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Cohomology Operations and π-Strongly Homotopy Commutative Hopf Algebra

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Abstract: Steenrod operations are cohomology operations that are themselves natural transformations between cohomology functors. There are two distinct types of steenrod operations initially constructed by Norman Steenrod and called Steenrod squares and reduced p-th power operations usually denoted Sq and p^i respectively. Since their creation, it has been proved that these operations can be constructed in the cohomology of many algebraic structures, for instance in the cohomology of simplicial restricted Lie algebras, the cohomology of cocommutative Hopf algebras and the homology of infinite loop space. Later on J. P. May developped a general algebraic setting in which all the above cases can be studied. In this work we consider a cyclic group π of oder a fixed prime p and combine the π-strongly homotopy commutative Hopf algebra structure to the May's approach with the aim to build these natural transformations on the Hochschild cohomology groups. Moreover we give under some conditions a link of these natural transformations with the Gerstenhaber algebra structure.

Keywords: π-Strongly Homotopy Commutative Hopf Algebra, Cohomology Operations, Gerstenhaber Algebra

1. Introduction

One of the main computational tool of the homotopy theory is the Steenrod algebra defined by Henri Cartan in 1955 as an algebra of stable cohomology operations for a mod p cohomology and denoted by A_p, for a given prime number p[3]. The Steenrod operations were initially introduced by N. Steenrod in 1947 [15] for $p = 2$. Six years later, he sets out these natural transformations for odd prime p[16]. The relations between these operations were respectively determined by J. Adem [1] and H. Cartan[3]. In 1970 J. P. May generalized Steenrod operations and gave their complete algebraic traitment[10].

Using this general approach in conjunction with the π-strongly homotopy commutative algebra structure, Bitjong Ndombol and J. C. Thomas constructed in 2003 algebraic Steenrod operations on the negative Hochschild homology[12]. Their result was extended by the author in 2009 to the negative cyclic homology[17]. Later on the author introduced in [18], the notion the notion of π-strongly homotopy commutative coalgebra which is indeed an extention of the notion strongly homotopy commutative coalgebra[8].

In this work, we use May's general approach combined with the notion of π-stongly homotopy commutative algebra and coalgebra, and Hopf algebra up to homotopy to establish the following result:

Theorem
Let $A = \left(\{A^i\}_{i\geq 0}\,, A^i \xrightarrow{d} A^{i+1}\right)$ be a differential graded algebra over the prime field $\mathbb{F}_p$ such that:

1. $dimA^i < +\infty$, for all $i > 1$.
2. $A = (\{A^i\}_{i\geq 0}, d_A)$ is a π-strongly homotopy commutative augmented cochains algebra.
3. $A = (\{A^i\}_{i\geq 0}, d_A)$ is a Hofp algebra up to homotopy.

Then there exists algebraic Steenrod operations,

$$P^i : HH^*(A;A) \to \begin{cases} HH^{*+i}(A;A) \text{ if } p = 2 \\ HH^{*+(p-1)i}(A;A) \text{ if } p \text{ is odd} \end{cases}$$

satisfying

(a) $P^0(1) = 0$ if 1 denote the unit of the graded algebra $HH^*(A, A)$.

(b) $P^i_{|_{HH^k(A,A)}} = 0$, (respectively ξ) if $\begin{cases} i > k(\text{respectively } i = k) \text{ ,p=2} \\ i > 2k(\text{respectively } i = 2k) \text{ , } p \text{ is an odd prime} \end{cases}$

Moreover these operations are compatibles with the algebra homomorphisms commuting with the structural map θ.

These operations does not satisfy in general the Cartan formula and the Adam relation.

As it is well knwon from the work of Gerstenhaber[7], there is a Lie algebra structure on the supension of the Hochschild cohomology, $sHH^*(A, A)$, of differential graded algebra (A, d) with coefficients in itself. It is therefore natural to wonder whether there is a link between this Lie algebra structure and the above operations. The Answer is given through these statments.

Proposition

If $((A, d_A), \nu_A, \widetilde{\kappa}_A)$ *is an augmented* π*-Stongly homotopy commutative cochains Hopf algebra over a field* $\mathbb{F}_p$ *with caracteristic* $p = 2$*. Then for all* $m, n \in \mathbb{Z}$*, the composite:* $HH^m(A; A) \otimes HH^n(A; A) \stackrel{Sq^m \otimes Sq^n}{\longrightarrow} HH^{2m}(A; A) \otimes HH^{2n}(A; A) \stackrel{s \otimes s}{\longrightarrow} sHH^{2m}(A; A) \otimes sHH^{2n}(A; A) \stackrel{[-,-]}{\longrightarrow} sHH^{2(m+n)}(A; A)$ *is a trivial bilinear map.*

The paper is organized as follows: The second section is devoted to the review on some preliminaries and the construction of algebraic Steenrod operations. In the third section is given the proof of the main theorem. More precisely we define in this section the algebraic Steenrod operations on the Hochschild cohomology of a differential graded π-strongly homotopy commutative Hopf algebras (A, d_A) with coefficients in itself and provide a topological illustration on which the above operations can be constructed. In the last section, we show that these algebraic Steenrod operations are annihilated in some sense by the Gerstenhaber bracket.

