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The Study of the Concept of Q*Compact Spaces

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Abstract: The aim of this research is to extend the new type of compact spaces called Q* compact spaces, study its properties and generate new results of the space. It investigate the Q*-compactness of topological spaces with separable, Q*-metrizable, Q*-Hausdorff, homeomorphic, connected and finite intersection properties. The closed interval [0, 1] is Q* compact. So, it is deduced that the closed interval [0, 1] is Q*-compact. For example, if $(X,\tau) = \mathbb{R}$ and $A = (0,\infty)$ then A is not Q*-compact. A subset S of $\mathbb{R}$ is Q*-compact. Also, if (X,τ) is a Q*-compact metrizable space. Then (X,τ) is separable. (Y,τ_1) is Q*-compact and metrizable if f is a continuous mapping of a Q*-compact metric space (X, d) onto a Q*-Hausdorff space (Y,τ_1). An infinite subset of a Q*-compact space must have a limit point. The continuous mapping of a Q*-compact space has a greatest element and a least element. Eleven theorems were considered and their results were presented accordingly.

Keywords: Topological Paces, Semi Compact Spaces, Q*O Compact Space

1. Introduction

Borel proved in his 1894 Ph.D. thesis that a countable covering of a closed interval by open intervals has a finite subcover. It turns out that Borel's approach was similar to the approach Heine used to prove in 1872 that a continuous function on a closed interval was uniformly continuous (actually first proved, but unpublished for 60 years, by Dirichlet in 1852). In 1898, Lebesgue (and apparently someone named Cousins in 1895) removed "countable" from the hypothesis of Borel's result. Thus, the generalized theorem, which is now commonly called the Heine-Borel theorem.

Murugalingam and Lalitha (2010) introduced the concept of Q* sets [2]. Lalitha and Murugalingam (2011) further studied the properties of Q* closed and Q* open sets in affine space [3]. Padma (2015) introduced the concept of Q*O compact spaces and obtained very crucial results [7, 8] and applied results from [6]. Some important results on bitopological spaces are obtained in [1], [4], [5] and [11]. Let (X,τ) be a topological space. A subset S in X is said to be Q* closed in (X,τ) if S is closed and Int $(S) = \phi$. Its compliment S' is therefore Q* open [9, 10]. If every open cover of X has a finite sub cover then X is called a compact space. (X,τ) is said to be separable if it has a countable dense subset. Let X be a set and $\Im$ a family of subsets of X. Then $\Im$ is said to have Finite Intersection Property if for any finite number $F_1, F_2, ..., F_n$ of members of $\Im$, $F_1 \cap ... \cap F_n \neq \phi$ [9].

2. Preliminaries

This section gives an overview of the basic definitions of a compact space, Q*-compact which is the new type of a compact space.

Definition: A subset A of a topological space (X,τ) is said to be compact if every open covering of A has a finite subcovering. If the compact subset A equals X, then (X,τ) is said to be a compact space.

Definition: Let (X,τ) be a topological space. Then it is said to be connected if the only clopen subsets of X are X and ϕ.

Definition: A subset A of a topological space (X,τ) is said to be Q*O - compact space if every $\tau - Q*$ open cover of X has a finite sub cover.

Definition: Let (X,τ) be a topological space. Then it is said to be Q*- connected if the only Q* open subsets of X are X and ϕ.

3. Results on Generalization of Q*O Compact Space

Theorem: The closed interval [0, 1] is Q*-compact.

Proof: Let G_α, $\alpha \in \Lambda$ be any open covering of $[0, 1]$. Then for each $x \in [0,1]$, there is a G_α such that $x \in G_\alpha$. As G_α is open in about x, there exist an interval U_x, open in $[0, 1]$ such that $x \in U_x \subseteq G_\alpha$

Now define a subset S of [0, 1] as follows:

$$S = \{z: [0,z] \text{ can be covered by a finite number of sets } U_x\}$$

[So $z \in S \Rightarrow [0,z] \subseteq U_{x_1} \cup U_{x_2} \cup ... \cup U_{x_n}$ for some $x_1, x_2, ..., x_n$

Now let $x \in S$ and $y \in U_x$. Then as U_x is an interval containing x and y, $[x,y] \subseteq U_x$. (Here we are assuming without loss of generality that $x \le y$). So $[0,y] \subseteq U_{x_1} \cup U_{x_2} \cup ... \cup U_{x_n} \cup U_x$ and hence $y \in S$.

For each $x \in [0,1], U_x \cap S = U_x$ or ϕ.

This implies that $S = \bigcup_{x \in S} U_x$ and $[0,1] \mid S = \bigcup_{x \notin S} U_x$

Thus, S is open in [0,1] and S is closed in [0,1]. But [0,1] is connected. Therefore $S = [0,1]$ or ϕ

However, $0 \in S$ and so $S = [0,1]$; that is, [0,1] can be covered by a finite number of U_x. So $[0,1] \subseteq U_{x_1} \cup U_{x_2} \cup ... \cup U_{x_m}$. But each U_x is contained in a $G_\alpha, \alpha \in \Lambda$.

