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Proving the Collatz Conjecture with Binaries Numbers

Olinto de Oliveira Santos

CETEP- Territorial Center for Education Professional of the Coast of the Discovery, Eunápolis, Brazil

Email address:
profolinto13@gmail.com

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Abstract: The objective of this article is to demonstrate the Collatz Conjecture through the Sets and Binary Numbers Theory, in this manner: $2^n + 2^{n-1}+...1$. This study shows that there are subsequences of odd numbers within the Collatz sequences, and that by proving the proposition is true for these subsequences, it is subsequently proven that the entire proposition is correct. It is also proven that a sequence which begins with a natural number is generated by a set of operations: Multiplication by 3, addition of 1 and division by 2^n. This set of operations shall be called "Movement" in this study, and may be increasing when n=1, and decreasing for $n \geq 2$. The numbers in 2^n form generate decreasing sequences in which the 3n+1 operation does not occur. One of the important discoveries is how to generate numbers in which the 3n+1 operation only occurs once and how to generate numbers with a minimum quantity of increasing movements that are the numbers of greater "orbits" (Longer sequences that take longer to reach the number one). The conclusion is that, as the decreasing numbers dominate as compared to the increasing ones, the statement that the sequence is always going to reach the number 1 is true.

Keywords: Binary Numbers, Collatz Conjecture, Hail Sequences

1. Introduction

In this article, the Theory of Sets and Binary Numbers will be used, with the ED [Portuguese acronym] (Written by Definition) method, to investigate the Collatz conjecture through the results obtained. It is shown that the orbit of each number is determined by its binary form. This paper also demonstrates how to obtain numbers in which the 3n+1 operation does not take place at all, takes place only once, or in which this operation appears at least "n" times.

The Collatz Conjecture, or 3n+1 problem, was formulated in 1937 by German mathematician, Lothar Collatz. It is a mathematical assumption which is thought to be true, but has yet to be proven or rejected.

The Collatz Conjecture asserts that, by performing the following operations: begin with a natural number. If this number is even, divide by 2. If it is odd, multiply by 3 and add 1. After this, a new number is obtained and the process repeated. Lothar Collatz conjectured that, by pursuing these operations, one will always arrive at the number 1.

Observe the example of number 10, in which we arrive at the sequence: 10, 5, 16, 8, 4, 2, 1. This sequence is called the Hailstone Sequence, and has its operations closed whenever it reaches the number 1. The elements that make up this sequence are called the orbit. As such, the orbit of number 10 has 7 elements.

If the operations continue after reaching the number 1, the result is an endless repetitive sequence: 1, 4, 2, 1, 4, 2, 1...

Portuguese researcher, Tomás Oliveira e Silva, explored a number of assumptions, starting at number 1 and surpassing the number 19×2^{58}. He did not come across any case in which the result did not reach 1.

Based on this Conjecture, Filho [1] proposes the Collatz Function C (n) f:N→N, defined as follows:

$$C(n) = \begin{cases} \frac{n}{2} \text{ if n is even} \\ 3n + 1 \text{ if n is odd} \end{cases} \quad (1)$$

According to Filho [1], the sequence of numbers generated through this process until reaching the number 1 is called Orbit Number *n*. This is a sequence obtained by recurrence. According to Lima et al. [8], "A sequence is recursively defined when obtained by a rule that allows for the calculation of any term in function of the immediate precursor (s). For Lima et al. [ ], "For a sequence to be perfectly determined, knowledge of the first terms is also necessary."

Cz (n) refers to the orbit of a natural number, $n > 1$, the sequence obtained by recursively applying the Collatz function, starting from the natural number *n*, by successively applying the function C (n) until the sequence reaches the number 1.

Cz (n) is a way of identifying the orbits without ambiguities among other mathematical subjects. Using the first and last letter of the name Collatz allows for easy identification of the orbits in any language. Observe three orbits:

Cz (6) = {6, 3, 10, 5, 16, 8, 4, 2, 1}

Cz (7) = {7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1}

Cz (8) = {8, 4, 2, 1}

This problem contains a challenge which is to understand the orbit of each number. How can it be explained that numbers of such similar values produce such different sequences? The most likely possibility is that the orbit be explained in conjunction with the demonstration of the Conjecture. Another possibility is that, after the demonstration of the Conjecture, the orbit becomes the next challenge. However, perhaps to the surprise of mathematicians researching this problem, at the end of this chapter we shall see that what determines the orbit of each number is its own binary form.

This research began with the reading of an article in "Revista Cálculo", Osone [3], in which the conjecture was disclosed. After reading the article, I searched the Internet for earlier attempts to prove the conjecture. This was when the work of American mathematician Lagarias [4] stood out, in which he organized all advances related this conjecture up to that point (2013). The next step was to seek tools that could be used to solve the problem in the literature on Numbers Theory, from such authors as Carvalho [5], Chaves [6], Coutinho [7], Neto [8], Hefez [9] and Scheinerman [10].

The work of both Lesieutre [11] and Carnielli [12], which generalizes the conjecture for other numbers, particularly divisors other than 2, was also consulted. However, generalization proposals were not used in this study though other of the authors' ideas have been considered.

In his book about the last Fermat theorem, Singh [13] shows how conjectures are important for the development of Mathematics, as studies that do not reach the objective produce important contributions to science. What actually took place during this research was the discovery of a new Diophantine equation and a Cryptographic system that is useful for entertaining students and motivating the study of Numbers Theory. These discoveries are described in Santos' book [14].

For Stewart [15], the conjectures can be formulated in a way that is both simple and easy to understand for those that have mastered Basic Mathematics. Others are so complex that only specialists are able to understand their formulation. However, the advantage of the simple-formula conjectures is that they are "democratic"; they invite students of any age to study Mathematics. Andrew Wiles began trying to solve the last Fermat Theorem while he was still a child, and fulfilled his dream after receiving his doctorate. That's why it is interesting to present the simple formulation of the conjectures, such as the Collatz, to the students.

Sing [13] and Stewart [14] were important in the choice of strategies and also in this author's motivation, showing that, by describing the attempts of other researchers, with their successes, failures and advances, it was worthwhile to try solving the Collatz Conjecture. And, after the choice of strategies and research of previous results were done, an attempt was made at a demonstration, the results of which are presented below.

2. Methodology

2.1. Cz (n) Properties

2.1.1. Proposition

If a natural number, n, is presented as 2^n, the CZ (n) set does not have the 3n+1 operation.

Demonstration:

As 2^n:2 is even:

Cz ($2^{n)}$={2^n, 2^{n-1}, 2^{n-2},...2^o }, logo Cz (2^n) = {2^n, 2^{n-1}, 2^{n-2},...1 }.