2. Preliminaries

2.1. Recollection on the Bar and Cobar Construction

1. Recall that DA(resp. DC, resp. DM) denotes the category of augmented differential graded algebras(resp. coaugmented differential graded coalgebras, resp. differential graded modules). An object $A \in ObjDA$ is a $\mathbb{K}$-graded vector space $A = \{A^k\}_{k\geq 0}$ equipped with the structure maps $A^k \xrightarrow{d_A} A^{k+1}$, $\oplus_{k+l=n} A^k \otimes A^l \xrightarrow{m_A} A^n$, $\mathbb{K} \xrightarrow{\eta_A} A$, $A \xrightarrow{\varepsilon_A} \mathbb{K}$ and an exact sequence $0 \longrightarrow IA \xrightarrow{i_A} A \xrightarrow{\varepsilon_A} \mathbb{K}$.
   Similarly, An object $C \in ObjDC$ is a $\mathbb{K}$-graded vector space $C = \{C^k\}_{k\geq 0}$ endowed with the structure maps $C^k \xrightarrow{d_C} C^{k+1}$, $C^n \xrightarrow{\Delta_C} \oplus_{k+l=n} C^k \otimes C^l$, $\mathbb{K} \xrightarrow{\eta_C} C$, $C \xrightarrow{\varepsilon_C} \mathbb{K}$ and an exact sequence
   $$\mathbb{K} \xrightarrow{\eta_A} C \xrightarrow{j_A} jC \longrightarrow 0$$
2. Let p be a fixed prime and $\pi = \{1, \tau, \cdots \tau^{p-2}, \tau^{p-1}\}$ a finite group of order p. We work throughout this paper on the prime field $\mathbb{F}_p$ equipped with the trivial action of π and the ring group $\mathbb{F}_p[\pi]$ is an augmented algebra. All the modules and linear maps will be over the field $\mathbb{F}_p$; unadorned $\otimes$ means $\otimes_{\mathbb{F}_p}$ and $Hom_{\mathbb{F}_p}(-,-)$ will be simply denoted $Hom(-,-)$.
3. The k^{th} suspensions of a graded module V is a graded module $s^kV = \{(s^kV)^i\}_{i\in\mathbb{Z}}$ defined by $(s^kV)^j = V^{j+k}$ i.e. $|s^kv| = |v| - k$. The suspensions of the complex (V, d) is the complex $s(V, d) = (sV, -d \circ s)$. If we need the lower degree we will use the classical Kronecker convention: an object V with lower negative graduation has upper non negative graduation ($V_{-i} = V^i$).
4. Let's denote by $DA \xrightarrow{B} DC$(resp.$DC \xrightarrow{\Omega} DA$) the normalized bar construction(resp. the normalized cobar construction). B and Ω are two adjoint functors to each other. The standard generators of BA(resp. ΩC) are denoted $[a_1|a_2|a_3|\cdots|a_k]$(resp. $\langle c_1|c_2|c_3|\cdots|c_k\rangle$) and $[\ ] = 1 \in B^0A$ (resp. $\langle\ \rangle = 1 \in \Omega^0 C \cong \mathbb{K}$).
5. Recall that $f, g \in DA(A, A')$(resp. $DC(C, C')$) are homotopic in DA (resp. DC) if there exists a linear map $A \xrightarrow{h} A'$(resp. $C \xrightarrow{h} C'$) of degree -1 such that $f - g = d_{A'} \circ h + h \circ d_A$ and for all $x, y \in A$, $h(x.y) = h(x)g(y) + (-1)^{|x|} f(x)h(y)$(resp. $f - g = d_{A'} \circ h + h \circ d_A$). If f, g are homotopic in DA(resp. DC) we write $f \simeq_{DA} g$(resp.$f \simeq_{DC} g$).
6. From the adjunction mention above, one can respectively deduces that:
   (a) For any cochain algebra (A, d_A), there exists a natural quasi-isomorphism $\alpha_A \in DA(BA, A)$ corresponding by adjunction to the map $Id_{BA} \in DC(BA, BA)$. More precisely α_A defined by:
   $$\alpha_A(\langle[a_1|a_2|a_3|\cdots|a_k]\rangle) = \begin{cases} 0 \text{ if} k > 1 \\ a_1 \text{ if } k = 1 \\ 1 \text{ if } k = 0 \end{cases} \quad (1)$$
   is an homotopy equivalence with inverse $i_A \in DM(A, \Omega BA)$ defined by:
   $$i_A(a) = \begin{cases} \eta_A \circ \varepsilon_A(a) \text{ if } \varepsilon_A(a) \neq 0 \\ \langle[a]\rangle \text{ if not} \end{cases} \quad (2)$$
   and satisfying: $\alpha_A \circ i_A = Id_A$, $Id_{\Omega BA} - i_A \circ \alpha_A = d_{\Omega BA} \circ h + h \circ d_{\Omega BA}$, for some $\Omega BA \xrightarrow{h} \Omega BA$ such that $\alpha \circ h = 0$, $h \circ i_A = 0$, $h^2 = 0$
   (b) similarly for any cochain coalgebra (C, d_C), there exists a natural quasi-isomorphism $\beta_C \in DC(C, B(\Omega C))$ corresponding by adjunction to the map $Id_{\Omega C} \in DC(\Omega C, \Omega C)$. β_C is in fact an homotopy equivalence with inverse $\rho_C \in$