Hence $[0,1] \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup ... \cup G_{\alpha_m}$ and it has been established that [0,1] is Q*-compact.

Example: If $(X,\tau) = \mathbb{R}$ and $A = (0,\infty)$ then A is not Q*-compact

Proof: For any integer α, Let G_α, be any interval $(0,\alpha)$. Then clearly, $A \subseteq \bigcup_{\alpha=1}^{\infty} G_\alpha$. But there do not exist $\alpha_1, \alpha_2, ..., \alpha_n$ such that $A \subseteq (0,\alpha_1) \cup (0,\alpha_2) \cup ... \cup (0,\alpha_n)$. Therefore, A is not Q*-compact.

Corollary: For a and b in $\mathbb{R}$ with $a < b$, [a, b] is Q* compact, while (a, b) is not Q*-compact.

Proof: The space [a, b] is homeomorphic to the Q*-compact [0, 1] and so, and by Padma (2015), is Q*- compact.

The space (a,b) is homeomorphicto $(0,\infty)$. If (a,b) were Q*-compact, then $(0,\infty)$ would be Q*-compact, but by example 4.1 $(0,\infty)$ is not Q*-compact. Hence (a,b) is not Q*-compact.

Example: Suppose $X = \{e,f,g,h\}, \tau = \{\phi, X, \{e,f,g\}, \{e,f,h\}\}$, Let $S = \{e,g,h\}$. Now $A \subset \{a,d\} \cup \{b,c\}$. By definition, S is compact set. But S is not a Q*O - compact set because S is not Q* closed since its complement $\{b\}$ is not Q* open.

Remark: Every Q*O - compact space is compact, but the converse is not necessarily true.

Theorem: A subset S of $\mathbb{R}$ is Q*-compact if and only if S is closed and bounded.

Proof: First suppose that S is Q*-compact. To see that S is bounded is fairly simple: Let $I_n = (-n,n)$

Then $\bigcup_{n-1}^{\infty} I_n = \mathbb{R}$

Therefore S is covered by the collection of $\{I_n\}$. Hence, since S is Q*-compact, finitely many will suffice,

$$S \subseteq \left(I_{n_1} \cup ... \cup I_{n_k}\right) = I_m$$

Where $m = max\{n_1, ..., n_k\}$. Therefore $|x| \le m$ for all $x \in S$, and S is bounded.

Now showing that S is closed. Suppose not. Then there is some point $p \in (cl(S)) \mid S$. For each n, define the neighborhood around p of radius $1/n$, $N_n = N(p, 1/n)$. Take the complement of the closure of $N_n U_n = R \mid cl(N_n)$ is open (since its complement is closed), and we have

$$\bigcup_{n-1}^{\infty} U_n = \mathbb{R} \mid \bigcap_{n-1}^{\infty} cl(N_n) = \mathbb{R}\{p\} \supseteq S$$

Therefore, $\{U_n\}$ is an open cover for S. Since S is Q*-compact, there is a finite subcover $U_{n_1}, ..., U_{n_k}$ for S. Furthermore, by the way, they are constructed, $U_i \subseteq U_j$ if $i \le j$. It follows that $S \subseteq U_m$ where $m = max\{n_1, ..., n_k\}$. But then $S \cap N(p, 1/m) = \phi$, which contradicts our choice of $p \in cl(S) \mid S$.

Conversely, there is need to show that if S is closed and bounded, then S is Q*-compact. Let $\Im$ be an open cover for S. For each $x \in \mathbb{R}$, define the set

$$S_x = S \cap (-\infty, x],$$

and let

$$B = \{x : S_x \text{ is covered by a finite subcover of } \Im\}.$$

Since S is closed and bounded, hypothesis tells us that S has both a maximum and a minimum. Let $d = minS$. Then $S_x = \{d\}$ and this is certainly covered by a finite subcover of $\Im$. Therefore, $d \in B$ and B is nonempty. If it is shown that B is not bounded above, then it will contain a number p greater than max S. But then, $S_p = S$ so we can conclude that S is covered by a finite subcover, and is therefore Q*-

compact.

Toward this end, suppose that B is bounded above and let $m = supB$. We shall show that $m \in S$ and $m \notin S$ both lead to contradictions.

If $m \in S$, then since $\Im$ is an open cover of S, there exists F_0 in $\Im$ such that $m \in F_0$. Since F_0 is open, there exists an interval $[x_1, x_2]$ in F_0 such that $x_1 < m < x_2$. Since $x_1 < m$ and $m = supB$, there exists $F_1, ..., F_k$ in $\Im$ that cover S_{x_1}. But then $F_0, F_1, ..., F_k$ cover S_{x_2}, so that $x_2 \in B$. But this contradicts $m = supB$.