2.1.2. Proposition

If **n** is a natural number with n=3 or n≥5, and n≠8, and Cz (16) is contained in Cz (n).

Demonstration:

By definition, the last term of the sequence produced by Collatz operations is 1. As 3n+1>1, the previous operation was a division by 2, thus the penultimate number is 2. Similarly, as 3n+1>2, the number preceding 2 is 4.

For 3n+1= 4 => n=1. This is ridiculous because 1 is the last number in the sequence, so the term preceding 4 is 8.

As 3n+1=8 => n= 7/3, and 7/3 is not a natural number, the number preceding 8 is 16.

If n=3, Cz (3)= {3, 10, 5, 16, 8, 4, 2, 1}, therefore the property holds true.

If n=4, Cz (4)={4, 2, 1}, the property does not hold true as 16 and 8 are missing.

If n=8, Cz (8)={8, 4, 2, 1}, the property does not hold true once again as 16 is missing.

Based upon the arguments above, the proposition is true.

The Cassini [16] graph demonstrates this proposition.

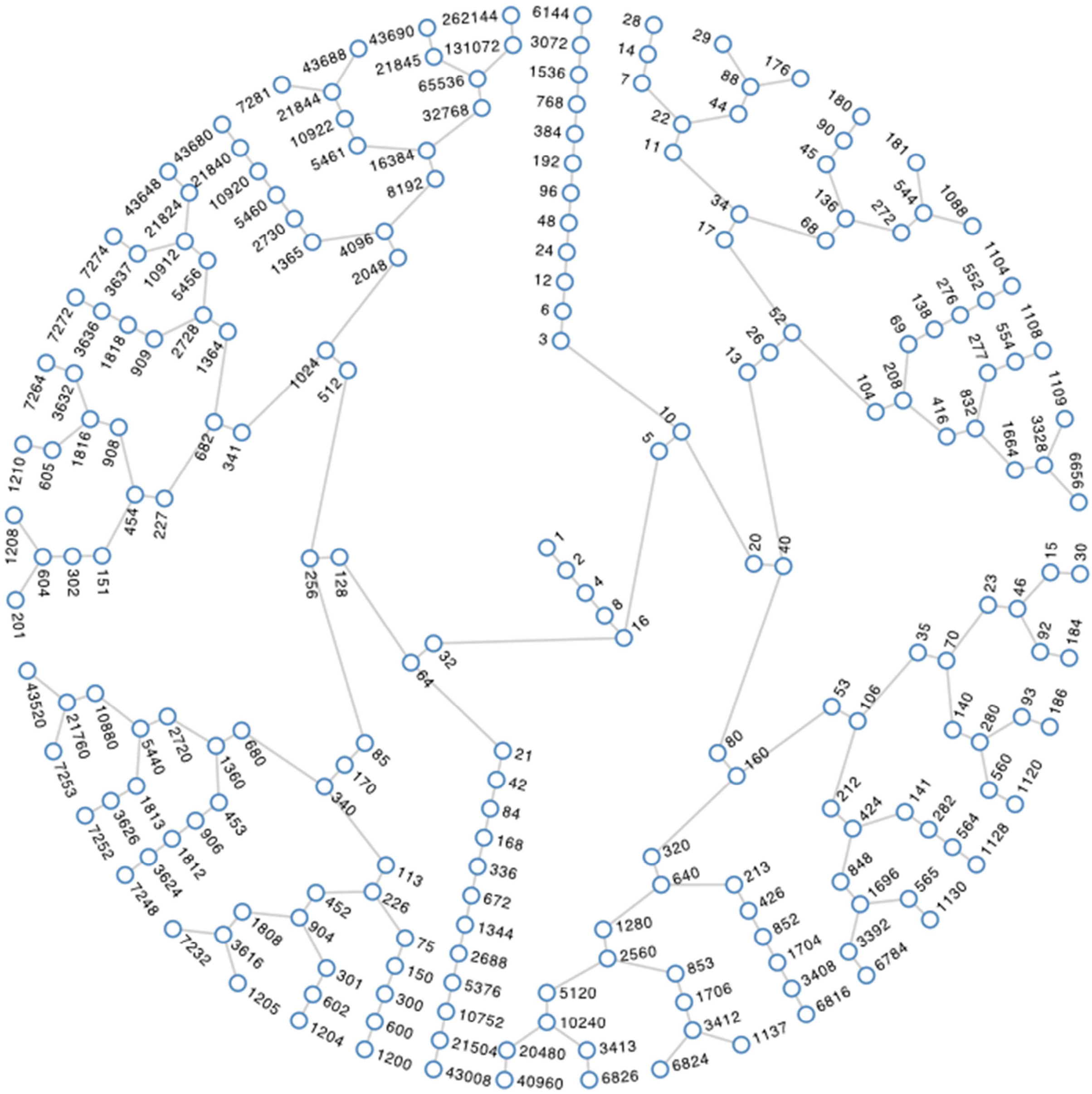

Figure 1. *Graph created by Jason Davies, programmer, shown in Cassini's work.*

2.1.3. Proposition

The last elements of the Collatz sequence prior to the number 1 form a Collatz subsequence.

This property allows the Collatz sets to be written more elegantly, and also tests whether the conjecture is true for an untested number more quickly. This property appears in the work of other mathematicians, such as Lagarias (2013), in the form of an organizational chart.

Demonstration:

Cz (a) = {a, a_1, a_2, a_3,...,16, 8, 4, 2, 1}.

Using associativity, beginning with the term a_3 results in:

Cz (a) = {a, a_1, a_2, Cz (a_3)}

Example:

Cz (76x2^{58})= {76x 2^{58}, 38x2^{58}, Cz (19x2^{58})}

2.1.4. Proposition

If the intersection of two Collatz sequences is not empty and is not contained in Cz (16), they have at least one Collatz subset beyond Cz (16).

Demonstration:

Cz (a) = {a, a_1, a_2, a_3,...,16, 8, 4, 2, 1}.

Cz (b) = {b, b_1, b_2, b_3, b_4...,16, 8, 4, 2, 1}.

Assuming that a_3=b_4, therefore, the result is:

Cz (a) = {a, a_1, a_2, Cz (a_3)}

Cz (b) = {a, b_1, b_2, b_3, Cz (a_3)}

This property can reduce the work inherent in operations with Collatz sets.