$DM(B\Omega C, C)$ respectively defined by:

$$\rho_C([\langle c_1|c_2|c_3|\cdots|c_k\rangle]) = \begin{cases} 0 \text{ if } k > 1 \\ c_1 \text{ if k = 1} \\ 1 \text{ if k = 0} \end{cases} \quad (3)$$

$$\beta_C(x) \begin{cases} \eta_C \circ \varepsilon_C(x) \text{ if } x \in im\eta_C \\ \\ [\langle x\rangle] \text{ if not} \end{cases} \quad (4)$$

2.2. Hochschild Cohomology

Let $M = (\{M^i\}_{i\in\mathbb{Z}}, d_M)$ and $N = (\{N^i\}_{i\in\mathbb{Z}}, d_N)$ be two differential graded modules. The Hom complex $Hom(M, N)$ is the differential graded module $(\{Hom^i(M, N)\}_{i\in Z}, D)$ with $Hom^i(M, N) = \prod_i Hom(M^k, N^{k+i})$ equipped with the differential D defined by $Df = d_N \circ f - (-1)^{|f|} f \circ d_M$. The Commutator $[f, g] = f \circ g - (-1)^{|f||g|} g \circ f$ gives to the differential graded $\mathbb{K}$-module End$(M) = Hom(M, M)$ the structure of differential graded Lie algebra.

Let (V, d_0) be a differential graded module and $T^c sV$ the tensor coalgebra generated by sV. The differential d_0 uniquely extended to a coderivation of $T^c sV$ denoted by d so that $(T^c sV, d)$ is a differential graded coalgebra. We set:

$$[sv_1|sv_2|\cdots|sv_k] = \begin{cases} [] & \text{if} \quad k = 0 \\ \\ sv_1 \otimes sv_2 \otimes \cdots \otimes sv_k & \text{if} \quad k \geq 1 \end{cases} \quad (5)$$

and

$$|[sv_1|sv_2|\cdots|sv_k]| = \epsilon_{k+1} := \begin{cases} 0 & \text{if} \quad k = 0 \\ \\ |sv_1| + |sv_2| + \cdots + |sv_k| & \text{if} \quad k \geq 1 \end{cases} \quad (6)$$

Thus d is explicitly defined by

$$d([sv_1|sv_2|\cdots|sv_k]) = \begin{cases} 0 & \text{if} \quad k = 0 \\ \\ -\sum_{i=1}^{i=k} (-1)^{\epsilon_i} [sv_1|sv_2|\cdots|d_0(sv_i)|\cdots|sv_k] & \text{if} \quad k \geq 1 \end{cases} \quad (7)$$

Fixe a differential graded algebra $A = (\{A^i\}_{i\in\mathbb{Z}}, d)$ and a (left) differential graded A-module $M = (\{M^i\}_{i\in\mathbb{Z}})$. It is well known that the linear map $Hom((sA)^{\otimes k}, M) \xrightarrow{b} Hom((sA)^{\otimes k+1}, M)$ $k \geq 1$ defined by $b(f)([]) = 0$ if k = 0.

$$\begin{aligned}(-1)^{|f|} b(f)([sa_1|sa_2|\cdots|sa_k]) &= (-1)^{|a_1||f|} a_1 f([sa_2|\cdots|sa_k]) \\ &- \sum_{i=2}^{i=k} (-1)^{\epsilon_i} f([sa_1|sa_2|\cdots|a_{i-1}a_i|\cdots|sa_k]) \\ &+ (-1)^{\epsilon_k} f([sa_1|sa_2|\cdots|sa_{k-1}]) a_k if \quad k \geq 2.\end{aligned} \quad (8)$$

satisfies $(D + b)^2 = 0$, where $D(f) = d_M \circ f - (-1)^{|f|} f \circ d$.

The complex $(Hom(T^c sA, M), D + b)$ is called the Hochschild complex of the differential graded algebra A with coefficients in M. The homology of this complex, denoted by $HH^*(A, M)$ is called the Hochschild cohomology of A with coefficients in M. Moreover the $(Hom(T^c sA, M), D + b)$ is the differential graded algebra for the usual product:

$$(f \cup g)([sa_1|sa_2|\cdots|sa_k]) = \sum_{i=0}^{i=k} (-1)^{|g|\epsilon_{i+1}} f([sa_1|\cdots|sa_i]) g([sa_{i+1}|\cdots|sa_k]) \quad (9)$$

Therefore the Hochschild cohomology is a graded algebra.

2.3. Algebraic Steenrod Operations

Let $(W, \delta) = \{(W_i, \delta_i)\}_{i\in\mathbb{N}}$, be a right projective π-chain complex such that $W \xrightarrow{\varepsilon_W} \mathbb{F}_p$ is the projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[\pi]$ and consider the cochain algebra $(A, d_A) = (\{A_i\}_{i\in\mathbb{N}}, d_A)$ whose product is m_A such that π acts trivially on A. One has:

1. The cochain complex $(Hom^*(W, A), D)$:

$$\begin{array}{ccc} Hom^k(W, A) = \prod_{i\geq 0} Hom(W_i, A^{k-i}) & \xrightarrow{D} & Hom^{k+1}(W, A) = \prod_{i\geq 0} Hom(W_i, A^{k-i+1}) \\ f & \longmapsto & D(f) = d_A \circ f - (-1)^{|f|} f \circ \delta_W. \end{array},$$

   is a π-cochain complex through the action: $(\sigma \circ f)(w) = f(\sigma w)$.
2. The evaluation map defined by $Hom(W, A) \xrightarrow{ev_0} A$ is a π-chain complexes homomorphism.

