $\eta(s, t) = 1$ for $(s, t) \in \overline{B_{\frac{\varepsilon}{2}}(1, 1)}$ and $\eta(s, t) = 0$ for $(s, t) \in B_\varepsilon^c(1, 1)$. We perturb the original curve $su^+ + tu^-$, for $s, t \geq 0$,

$$\gamma(s, t) = \begin{cases} su^+ + tu^-, & \text{if } (s, t) \in B_\varepsilon^c(1, 1), \\ su^+ + tu^- + \varepsilon\eta(s, t)\psi, & \text{if } (s, t) \in B_\varepsilon(1, 1). \end{cases} \quad (37)$$

Noting that $\gamma(s, t)$ is a continuous curve in $(X, \|\cdot\|)$, and eventually choosing a smaller ε , we obtain that $\gamma(s, t)^\pm \neq 0$ for $(s, t) \in B_\varepsilon(1, 1)$. We have the following claim.

Claim 3.1.

$$\sup_{s, t \geq 0} I(\gamma(s, t)) < m. \quad (38)$$

If $(s, t) \in B_\varepsilon^c(1, 1)$, according to Lemma 2.2, $I(\gamma(s, t)) < m$. If $(s, t) \in B_\varepsilon(1, 1)$, by using the mean value theorem, we can find a suitable $\bar{\sigma} \in (0, \varepsilon)$ such that

$$\begin{aligned} \gamma(s, t) &= (su^+ + tu^-) + \langle I'(su^+ + tu^- + \bar{\sigma}\eta(s, t)\psi), \eta(s, t)\psi \rangle \\ &\leq I(su^+ + tu^-) - \frac{1}{2}\eta(s, t) \\ &< m, \end{aligned} \quad (39)$$

Where in the first inequality we have used (36).

However, observing that Lemma 2.2, it holds that for $(s, t) \in (1 - \frac{\varepsilon}{2}, 1) \times (1 - \frac{\varepsilon}{2}, 1)$,

$$\langle I'(su^+ + tu^- + \varepsilon\psi), u^+ \rangle > 0 \quad (40)$$

and

$$\langle I'(su^+ + tu^- + \varepsilon\psi), u^- \rangle > 0. \quad (41)$$

Similarly, for $(s, t) \in (1, 1 + \frac{\varepsilon}{2}) \times (1, 1 + \frac{\varepsilon}{2})$, we have

$$\langle I'(su^+ + tu^- + \varepsilon\psi), u^+ \rangle < 0 \quad (42)$$

and

$$\langle I'(su^+ + tu^- + \varepsilon\psi), u^- \rangle < 0. \quad (43)$$

Thus, there exists a unique pair $(s_0, t_0) \in (1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}) \times (1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2})$ such that

$$s_0u^+ + t_0u^- + \varepsilon\eta(s_0, t_0)\psi \in M, \quad (44)$$

which contradicts the claim. \square

4. Conclusions

In this research, under the exponential critical growth condition (f_2) and other suitable conditions, we prove that the planar Schrödinger-Newton system with a Coulomb potential admits a least-energy sign-changing solution. Our basic leitmotiv is to seek the minimizer of the energy functional on the sign-changing Nehari manifold. So, at first, we show the sign-changing Nehari manifold (see (6)) is well defined. Furthermore, we show that we can define the minimization problem. In addition, we prove that minimizer is achieved. Finally, we show that the minimizer is a critical point. Clearly,

the minimizer is sign-changing. Also, a example satisfying our assumptions is given in our research.

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