Example:

Cz (5)={**5**, 16, 8, 4, 2, 1}

Cz (3)={3, 10, Cz (5)}

Cz (17)={17, 52, 26, 13, 40, 20, Cz (10)}

2.1.5. Proposition

If n is an odd natural number, Cz (n) is a subset of natural

Cz (2^pn) $\forall p$, i.e., Collatz sets of odd numbers are subsets of even numbers.

Demonstration:

In accordance with the fundamental theorem of arithmetic, a natural even number may be written as a product of primes including 2^p. As such, if K is a natural even number, it can be written as follows: K=2^px3^qx5^tx....V^z. It is understood that 3^qx5^tx....V^z = M=> M is a number formed by the product of odd numbers, and so, it is odd. Applying the Collatz operations, the result is Cz (K)={2^pM, 2^{p-1}M,..., 2^o M,....., 2, 1} => Cz (K)={2^pM, 2^{p-1}M,...,Cz (M)}.

This property bring about an interesting conclusion: that to prove or disprove the conjecture, one just needs work with odd numbers.

2.1.6. Proposition

If two odd numbers are used, A and B, where B is a multiple of A, it cannot be said that Cz (A) is a subset of Cz (B).

Example:

Cz (3)={3, 10, 5, 16, 8, 4, 2, 1}

Cz (21)={21, 64, 32, 16, 8, 4, 2, 1}

2.2. Property of the Numbers in Base Two, as Follows: 2^n $+2^{n-1}$+...+1

2.2.1. Proposition

$2^p+2^p=2\text{x }2^p=2^{p+1}$

Example:

$2^3+2^3=2^4$

2.2.2. Proposition

$2^n>2^{n-1}+2^{n-2}+...+1$

Demonstration:

$2^n>2^n-1$

$2^n>2\text{x}2^{n-1}-1 => 2^n>2^{n-1}+2^{n-2}+2^{n-2}-1$

Similarly:

$2^n>2^{n-1}+2^{n-2}+2^{n-3}-2^{n-3}-1$

Repeating the operation n times we results in:

$2^n> 2^{n-1}+2^{n-2}+...+2+1+1-1$

$2^n> (2^{n-1}+2^{n-2}+...+2+1)$

As such, the statement is true.

2.2.3. Proposition

$(2^{n-1}+2^{n-2}+...+2+1)+1= 2^n$

Demonstration:

$2+1+1 = 2+2 = 2^2$

$2^2+2^2 = 2^3$

$2^3+2^3 =2^4$

Continuing in a similar fashion results in:

$2^{n-1}+2^{n-2}+2^{n-2} = 2^{n-1}+2^{n-1} = 2^n$

2.3. Collatz Operations with Binaries

The advantage of doing Collatz operations in base 2 is that, in this format, it is possible to see numerous hidden properties in base 10. This method of writing the binary will be referred to as ED [Portuguese acronym - Writing by Definition], and each power of 2^n is called a "term". Before beginning the Collatz operations, the base number 2 is transformed into its ED equivalent.

With this form of writing numbers in any base, Santos [14], a variety of properties and information are obtained, which are not visible in traditional Hindu-Arabica script.

Consider the example of the number 17 = 2^4+1:

17x3 +1= (2^4 +1) x (2+1) +1 = 2^4x2 +2^4x1+2x1+1x1+1

$2^5+2^4+2+1+1 = 2^5+2^4+2+2 =2^5+2^4+2^2$

It is possible to continue the Collatz operations until reaching the number 1 and determining all of the orbit numbers or, if seeking just the next odd number, to divide everything by the lowest power. In the following example, we shall determine all of the terms:

$(2^5+2^4+2^2)$: 2 = 2^4+2^3+2

(2^4+2^3+2):2 = 2^3+2^2+1

(2^3+2^2+1) x (2+1)+1 = $2^4+2^3+2^3+2^2+2+1+1$ = $2^4+2^4+2^2+2+2$=

$2^5+2^2+2^2 = 2^5+2^3$

(2^5+2^3):2 = (2^4+2^2)

(2^4+2^2):2 = 2^3+2

(2^3+2):2 = 2^2+1

(2^2+1) x (2+1)+1 = $2^3+2^2+2+1+1$= 2^3+2^2+2+2 = $2^3+2^2+2^2$= 2^3+2^3= 2^4

2^4:2 =2^3

2^3:2 = 2^2

2^2:2 = 2

2:2 = 1

Cz (2^4 +1) = {2^4 +1, $2^5+2^4+2^2$, $2^4+2^3+2^1$, 2^3+2^2+1, 2^5+2^3, 2^4+2^2, 2^3+2, 2^2+1, $2^{4,}$ 2^3, 2^2, 2, 1}.

Example 2

Let's employ the algorithm to determine only the odd terms, beginning from 13 = 2^3+2^2+1.

(2^3+2^2+1) x (2+1)+1 =$2^4+2^3+2+2^3+2^2+1+1$=2^5+2^3

(2^5+2^3): 2^3 = 2^2+1

(2^2+1) x (2+1)+1= 2^3+2+2^2+1+1= 2^4

2^4: 2^4 = 1

We shall use the name ICz (n) in reference to the subset of odd numbers from a Collatz set, which gives us:

ICz (13) = {13, 5, 1}

or

ICz (2^3+2^2+1) = {2^3+2^2+1, 2^2+1, 1}.

The ICz (n) set makes the number of times that the 3n+1 operation occurs explicit. In the case of ICz (13), this operator is used 2 times. We shall call each operation 3n+1, followed by division by 2^n, by movement, and that the ICz 2^n exponent of the degree of movement. As such, ICz (13) has a third-degree movement and a fourth-degree movement.

Therefore, ICz (n) is a subsequence of odd Cz (n) numbers.

Examples;

ICz (3) = {3, 5, 1}

ICz (26) = {13, 5, 1}

ICz (8) = {1}

The advantages of ICz (n) with respect to Cz (n) may be observed through the graphing of each sequence. As an example, we consider the Cz (7) with ICz (7) graphs:

Cz (7) Graph

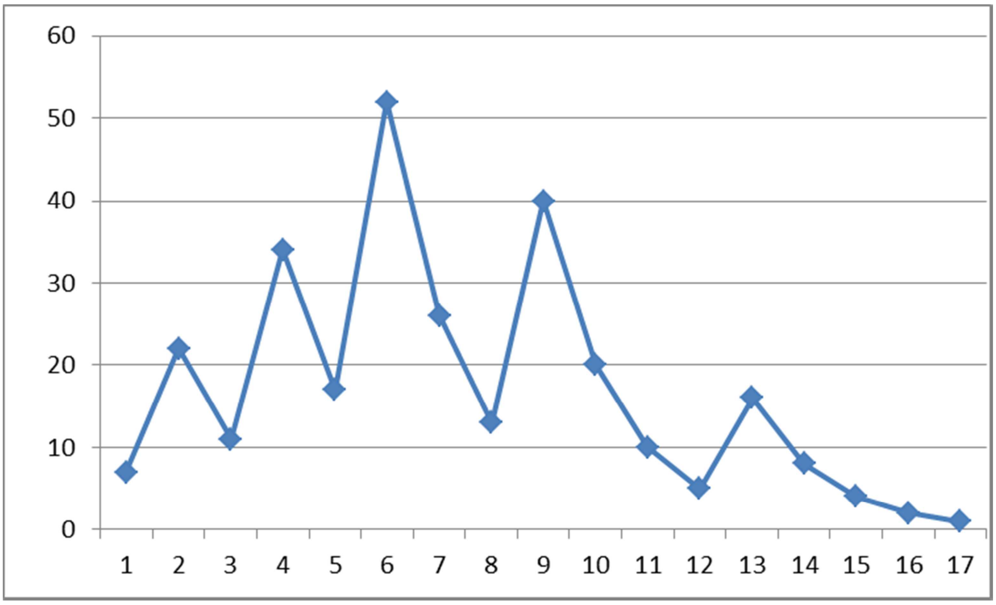

Figure 2. *Cz (7) Graph.*

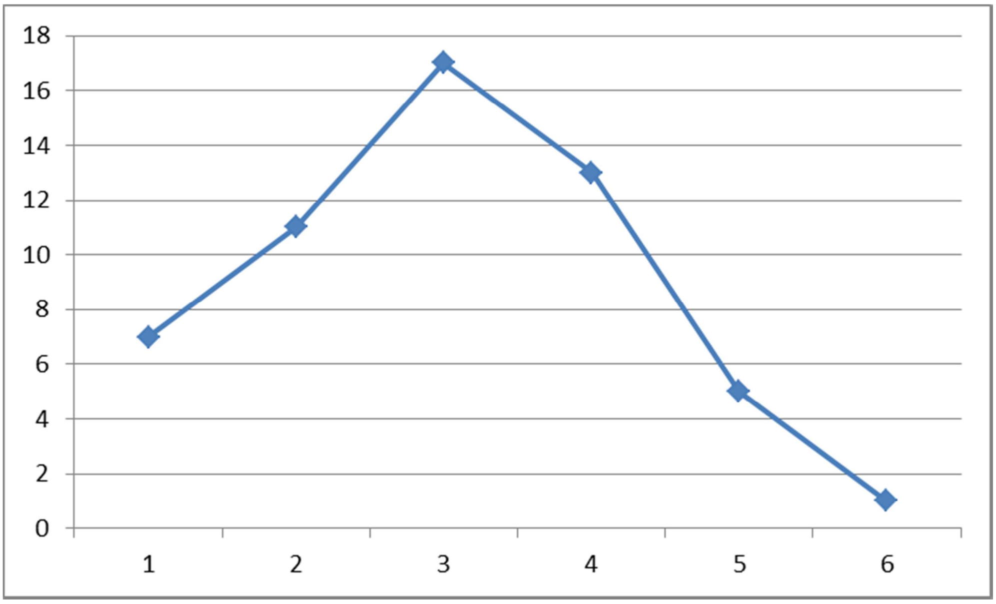

Figure 3. *ICz (7) Graph.*

Note how the ICz (7) graph is much more regular, improving observation of each number's orbit.

2.4. Collatz Movements

As has already been defined, Collatz Movement is the 3n+1 operation which is then divided by 2^n in order to obtain the ICz (n) set. This movement will also be classified in accordance with its divider, 2^n. The exponent "n" will be the degree of movement. As such, if the division is by 2, the movement will be of the first degree. If divided by 2^3, it will be a third-degree movement. Therefore, ICz (13) has a third-degree movement as well as a fourth-degree movement.

Second-degree movement is what results most frequently and, hence, is known as "Monotonous Motion".

2.4.1. Analysis of Collatz Movement

Collatz movement can be divided into three phases: multiplication, addition and division.

In the multiplication stage, each 2^n term produces a new 2^{n+1} term that will be known as the "New Term", and which causes the quantity of the terms to double.

$(2^n +1) \times (2+1) +1 = 2^{n+1}+2^n+2+1+1$

In addition, the similar terms are summed up and the quantity of the terms decreases. Therefore, we have:

$2^{n+1}+2^n+2+1+1= 2^{n+1}+2^n+2+2=2^{n+1}+2^n+2^2$

In the division, there is a reduction in the exponent of each term:

$(2^{n+1}+2^n+2^2):2^2 = 2^{n-1}+2^{n-2}+1$

2.4.2. Proposition

The first-degree movement is increasing. All other movements are decreasing.

Demonstration:

We take a natural number, “n”, which is greater than two. As such, we have:

$3n+1>2n$

Dividing the two terms by 2, we have:

$(3n+1):2 > n$.

Note that $(3n+1):2$ is the first-degree movement.

Supposing that “p” is a natural number which is greater than, or equal to, one, then we have:

$3n+1< 2^p x2n$

And so, dividing the two terms by $2^p x2$ we have:

$(3n+1):2^p x2 < n \Rightarrow (3n+1):2^{p+1} < n$

Once again, note that $(3n+1):2^{p+1}$ is movement of a degree which is greater than, or equal to, two.

Example:

$11 = 2^3+2+1$

$(2^3+2+1) \text{ x } (2+1) + 1= 2^4+2^3+2^2+2+2+2 = 2^5 + 2$

$(2^5 + 2):2 = 2^4+1$ (First-degree movement)

$(2^4+1) \text{ x } (2+1)+1 =2^5+2^4+2+2= 2^5+2^4+2^2$

$(2^5+2^4+2^2):2^2 = 2^3+2^2+1$ (Second-degree movement)

$(2^3+2^2+1) \text{ x } (2+1)+1 = 2^4+2^3+2^3+2^2+2+2 = 2^5+2^3$

(2^5+2^3)☹$2^3) = (2^2+1)$ (Third-degree movement)

$(2^2+1) \text{ x } (2+1)+1= 2^3+2^2+2+2 = 2^4$

$2^4:2^4 = 1$ (Fourth-degree movement)

Icz (11) = {11, 17, 13, 5, 1}

Movements with a degree higher than second cause a tremendous reduction in the terms and the module of the Icz (n) sequence numbers. Therefore, they will be known as “Strong Reductions”.