3. Let $W \xrightarrow{\psi_W} W \otimes W$ be any diagonal approximation, we have a non associative differential graded algebra structure on $Hom(W, A)$ given by the usual cup-product defined as follows:

$$\begin{array}{ccc} Hom^k(W,A) \otimes Hom^l(W,A) & \xrightarrow{\cup} & Hom^{k+l}(W,A) \\ f \otimes g & \longmapsto & f \cup g = m_A \circ f \otimes g \circ \psi_W \end{array}$$

4. π acts diagonally on $A^{\otimes p}$ and $W \otimes A^{\otimes p}$. Moreover if we denote by $m_A^p(resp. H(m_A)^p)$ the iterated product $a \otimes a \otimes \cdots \otimes a \longmapsto a_1(a_2(\cdots a_p)\cdots)$ (resp. the induced iterated product on $H(A)$ by m_A^p). Then the natural map $H(W \otimes A^{\otimes p}) \xrightarrow{\cong} H(A)^{\otimes p} \xrightarrow{H(m_A)} H(A)$ lifts to a π-linear chain map $W \otimes A^{\otimes p} \xrightarrow{\varrho} A$ which induces the π-linear chains map

$$\begin{array}{ccc} A^{\otimes p} & \xrightarrow{\tilde{\varrho}} & Hom(W,A) \\ u & \longmapsto & \left\{ \begin{array}{ccc} W & \xrightarrow{\tilde{\varrho}(u)} & A \\ w & \longmapsto & \tilde{\varrho}(u)(w) = (-1)^{|u||w|}\varrho(w \otimes u) \end{array} \right. \end{array},$$

such that $ev_0 \circ \tilde{\varrho} = m_A^p$ and $H(ev_0) \circ H(\tilde{\varrho}) = H(m_A)^p$. Hence for any $i \in \mathbb{Z}$ and $x \in H^n(A)$ there exists a well defined class

$$P^i(x) \in \left\{ \begin{array}{lll} H(A)^{n+i} & \text{if} & p = 2 \\ H(A)^{n+2i(p-1)} & \text{if} & p > 2 \end{array} \right. \tag{10}$$

such that:

(a)
$$P^i(1_A) = 0 \quad \text{if } i \neq 0$$

(b)
$$\text{if } p = 2, \left\{ \begin{array}{ll} P^i(x) = 0 & \text{if } i > n \\ P^i(x) = x^2 & \text{if } i = n \end{array} \right.$$

(c)
$$\text{if} p > 2, \left\{ \begin{array}{ll} P^i(x) = 0 & \text{if } 2i > n \\ P^i(x) = x^p & \text{if } 2i = n \end{array} \right.$$

Moreover these operations are compatibles with the algebra homomorphisms commuting with the structural map θ.

5. If we assume that $H(\tilde{\varrho})$ respects the product then the algebraic Steenrod operations defined by $\tilde{\varrho}$ satisfy the Cartan formula:

$$P^i(xy) = \sum_{i+j=k} P^i(x)P^j(y), \text{for } x, y \in H^\star(A) \tag{11}$$

The above cohomological operations does not satisfy in general the Cartan formula and the Adam relation.

3. Proof of the Theorem

3.1. π-strongly Homotopy Commutative Coalgebra Structure on BA

3.1.1. π-strongly Homotopy Commutative Algebra and Coalgebra Structures (For short π-shc-algebras and π-shc-coalgebras)

1. A strongly homotopy commutative algebras(shc algebras for short)is a triple $((A, d_A), \mu_A)$ where $(A, d_A) \in ObjetDA$ and $\mu_A \in DA(\Omega B(A^{\otimes 2}), \Omega BA)$ and satisfying the following:
   (a) $\alpha_A \circ \mu_A \circ i_{A\otimes A} = m_A$ with m_A the product in A.
   (b) $\alpha_A \circ \mu_A \circ \Omega(id_A \otimes \eta_A) \circ i_A = \alpha_A \circ \mu_A \circ \Omega(\eta_A \otimes id_A) \circ i_A = id_A$.
   (c) $\mu_A \circ \Omega B(\alpha_A \otimes id_A) \circ \Omega B(\mu_A \otimes id_A) \circ \chi_{(A\otimes A)\otimes A} \simeq_{DA} \mu_A \circ \Omega B(\alpha_A \otimes id_A) \circ \Omega B(\mu_A \otimes id_A) \circ \chi_{A\otimes(A\otimes A)}$.
   (d) $\mu_A \circ \Omega BT \simeq_{DA} \mu_A$
   Where T denotes the interchange map defined as follows: for all $x, y \in A$, $T(x \otimes y) = (1)^{|x||y|} y \otimes x$ and $\Omega B \Omega B((A \otimes A) \otimes A) \xleftarrow{\chi_{(A\otimes A)\otimes A}} |\Omega B(A \otimes A \otimes A) \xrightarrow{\chi_{A\otimes(A\otimes A)}} \Omega B(A \otimes \Omega B(A \otimes A))$ two natural differential graded algebras homomorphisms defined in [11].