2.4.3. Proposition

Monotonous movement occurs whenever the penultimate term has a degree which is greater than second. If the degree of the penultimate term is second, we have a strong reduction. If the degree is first, we shall have an increasing movement.

Demonstration:

$(2^p+2+1) \text{ x } (2+1)+1 = 2^{p+1}+2^p+2^2+2+2+1+1= 2^{p+1}+2^p+2^2+2$

$(2^{p+1}+2^p+2^2+2):2 = 2^p+2^{p-1}+2+1$ (Increasing Movement).

We take 2^p+2^2+1, so we have:

$(2^p+2^2+1) \text{ x } (2+1)+1 = 2^{p+1}+2^p+2^3+2^2+2+1+1 = 2^{p+1}+2^p+2^4$

$(2^{p+1}+2^p+2^4):2^4 = 2^{p-3}+2^{p-4}+1$ (Strong Reduction)

Lastly, we take 2^p+1, with $p>2$:

$(2^p+1) \text{ x } (2+1)+1= 2^{p+1}+2^p+2+1+1 = 2^{p+1}+2^p+2^2$

$(2^{p+1}+2^p+2^2):2^2 = 2^{p-1}+2^{p-2}+1$ (Monotonous Movement)

2.4.4. Proposition

The smallest strong reduction occurs when the last three terms are 2^3+2^2+1, and the preceding term has an exponent which is greater than 4.

Demonstration:

$(2^p+2^3+2^2+1) \text{ x } (2+1)+1 = 2^{p+1} +2^p+2^4+2^3+2^3+2^2+2+1+1== 2^{p+1}+2^p+2^5+2^3$

$(2^{p+1}+2^p+2^5+2^3):2^3 = 2^{p-2}+2^{p-3}+2^2+1$ (Strong 3rd-degree reduction).

2.4.5. Theorem

The numbers of a base 2 movement have decreasing exponent pairs.

Demonstration:

We take the number $N=2^{2p}+2^{2p-2}+2^{2p-4}+...+2^6+2^4+2^2+1$. In the multiplication stage, we have:

$(2^{2p}+2^{2p-2}+2^{2p-4}+...+2^4+2^2+2^0) \text{ x } (2+1) + 1=$ (note: observe that $1 = 2^{nd}$).

$2^{2p+1}+ 2^{2p}+2^{2p-1} +2^{2p-2}+2^{2p-3}+ 2^{2p-4}+...+2^5 +2^4+2^3+2^2+2+1+1 =$

Using the property of the binaries, in the addition phase we have:

$2^{2p+1}+ 2^{2p}+2^{2p-1} +2^{2p-2}+2^{2p-3}+ 2^{2p-4}+...+2^5 +2^4+2^3+2^2+2+1+1 = 2^{2p+2}$

$(2^{2p+2}):2^{2p+2} = 1$

Therefore, the number has a movement of the 2p+2 degree

Example:

$5 = 2^2 + 2^0$, Cz (5) = {5, 16, 8, 4, 2, 1}

$21 = 2^4 + 2^2 + 2^0$, Cz (21) = {21, 64, 32, 16, 8, 4, 2, 1}

The odd numbers of a movement are obtained by series:

$$Sn = \frac{2^{2n}-1}{3} \tag{2}$$

Note that these numbers are formed by the sum of the terms of a $a_n = 2^{2n}$ Geometric Progression

$$21 = 2^4 + 2^2 + 2^0 => 2^4: 2^2 = 2^2: 2^0 => 2^2=2^2$$

Using the sum of the finite PG, with $a_n = 1$, we have:

$$Sn = \frac{2^{2n} - 1}{2^2 - 1}$$

$$Sn = \frac{2^{2n} - 1}{3}$$

2.5. Augmentative Movement

2.5.1. Proposition

If the last two terms of a number are 2+1, the next movement of said number is augmentative.

Demonstration:

$(2^n+2+1) \text{ x } (2+1)+1=2^{n+1}+2^n+2^2+2+2+1+1$

$2^{n+1}+2^n+2^3+2$

Note that, as the lesser term is two, there will be a division by two, determining a 1st degree movement, known as augmentative movement.

2.5.2. Theorem

If the exponents of a number's "N" terms are in descending order, this number will have at least "N" augmentative movements which will be the first n movements of ICz.

Demonstration:

We take the number $2^n+2^{n-1}+2^{n-2}....2^3+2^2+2+1$.

$(2^n+2^{n-1}+2^{n-2}....2^3+2^2+2+1) \text{ x } (2+1)+1 =$

After the multiplication phase, we shall have:

$2^{n+1}+2^n+2^n+2^{n-1}+2^{n-1}+2^{n-2}....2^4+2^3+2^3+2^2+2^2+2+2+1+1$

Adding the new terms in bold to the last two terms, and using the property of the binaries, we shall have:

$2^{n+1}+2^n+2^{n-1}....2^3+2^2+1+1= 2^{n+2}$

Let's add this new term to others, and we shall have:

$2^{n+2} +2^n+2^{n-1}+2^{n-2}....2^3+2^2+2$

As the smallest term is two, this number is divisible by two. By dividing, we have:

$2^{n+1}+2^{n-1}+2^{n-2}+2^{n-3}....2^2+2+1.$

Upon repeating the Collatz operations, we have:

$(2^{n+1}+2^{n-1}+2^{n-2}+2^{n-3}+....+2^2+2+1) \times (2+1)+1 =$

$2^{n+2}+2^{n+1}+2^n+2^{n-1}+2^{n-1}+2^{n-2}... +2^3+2^2+2^2+2+2+1+1$

Summing up the new terms in bold with the second term and the last two terms, we have:

$2^{n+2}+2^{n+1}+2^n+... +2^3.+2^2+2+1+1. = 2^{n+3}$

Applying the division, we shall have:

$(2^{n+3}+2^{n-1}+2^{n-2}...2^3+2^2+2):2 = 2^{n+2}+2^{n-2}+2^{n-3}...2^2+2+1$

Note that the operations do not change the last two terms, which continue to be 2+1. Moreover, as long as they are not changed, the movements will be augmentative. Also note that with each new movement, the terms with exponents in decreasing order decrease after the second movement has moved past n to n-2. If we continue applying the augmentative movements, after n movements the penultimate term will be affected and there will be another 1st degree movement. As such, a monotonous movement will take place, and so the statement is correct.

i) Corollary: The numbers of the $2^n+2^{n-1}+2^{n-2}...2+1$ form may be written as $2^{n+1}-1$.

Demonstration:

Using the property of the binary, we have:

$(2^n+2^{n-1}+2^{n-2}....2^3+2^2+1+)+1= 2^{n+1}$

$(2^n+2^{n-1}+2^{n-2}....+2^3+2^2+1+)= 2^{n+1}-1$

ii) Corollary: Numbers in which the last n terms have exponents in descending order, also have n initial augmentative movements.

Demonstration:

Supposing that p-n>2, we have:

$(2^p+2^n+2^{n-1}+2^{n-2}....2^3+2^2+2+1) \times (2+1)+1 =$

$(2^{p+1}+2^p+2^{n+1}+2^n+2^{n-1}+2^{n-1}+2^{n-2}+...2^3+2^2+2):2 = 2^{p+1}+2^p + 2^{n+2}+2^{n-2}+2^{n-3}...2^2+2+1$

Note that there was no change in the Collatz movements which continue to be augmentative and to reduce the quantity of ordered terms. Therefore, the statement is valid.

When the exponents of two or more of a number's terms with a difference of just one unit in relation to the previous or following term, these terms will be called "aligned terms" or "term alignment".

2.6. Augmentative Movement x Monotonous Movement

Suppose an odd number, K>30, that, in ICz (K), initially undergoes two consecutive augmentative movements followed by three monotonous movements:

K x 3 + 1 = 3K + 1

(3K + 1): 2 = 1,5K + 0,5

(1,5K + 0,5) x 3 + 1 = 4,5K + 2,5

(4,5K + 2,5):2 = 2,25K +1,25

(2,25K +1,25) x 3+1 = 6,75K + 4,75

(6,75K + 4,75):4 = 1,6875 K+ 1,1875

(1,6875 K+ 1,1875) x 3 +1 = 5,0625K + 4,5625

(5,0625K + 4,5625):4 = 1,2656K + 1,1406

(1,2656K + 1,1406) x 3 + 1 = 3,7968K + 4,4219

(3,7968K + 4,4219):4 = 0,9492K + 1,1055

Note that 1 – 0,9492 = 0,0508. Therefore, K = 0,9492K+ 0,0508K, and as K>30, it follows that:

0,0508K > 0,0508x30

0,0508K > 1,524 > 1,1055

K > 0,9492K + 1,1055

As the orbits of numbers less than 30, and even that of 30 itself, are well known, we shall state only that if a number is greater than 30 and suffers two augmentative movements followed by three monotonous movements, it will have a smaller module than in the beginning. In other words, it takes three monotonous movements to cancel out two augmentative movements.

2.7. The Effects of Multiplication and Addition on the Terms

As a new term with an exponent that is greater than the original term is created in the multiplication step, the terms will become "closer", i.e., the difference between its terms decreases and, in the addition, they can join to form a single term. This result of multiplication and addition operations will be called "Fusion", and is independent of the kind of movement.

There are two types of fusion. The first is when the exponents of the terms have one unit of difference.

$(2^n+2^{n-1}+1) \times (2+1) + 1=$

$2^{n+1}+2^n+2^n+2^{n-1}+2+1+1=$

Note the new terms in bold.

$(2^{n+1}+2^n+2^n)+2^{n-1}+(2+1+1) =$

Note that three terms become one prior to the division.

$(2^{n+2}+2^{n-1}+2^2):(2^2) = 2^n+2^{n-3}+1$

The other fusion is when there are two units of difference between the exponents of the terms which precede a term with a single unit.

$(2^{n+4}+2^{n+2}+2^n+2^{n-1}+2+1) \times (2+1)+1 =$

$2^{n+5}+2^{n+4}+2^{n+3}+2^{n+2}+2^{n+1}+2^n+2^n+2^{n-1}+2^2+2+2+1+1 =$

$(2^{n+5}+2^{n+4}+2^{n+3}+2^{n+2}+2^{n+1}+2^n+2^n)+2^{n-1}+(2^2+2+1+1)+2 =$

$(2^{n+6}+2^{n-1}+2^3+2):2 =$

$2^{n+5}+2^{n-2}+2^2+1$

Note that even as an augmentative movement, the effect of multiplication and addition on the terms was the same, a reduction in the number of terms, while the difference between the exponents of the following terms increased. This effect is fundamental in ensuring that the Collatz proposition is true, for these operations because, by distancing the terms, they prevent them from forming alignments that would lead to new augmentative movements.

A very important consequence of these operations is that the first terms undergo more fusions than the last ones because, with every multiplication, a new term is created that has one more unit than the term suffering multiplication. Additionally, every two moves, this term undergoes a fusion which creates a term with an even larger. The process repeats until the new terms undergo fusion with the terms ahead of them.

Another important effect of the multiplication and addition operations is the change in parity. This explains the sudden changes in their orbits.

Supposing that "n" is an even number, then n+1 and n-1 are odd. Similarly, if "n" is odd, then n+1 and n-1 are even. Applying Collatz operations to the number 2^n+2^2+2+1, we have:

$(2^n+2^2+2+1) \times (2+1)+1=$
$2^{n+1}+2^n+2^3+2^2+2^2+2+2+1+1=$
$(2^{n+1}+2^n+2^4+2^2+2):(2) =$
$2^n+2^{n-1}+2^3+2+1$

Note that the first two terms already have different parities.

Continuing on, we have:

$(2^n+2^{n-1}+2^3+2+1) \times (2+1) + 1 =$
$2^{n+1}+2^n+2^n+2^{n-1}+2^4+2^3+2^2+2+2+1+1 =$
$(2^{n+2}+2^{n-1}+2^5+2):(2) =$
$2^{n+1}+2^{n-2}+2^4+1$

The combination of multiplication and addition operations ensures that, even when combining decreasing and increasing movements, the difference between the exponents must increase, especially among first and last terms. This prevents the number that initiated the sequence from repeating, which would negate the conjecture.