A shc algebra $((A, d_A), \mu_A)$ is said to be a π-shc algebras if there exists a π-linear homomorphism of differential graded algebras $\Omega B(A^{\otimes p}) \xrightarrow{\widetilde{\kappa}_A} Hom(W, A)$ such that $ev_0 \circ \widetilde{\kappa}_A \simeq_{DA} \alpha_A \circ \mu_A^{(p)}$(see [13] for more details)

2. A strongly homotopy commutative coalgebras(shc coalgebras for short) is a triple $((C, d_C), \nu_C)$, where $(C, d_C) \in ObjetDC$and $\nu_C \in DC(B\Omega C, B\Omega(C^{\otimes 2}))$ and satisfying the following:
   (a) $P_{C\otimes C} \circ \nu_C \circ \beta_C = \Delta_C$, with Δ_C is the coproduct in C.
   (b) $P_C \circ B\Omega(\varepsilon_C \otimes id_C) \circ \nu_C \circ \beta_C = id_C = P_C \circ B\Omega(id_C \otimes \varepsilon_C) \circ \nu_C \circ \beta_C$.
   (c) $\chi_{C\otimes(C\otimes C)} \circ B\Omega(id_C \otimes (\nu_C \circ \beta_C)) \circ \nu_C \circ \beta_C \simeq_{DA} \chi_{(C\otimes C)\otimes C} \circ B\Omega((\nu_C \circ \beta_C) \otimes id_C) \circ (\nu_C \circ \beta_C)$.
   (d) $B\Omega(T) \circ \nu_C \circ \beta_C \simeq_{DC} \nu_C \beta_C$ (See [18] for more details).

   A shc-coalgebra (C, d_C, ν_C) is said to be a π-shc coalgebra if there exists a π-linear differental graded coalgebras homomorphism $Hom(C, W^\sharp) \xrightarrow{\kappa_C^\sharp} B\Omega(C^{\otimes p})$ such that: $\kappa_C^\sharp \circ coev_0 \simeq_{DC} \nu_C^{(p)} \circ \beta_C$. Where $coev_0$ denotes the coevaluation map(see [18])

3.1.2. π-Strongly Homotopy Comutative Coalgebra Structure on BA

Let $((A, d_A), \mu_A, \widetilde{\kappa}_A)$ be an augmented π-shc differential graded algebra which is also assumed to be a Hopf algebra up to homotopy. That is in addition to the π-shc algebra structure, (A, d_A) is also endowed with a differential graded algebras homomorphism $A \xrightarrow{\delta_A} A \otimes A$ such the following holds:

1. δ_A is associative up to homotopy.
2. $(Id_A \otimes \delta_A) \circ \delta_A \simeq_{DA} (\delta_A \otimes Id_A) \circ \delta_A$
3. δ_A is commutative up to homotopy: $\delta_A \circ T \simeq_{DA} \delta_A$
4. $((A, d_A), \delta_A)$ has a counit $A \xrightarrow{\varepsilon_A} \mathbb{K}$satisfying $(\varepsilon_A \otimes Id_A) \circ \delta_A = Id_A = (Id_A \otimes \varepsilon_A) \circ \delta_A$

Remark that the normalized bar construction of such differential graded algebras, BA has a shc coalgebra structure given by the composite $\nu_{BA} := B(q) \circ B\Omega B(\delta_A)$. Where q denotes the homotopy inverse of the quasi-isomorphism $B\Omega(sh)$, with sh, the shuffle map. Indeed figure 1 bellow is a commutative diagram.

$$\begin{tikzcd}
B(\Omega(BA)) \arrow[r, "B(\Omega(B(\delta_A)))"] & B(\Omega(B(A\otimes A))) \arrow[r, "B(q)"] & B(\Omega(BA\otimes BA)) \\
BA \arrow[u, "\beta_{BA}"] \arrow[r, "B(\delta_A)"] & B(A\otimes A) \arrow[u, "\beta_{B(A\otimes A)}"] \arrow[r, "q"] & (BA\otimes BA) \arrow[u, "\beta_{(BA\otimes BA)}"]
\end{tikzcd}$$

Figure 1. *The Shc structure construction.*

More over BA is a π-strongly homotopy commutative coalgebra. In fact the map $\beta_{BA} \in DC(BA, B\Omega BA)$ is a final object in the category of the trivialzed coextensions(see [18]) and $BA \overset{coev_0}{\longrightarrow} Hom(BA, W^\sharp)$, the coevaluation map , an object in the same category. We deduce from the universal property that there exists a unique differential graded coalgebras homorphism $Hom(BA, W^\sharp) \xrightarrow{\theta_A} B\Omega BA$ such that the diagram of figure 2 commutes.

$$\begin{tikzcd}
BA \arrow[r, "\cong"] \arrow[d, "\beta_{BA}"'] & B(A) \arrow[d, "coev_0"] \\
B\Omega(BA) & Hom(BA, W^\sharp). \arrow[l, "\theta_A"]
\end{tikzcd}$$

Figure 2. *The construction of the map θ_A.*

In the other hand, we obtain by using the fact that BA is a shc-differential graded coalgebra that the diagram of figure 3 commutes:

Set $\kappa^\sharp_{BA} := \nu_A \circ \theta_A$. We have $\kappa^\sharp_{BA} \circ coev_0 = \nu_A \circ \theta_A \circ coev_0$. Hence BA is endowed with the a π-shc coalgebra structure given by $\kappa^\sharp_{BA}$. Finally we obtain the diagram of figure 4 commutative up to homotopy.

Proposition 1.

Proposition 1. Let $((A, d_A), \nu_A, \widetilde{\kappa}_A)$ be an augmented π-shc differential graded Hopf algebra and (W, ψ_W) a coassociative graded coalgebra. There exists a natural chain map ϕ_A such that the diagram of figure 5 commutes:

More over, the map ϕ_A is a π-linear differential map with respect to the products.

Proof of proposition 1 *Proof*

$$\begin{tikzcd}
BA \arrow[rr, "\Delta_{BA}"] \arrow[d, "coev_0"'] & & B(A) \otimes BA \\
Hom(BA, W^\sharp) \arrow[r, "\theta_A"'] & B\Omega(BA) \arrow[r, "\nu_{BA}"] & B(\Omega(BA \otimes BA)) \arrow[u, "\rho_{BA \otimes BA}"]
\end{tikzcd}$$

Figure 3. *The construction of the map $\kappa^\sharp_{BA}$.*

$$\begin{tikzcd}
Hom(B(\Omega(BA \otimes BA)); \Omega B(A \otimes A)) \arrow[r, "\overline{\kappa}:=Hom(\kappa^\sharp_{BA}, \widetilde{\kappa}_A)"] \arrow[d, "Hom(\beta_{(BA \otimes BA)}, \alpha_A)"'] & Hom(Hom(BA, W^\sharp), Hom(W, A))) \arrow[dd, "Hom(coev_0, ev_0)"] \\
Hom(BA \otimes BA, A \otimes A) \arrow[d, "\cong"'] & \\
Hom(BA, A)^{\otimes 2} \arrow[r, "\cup"] & Hom(BA, A)
\end{tikzcd}$$

Figure 4. *The π-shc stucture construction.*

Consider firstly the map φ defined by:

$$\begin{array}{ccc}
Hom(W \otimes BA; \mathbb{K})^\sharp & \stackrel{\varphi}{\longrightarrow} & Hom(W^\sharp; BA) \\
w \otimes [a_1|a_2| \cdots |a_s] & \longmapsto & \varphi(w \otimes [a_1|a_2| \cdots |a_s])
\end{array}$$

such that for any $w'^\sharp \in W^\sharp$, $\varphi(w \otimes [a_1|a_2| \cdots |a_s])(w'^\sharp) = w'^\sharp(w) \cdot ([a_1|a_2| \cdots |a_s])$. Define the map θ as follows:

$$\begin{array}{ccc}
Hom(W \otimes BA, Hom(W, A)) & \stackrel{\theta}{\longrightarrow} & Hom(W, Hom(BA, A)) \\
\beta & \longmapsto & \theta(\beta)
\end{array}$$

such that $\begin{array}{ccc} W & \stackrel{\theta(\beta)}{\longrightarrow} & Hom(BA, A) \\ w & \longmapsto & \theta(\beta)(w) \end{array}$, with $\begin{array}{c} BA \stackrel{\theta(\beta)(w)}{\longrightarrow} A \\ x = [a_1|a_2| \cdots |a_s] \longmapsto \theta(\beta)(w)(x) \end{array}$ satisfying: $\theta(\beta)(w)(x) = \beta(w \otimes x)(e_0)$.

Set $\phi_A := \theta \circ Hom(\varphi, Hom(W, A))$. We obtain: $\phi_A(\gamma) = \theta \circ Hom(\varphi; Hom(W; A))(\gamma) = \theta(\gamma \circ \varphi)$
The evaluation map gives us the map defined as follows:

$$BA \stackrel{ev_0(\theta(\gamma \circ \varphi))}{\longrightarrow} Ax = [a_1|a_2| \cdots |a_s] \longmapsto (ev_0(\theta(\gamma \circ \varphi)))(x) = (\theta(\gamma \circ \varphi)(x))(e_0).$$

$$\begin{tikzcd}
Hom(Hom(W, (BA)^\sharp)^\sharp, Hom(W, A)) \arrow[rr, "\phi_A"] \arrow[dr, "Hom(Coev_0, ev_0)"'] & & Hom(W, Hom(BA, A)) \arrow[dl, "ev_0"] \\
& Hom(BA, A) &
\end{tikzcd}.$$

Figure 5. *The construction of the map ϕ_A.*

From the definitions of φ and the coevaluation map $coev_0$, we deduce the commutativity of the above diagram.

To check the π- linearity of the map ϕ, let (A, d_A) be a π-module under the trivial action of π on A. An consider the diagonal action on $W \otimes BA$ defined as follows: for all $\sigma \in \pi$ and $w \otimes x \in W \otimes BA$, $(w \otimes x) \cdot \sigma = w \cdot \sigma \otimes \sigma^{-1} \cdot x$.