Example:

Highlighting the original terms in bold, we have:

$(2^{n+5}+2^n+2+1) \times (2+1)+1=$
$2^{n+6}+2^{n+5}+2^{n+1}+2^n+2^2+2+2+1+1=$
$(2^{n+6}+2^{n+5}+2^{n+1}+2^n+2^3+2):(2) =$
$(2^{n+5}+2^{n+4}+2^n+2^{n-1}+2^2+1) \times (2+1) + 1 =$
$2^{n+6}+2^{n+5}+2^{n+5}+2^{n+4}+2^{n+1}+2^n+2^n+2^{n-1}+2^3+2^2+2+1+1 =$
$2^{n+7}+2^{n+4}+2^{n+2}+2^{n-1}+2^3+2^2+2+1+1 =$
$(2^{n+7}+2^{n+4}+2^{n+2}+2^{n-1}+2^4):(2^4) =$
$(2^{n+3}+2^n+2^{n-2}+2^{n-5}+1) \times (2+1) +1=$
$2^{n+4}+2^{n+3}+2^{n+1}+2^n+2^{n-1}+2^{n-2}+2^{n-4}+2^{n-5}+2+1+1=$
$(2^{n+4}+2^{n+3}+2^{n+1}+2^n+2^{n-1}+2^{n-2}+2^{n-4}+2^{n-5}+2^2): (2^2) =$
$(2^{n+2}+2^{n+1}+2^{n-1}+2^{n-2}+2^{n-3}+2^{n-4}+2^{n-6}+2^{n-7}+1):(2+1)+1 =$

Summing up the terms in bold, we have:

$2^{n+3}+2^{n+2}+2^{n+2}+2^{n+1}+2^n+2^{n-1}+2^{n-1}+2^{n-2}+2^{n-2}+2^{n-3}+2^{n-3}+2^{n-4}+2^{n-5}+2^{n-6}+2^{n-6}+2^{n-7}+2+1+1=$
$2^{n+4}+2^{n+2}+2^{n-1}+2^{n-2}+2^{n-3}+2^{n-7}+2+1+1=$
$2^{n+4}+2^{n+2}+2^{n-1}+2^{n-2}+2^{n-3}+2^{n-7}+2^2=$

Note that, at this instant, the original term is affected by new terms created by the previous original term, which even leads to a change in parity. Completing the movement, we have:

$(2^{n+4}+2^{n+2}+2^{n-1}+2^{n-2}+2^{n-3}+2^{n-7}+2^2):(2^2) =$
$2^{n+2}+2^n+2^{n-3}+2^{n-5}+2^{n-6}+2^{n-9}+1=$

2.8. Analysis of Monotonous Movement

Movement I
$(2^n+1) \times (2+1) + 1=$
$2^{n+1}+2^n+2+1+1=$
$2^{n+1}+2^n+2^2=$
$(2^{n+1}+2^n+2^2): 2^2 =$
$2^{n-1}+2^{n-2}+1$

Movement II
$(2^{n-1}+2^{n-2}+1) \times (2+1) +1 =$
$2^n+2^{n-1}+2^{n-1}+2^{n-2}+2+1+1 =$
$2^{n+1}+2^{n-2}+2^2 =$
$(2^{n+1}+2^{n-2}+2^2):(2^2) =$
$2^{n-1}+2^{n-4}+1 =$

Movement III
$(2^{n-1}+2^{n-4}+1) \times (2+1) + 1 =$
$2^n+2^{n-1}+2^{n-3}+2^{n-4}+2+1+1=$
$2^n+2^{n-1}+2^{n-3}+2^{n-4}+2^2 =$
$(2^n+2^{n-1}+2^{n-3}+2^{n-4}+2^2):(2^2) =$
$2^{n-2}+2^{n-3}+2^{n-5}+2^{n-6}+1$

Movement IV
$(2^{n-2}+2^{n-3}+2^{n-5}+2^{n-6}+1) \times (2+1) +1 =$
$2^{n-1}+2^{n-2}+2^{n-2}+2^{n-3}+2^{n-4}+2^{n-5}+2^{n-5}+2^{n-6}+2+1+1=$
$2^n+2^{n-2}+2^{n-6}+2^2=$
$(2^n+2^{n-2}+2^{n-6}+2^2): (2^2) =$
$2^{n-2}+2^{n-4}+2^{n-8}+1 =$

It can be noted that the exponent of the penultimate term decreases two units with each new monotonous movement, $\{2^n, 2^{n-2}, 2^{n-4}, 2^{n-6}, 2^{n-8}\}$. As such, if "n" is an even number, after n/2 -1 moves, the last two terms will be 2^2+1, and we shall have a strong reduction.

Example:
$(2^6+1) \times (2+1) + 1 =$
$2^7+2^6+2+1+1 =$
$(2^7+2^6+2^2):(2^2) =$
2^5+2^4+1

Movement II
$(2^5+2^4+1) \times (2+1) =$
$2^6+2^5+2^5+2^4+2+1+1 =$
$(2^7+2^4+2^2):(2^2) =$
2^5+2^2+1

Note that the next move will be a strong reduction.

Movement III
$(2^5+2^2+1) \times (2+1) + 1 =$
$2^6+2^5+2^3+2^2+2+1+1 =$
$(2^6+2^5+2^4):(2^4) =$
2^2+2+1
ICz (65) = {65, 49, 37, ICz{7}}

If "n" is an odd number, after (n-1)/2 movements, the last terms will be: 2+1. Therefore, we shall have one or more augmentative movements, depending on the alignment of the last terms.

Example:
$(2^5+1) \times (2+1) + 1 =$
$2^6+2^5+2+1+1 =$
$(2^6+2^5+2^2):(2^2) =$
2^4+2^3+1

Movement II
$(2^4+2^3+1) \times (2+1) =$
$2^5+2^4+2^4+2^3+2+1+1 =$
$(2^6+2^3+2^2):(2^2) =$
2^3+2+1

Note that the next move will be augmentative.

Movement III
$(2^3+2+1) \times (2+1) + 1 =$
$2^4+2^3+2^2+2+2+1+1 =$
$(2^5+2):(2) =$
2^4+1
ICz = {33, 25, 19, ICz{29}}

Note as well that, with every two movements, the difference between the second last and third last terms is only one unit, $2^{n-1} + 2^{n-2}$ e $2^{n-5} + 2^{n-6}$. This is key to understanding the orbits because it causes new alignments between the latter terms, which leads to new augmentative movements. This, in turn, makes the orbit of some numbers so complicated because the sequence decreases, and then increases again as the latter terms suffer repeated realignments. However, as has been shown in the previous item, the multiplication and addition operations produce terms with differing parity. As such, after an alternation of augmentative movements followed by monotonous movements, there will be a sequence of monotonous movements initiated by the penultimate term with an even exponent, which will end in a strong reduction. If, after the strong reduction, the penultimate term has and odd exponent, there will be monotonous movements followed by augmentative movement. Should it be even, there will be monotonous movements followed by a strong reduction.

Example:

$9 = 2^3 + 1$

$(2^3 + 1) \times (2+1) + 1 =$

$2^4+2^3+2+1+1 =$

$(2^4+2^3+2^2):(2^2) =$

2^2+2+1

Note that the terms are aligned, and we shall have two augmentative movements.