From these actions we deduce the structure of π-module on $Hom(W \otimes BA; Hom(W, A))$ by:

$$\begin{aligned}(\sigma \cdot \beta)(w \otimes x) &= \beta((w \otimes x) \cdot \sigma) = \beta(w \cdot \sigma \otimes \sigma^{-1} \cdot x) \\ &= \beta(w \cdot \sigma \otimes x),\end{aligned}$$

Here after we have $\phi(\tau \cdot \gamma) = \tau \cdot \phi(\gamma)$, thus the map ϕ is a π-linear map. More over this map commutes with the differentials since the maps φ and θ commute with the differentials. Hence it is a π-chains map.

To end the proof of the proposition, it remains to show that the map ϕ respect the product. For this purpose consider $f, g \in Hom[Hom(W, (BA)^\sharp)^\sharp; Hom(W, A)]$. Since θ and φ are respectively algebras and coalgebras homomorphisms, we obtain for all $w \in W, x \in BA$,

$$\begin{aligned}\phi(f\overline{\cup}g)(w)(x) &= \phi[m_{Hom(W,A)} \circ (f \otimes g) \circ \Delta_{Hom(W,(BA)^\sharp)^\sharp}](w)(x) \\ &= \theta(m_{Hom(W,A)} \circ (f \otimes g) \circ \Delta_{Hom(W,(BA)^\sharp)^\sharp} \circ \varphi)(w)(x) \\ &= (m_{Hom(W,A)} \circ (f \otimes g) \circ (\varphi \otimes \varphi))(\psi_W(w) \otimes \Delta_{BA}(x))(e_0) \\ &= (m_A \circ ((f \circ \varphi) \otimes (g \circ \varphi)) \circ (\psi_W(w) \otimes \Delta_{BA}(x))(\psi_W(e_0)) \\ &= (\phi(f) \cup \phi(g))(w)(x).\end{aligned}$$

Here $\overline{\cup}$ denote the product on $Hom[Hom(W, (BA)^\sharp)^\sharp; Hom(W, A)]$ and ψ_W the Alexander-Whitney diagonal approximation. Therefore ϕ respects the products.

3.2. Cohomology Operations on the Hochschild Cohomology

Consider the map ϕ defined above. By cluing diagrams of Fig4 and Fig5 together, we obtain from May construction[10] a commutative diagram which defines natural algebraic Steenrod operations on $HH^*(A; A)$. In fact the map $\tilde{\theta} = \phi \circ \overline{\kappa} = \phi \circ Hom(\kappa^\sharp_{BA}, \widetilde{\kappa}_A)$ is a π-linear differential graded modules homomorphism. Thus this map defines on the Hochschild cochain complex of a π-shc differential graded Hopf algebra with coefficients in itself, $\mathfrak{C}^*(A; A) = Hom_{\mathbb{K}}(BA, A)$, the Dold quasi-algebra structure. Hence there exists on the Hochschild cohomology, $HH^*(A; A)$, algebraic Steenrod operations defined as follows: for all $h \in HH^n(A; A)$ such that $h = cl(f)$, one has:

$$\begin{cases} Sq^i(h) = cl[\theta \circ \tilde{\kappa}_A \circ i_{A^{\otimes 2}}(f \otimes f) \circ \beta_{(BA)^{\otimes 2}} \circ \kappa^\sharp_{BA} \circ \varphi](e_{n-i}) \text{ if } p = 2 \\ p^i(h) = \vartheta_n cl[\theta \circ \tilde{\kappa}_A \circ i_{A^{\otimes p}}(f^{\otimes p}) \circ \beta_{(BA)^{\otimes p}} \circ \kappa^\sharp_{BA} \circ \varphi](e_{(n-i)(p-1)}) \text{ if } p > 2 \end{cases} \tag{12}$$

4.1. Example

Let $(A, d_A) = (H^*(\mathbb{C}P^\infty, 0))$

In this section, we assume once more that $\mathbb{K} = \mathbb{F}_2$ is a prime field of characteristic $p = 2$.

Let $X = \mathbb{C}P^\infty$, the infinity projective complex space and $H^*(\mathbb{C}P^\infty)$ its singular cohomology. It is known from the long exact homotopy sequence that $\mathbb{C}P^\infty$ is the Eilenberg-Maclane space $K(\mathbb{Z}, 2)$; therefore this singular cohomology algebra is the polynomial algebra $\mathbb{F}_2[u] = T(u) = \wedge(u)$ with one generator u of degree 2. Remark that $((\wedge(u), 0), \mu_{\wedge(u)}, \kappa_{\wedge(u)})$ is a 1-connected π-shc differential algebra with the shc structure defined by: $\mu_{\wedge(u)} = \Omega(m_{\wedge(u)})$, where $m_{\wedge(u)}$ is the product in $\wedge(u)$; while the π- shc algebra structure is given by:

$$\Omega B((\wedge(u))^{\otimes 2}) \overset{\tilde{\kappa}_{\wedge(u)}}{\longrightarrow} Hom(W; \wedge(u));$$

for any $y =< c_1|c_2| \cdots |c_{r-1}|c_r >$ such $|y| = 2\alpha + q$,

$$\tilde{\kappa}_{\wedge(u)}(y)(e_k \cdot \tau) = \begin{cases} u^{\alpha-\beta} \text{ if } 2\beta = k - q \\ o \text{ if not} \end{cases} \tag{13}$$