$(2^2+2+1):(2+1) + 1 =$

$2^3+2^2+2^2+2+2+1+1 =$

$(2^4+2^2+1):(2) =$

$(2^3+2+1) \times (2+1)+1 =$

$2^4+2^3+2^2+2+2+1+1 =$

$(2^5+2):(2) =$

Note that the parity of the penultimate term changed to even. As such, we shall have monotonous movements followed by strong reduction.

$(2^4+1) \times (2+1) + 1 =$

$2^5+2^4+2+1+1 =$

$(2^5+2^4+2^{2)}:(2^2) =$

$(2^3+2^2+1) \times (2+1) + 1 =$

$2^4+2^3+2^3+2^2+2+1+1 =$

$(2^5+2^3):(2^3) =$

$(2^2+1) \times (2+1) =$

$2^3+2^2+2+1+1 =$

$(2^4):(2^4) = 1$

This capacity for realignment is the reason that some of them (the number 129, for example) have such fantastic orbits. Note the segments in which, after one or more decreasing movements, the last terms suffer alignment, and begin growing anew.

ICz (129) = {129, 97, 73, (55, 83, 125,) (47, 71, 107, 161), 121, (91, 137), (103, 155, 233,) (175, 263, 395, 593), 445, (167, 251, 377), (283, 425, 319, 479, 719, 1079, 1619, 2429), (911, 1367, 2051, 3077), 577, 433, 325, 61, 23, 35, 53, 5}

2.9. Analysis of the Augmentative Movement

We take a number with aligned terms: $2^n+2^{n-1}....2^3+2^2+1$. By applying a movement, we shall have:

$(2^n+2^{n-1}....2^3+2^2+1) \times (2+1) + 1 =$

After the multiplication phase, we shall have the new terms in bold:

$2^{n+1}+2^n+2^n+2^{n-1}+2^{n-1}+2^{n-2}....2^4+2^3+2^3+2^2+2^2+2+2+1+1$

Summing up, separately, the new terms in bold with the last two terms, we shall have:

$2^{n+1}+2^n+2^{n-1}....2^3+2^2+2+1+1= 2^{n+2}$

Let's add this new term to the others, and divide by 2:

$(2^{n+2} +2^n+2^{n-1}+2^{n-2}....2^3+2^2+2):2=$

$2^{n+1} +2^{n-1}+2^{n-2}+2^{n-3}....2^2+2+1$

Producing other movements, we shall have:

$(2^{n+1} +2^{n-1}+2^{n-2}+2^{n-3}....2^2+2+1) \times (2+1)+1 =$

$2^{n+2}+2^{n+1}+2^n+2^{n-1}+2^{n-1} {}^{+}2^{n-2}....2^4+2^3+2^3+2^2+2^2+2+2+1+1 =$

$(2^{n+3} + 2^{n-1}+2^{n-2}+2^{n-3}....2^2+2):(2) = 2^{n+2} + 2^{n-2}+2^{n-3}+2^{n-4}....2^2+2+1$

$(2^{n+2} + 2^{n-2}+2^{n-3}+2^{n-4}....2^2+2+1) \times (2+1) +1 =$

$2^{n+3} + 2^{n+2} +2^{n-1}+2^{n-2}+2^{n-2}+2^{n-3}....2^2+2+2+1+1$

$(2^{n+3} + 2^{n+2} +2^n+2^{n-2}+2^{n-3}....2^3+2^2+2):(2) = 2^{n+2} + 2^{n+1} +2^{n-1}+2^{n-3}+2^{n-4}....2^2+2+1$

$(2^{n+2} + 2^{n+1} +2^{n-1}+2^{n-3}+2^{n-4}....2^2+2+1) \times (2+1) +1=$

Using associativity to highlight the fusions, we have:

$(2^{n+3} + 2^{n+2} +2^{n+2)} + (2^{n+1} +2^n+2^{n-1}+2^{n-2}+2^{n-3} +2^{n-3}+2^{n-4}+....2^2+2^2+2+2+1+1) =$

$2^{n+4} + 2^{n+2} +2^{n-3}+2^{n-4}....2^2+2=$

$(2^{n+4} + 2^{n+2} +2^{n-3}+2^{n-4}....2^2+2):(2) =$

$2^{n+3} + 2^{n+1} +2^{n-4}+2^{n-5}....2^2+2+1$

Note that, with each new movement, the quantity of queued terms decreases. This process will occur "n" times until the penultimate term is affected. After that, we shall have one or more monotonous movements.

If after "n" movements the penultimate term has an even exponent, we shall have monotonous movements followed by of a strong reduction. Inversely, if the penultimate term is odd, we shall have one or more monotonous movements followed by an alignment that will cause an increasing sequence in the orbit.

However, the effects of multiplication and addition operations will cause the terms to separate. The difference between the exponents of the terms that fall out of alignment will constantly increase, making new alignments impossible. This, in turn, will decrease movements predominant. Note that the difference between the exponents of the terms that fell out of alignment increased with each new movement.

3. Results

The orbits of the sequences produced by the Collatz C (n) operations can be divided into increasing and decreasing sub-sequences. Additionally, each sub-sequence can be determined by the binary ED form of each number initiating the subsequence.

The IC (n) subsequence, which uses only odd numbers, allows for more regular observation of the sub-sequence's behavior than C (n).

The obtained results allow for the creation of sequences with a desired behavior, with more or fewer terms where the 3n+1operation doesn't appear, occurs only once or at least "n" times.

4. Discussion

Mathematicians who have studied the Collatz Conjecture over the past 80 years have come to the conclusion that the mathematical knowledge available to them, including computers, was not up to the task of demonstrating. It took a new kind of number, the binary ED, to unravel this puzzle.

The ED numbers also produced other discoveries which can be found in Santos' book. It has only been published in Portuguese, to date, and does not contain the results laid out in this article. This, in fact, is the great advantage of researching conjectures in any area of mathematics. In addition to being fun and challenging, it also produces new mathematical knowledge that can be used in other areas.

Demonstrations of the generalizations created by other researchers are still lacking, and should remain for other mathematicians. They should be all that much easier with the knowledge found in the results of this research.

5. Conclusion

The factor that ensures that the Collatz Conjecture is a Theorem, is that the decreasing movements predominate in relation to the increasing movements. Even in a number formed by terms of odd exponents and a large initial alignment, the effect of multiplication and addition operations will cause the terms to have exponents with increasing differences. Additionally, the fusions will change the parities of the exponents. This leads to monotonous movements followed strong reductions.

Even if someone produces a very complicated number, using all of the knowledge about sockets and binary numbers, this sequence will have decreasing orbits with monotonous movements and strong reductions that, together, will predominate in relation to the increasing movement. This will always lead to the sequence ending in the number 1.

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