Since $(\wedge(u), 0)$ is a π-shc differential graded Hopf algebra up to homotopy, then its bar-construction, $B((\wedge(u))$ is not only a Hopf algebra but it is also has a π-shc coalgebra structure. Thus $\overline{\kappa}_{\wedge(u)} = Hom(\kappa^\sharp_{B(\wedge(u))}, \tilde{\kappa}_{\wedge(u)})$ and from our main theorem above, we obtain the algebraic Steenrod operations on the Hochschild cohomology, $HH^*(H^*(\mathbb{C}P^\infty; \mathbb{F}_2); H^*(\mathbb{C}P^\infty; \mathbb{F}_2))$.

4. Proof of Proposition

In addition to the construction of these cohomological operations, we will develop in this section their relationship with the Lie algebra structure as follows.

4.1. Review on the Gerstenhaber Algebra Structure

A graded Gerstenhaber algebra is a commutative graded algebra $G = \{G^i\}_{i\in\mathbb{Z}}$ with a negative one degree bilinear map:

$$\begin{array}{ccc} cccG^i \otimes G^j & \overset{\{-,-\}}{\longrightarrow} & G^{i+j-1} \\ x \otimes y \longmapsto & & \{x, y\} \end{array}$$

such that:

1. The suspension of G is a graded Lie algebra with bracket

$$\begin{array}{ccc} (sG)^i \otimes (sG)^j & \xrightarrow{[-,-]} & (sG)^{i+j} \\ sx \otimes sy & \longmapsto & [sx, sy] := s\{x, y\} \end{array}$$

2. The cup product and the Lie bracket are related by the poisson rule so called the Leibniz rule:

$$\{x, \{yz\}\} = \{\{x, y\}, z\} + \{y, \{x, z\}\},$$

for all $x, y, z \in G$

In particular the Gerstenhaber algebra structure on the Hochschild cohomology is given as follows:

Let (A, d_A) be an associative supplemented differential graded algebra with unit. The Gerstenhaber bracket on the suspension of the Hochschild complex, $s\mathfrak{C}(A; A) \cong Hom(T(sA), sA)$ is:

$$\begin{array}{ccc} (s\mathfrak{C}(A; A))^i \otimes (s\mathfrak{C}(A; A))^j & \xrightarrow{[-,-]} & (s\mathfrak{C}(A; A))^{i+j} \\ sf \otimes sg & \longmapsto & [sf, sg] := (sf)\overline{\circ}(sg) - (-1)^{|sf||sg|}(sg)\overline{\circ}(sf). \end{array}$$

Where $(sf)\overline{\circ}(sg) = s(f \circ \beta_A(g))$ and $\begin{array}{ccc} \mathfrak{C}^\star(A; A) = Hom(T(sA), A) & \xrightarrow{\beta_A} & Coder(\widetilde{\mathbb{B}}A) \\ f & \longmapsto & \beta_A(f), \end{array}$ a $+1$ degree linear iomorphism given by: for all$[a_1|a_2|\cdots|a_n]$,

$$\beta_A(f)([a_1|a_2|\cdots|a_n]) = \sum_{0\leq i\leq j\leq n} (-1)^\lambda [a_1|a_2|\cdots|a_i|f([a_{i+1}|a_{i+2}|\cdots|a_j])|a_{j+1}|\cdots|a_n] \tag{14}$$

With $\lambda = |sf| \cdot (\sum_{1\leq r\leq i} |sa_r|)$ (see [7]). This bracket combined with cup product makes the Hochschild cohomology $HH^\star(A; A)$ into a graded Gerstenhaber algebra.

Moreover one has: $((sf)\overline{\circ}(sg)) \otimes ((sh)\overline{\circ}(sk)) = (-1)^{|sh||\beta_A(g)|}((sf) \otimes (sh)) \circ (\beta_A(g) \otimes \beta_A(k))$, for any $f, g, h, k \in \mathfrak{C}^\star(A; A)$.

4.2. Sketch of Proof

Let $((A, d_A), \nu_A, \widetilde{\kappa}_A)$ be an augmented π-shc differential graded Hopf algebra over a field $\mathbb{K}$ with caracteristic $P = 2$. Since $(HH(A, A), [-, -], \cup)$ is a gersttenhaber algebras, we deduce from the compatibility of the Lie bracket with the cup product through a simple but stedious computation that the Gerstenhaber backet annihilates the tensor product of the Steenrod squares ; that is for all $m, n \in \mathbb{Z}$, $[-, -] \circ (s)^{\otimes 2} \circ (sq^m \otimes sq^n)$ is the zero bilinear map.

5. Conclusion

Throughout this work, we extend not only the construction of cohomology operations, so called Steenrod operations in the Hochschild coholomogy of a large class of Hopf algebras, that of Hopf algebras up to homotopy by using the general algebraic appraoch of May, but we also give a link between these operations and the Gertenhaber algebra structure.

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