If $m \notin S$, then since S is closed there exists $\epsilon > 0$ such that $N(m, \epsilon) \cap S = \phi$. But then

$$S_{m-\epsilon} = S_{m+\epsilon}$$

Since $m - \epsilon \in B$ then $m + \epsilon \in B$, which again contradicts $m = supB$.

Therefore, either way, if B is bounded above, we get a contradiction. We conclude that B is not bounded above, and S must be Q*-compact.

Theorem: Let (X, τ) be a Q*-compact metrizable space. Then (X, τ) is separable.

Proof: Let d be a metric space on X which induces the topology τ. For each positive integer n, let S_n be the family of all open balls having centres in X and radius $\frac{1}{n}$. Then S_n is an open covering of X and so there is a finite subcovering $\mu_n = \{U_{n_1}, U_{n_2}, ..., U_{n_k}\}$, for some $k \in \aleph$. Let y_{n_j} be the centre of U_{n_j}, $j = 1, ..., K$, and $Y_n = \{y_{n_1}, y_{n_2}, ..., y_{n_k}\}$.

Put $Y = \bigcup_{n=1}^{\infty} Y_n$. Then Y is a countable subset of X. Now showing that Y is dense in (X, τ).

If V is any non-empty open set in (X, τ), then for any $v \in V$, V contains an open ball, B, of radius $\frac{1}{n}$, about v, for some $n \in \aleph$. As μ_n is an open cover of X, $v \in U_{n_j}$, for some j. Thus $d\left(v, y_{n_j}\right) < \frac{1}{n}$ and so $y_{n_j} \in B \subseteq V$. Hence, $V \cap Y \neq \phi$, and so Y is dense in X.

Theorem: Let (X, τ) be a topological space. Then (X, τ) is Q*-compact if and only if every family $\Im$ of closed subsets of X with the finite intersection property satisfies $\bigcap_{F \in \Im} F \neq \phi$.

Proof: Assume that every family $\Im$ of closed subsets of X with the finite intersection property satisfies $\bigcap_{F \in \Im} F \neq \phi$. Let μ be any open covering of X. Put $\Im$ equal to the family of complements of members of μ. So each $F \in \Im$ is closed in (X, τ). AS μ is an open covering in X, $\bigcap_{F \in \Im} F \neq \phi$. By our assumption, then $\Im$ does not have finite intersection property. So, for some $F_1, F_2, ..., F_n$ in $\Im$, $F_1 \cap F_2 \cap ... \cap F_n \neq \phi$. Thus $U_1 \cup U_2 \cup ... \cup U_n = X$, where $U_i = X \mid F_i$, $i = 1, ..., n$. So μ has a finite subcovering. Hence, (X, τ) is Q*-compact.

The converse statement is proved similarly.

Theorem: Let f be a continuous mapping of a Q*-compact metric space (X, d) onto a Q*-Hausdorff space (Y, τ_1). Then (Y, τ_1) is Q*-compact and metrizable.

Proof: Since every Q*-continuous image of a compact space is compact (Padma 2015), the space (Y, τ_1) is certainly compact and metrizable. As the map f is surjective, define the metric d_1 on Y as follows:

$$d_1(y_1, y_2) = inf\left\{d(a,b) : a \in f^{-1}\{y_1\} and b \in f^{-1}\{y_2\}\right\}, \text{ for all } y_1 \text{ and } y_2 \text{ in } Y.$$

To show that d_1 is indeed a metric. Since $\{y_1\}$ and $\{y_2\}$ are closed in the Q*-Hausdorff space (Y, τ_1), $f^{-1}\{y_1\}$ and $f^{-1}\{y_2\}$ are Q*- compact. So, the product $f^{-1}\{y_1\} \times f^{-1}\{y_2\}$, which is a subspace of the product space $(X, \tau) \times (X, \tau)$, is Q*-compact, where τ is the topology induced by the metric d.

Observing that $d : (X, \tau) \times (X, \tau) \to \mathbb{R}$ is a continuous mapping, then $d\left(f^{-1}\{y_1\} \times f^{-1}\{y_2\}\right)$, has a least element.

So there exist an element $x_1 \in f_{-1}\{y_1\}$ and an element $x_2 \in f_{-1}\{y_2\}$ such that

$$d(x_1, x_2) = inf\left\{d(a,b) : a \in f^{-1}\{y_1\}, b \in f^{-1}\{y_2\}\right\} = d_1(y_1, y_2)$$

Clearly if $y_1 \neq y_2$, then $f^{-1}\{y_1\} \cap f^{-1}\{y_2\} = \phi$. Thus $x_1 \neq x_2$ and hence $d(x_1, x_2) > 0$; that is $d_1(y_1, y_2) > 0$.

It is easily verified that d_1 has the other properties required of a metric, and so a metric on Y.

Let τ_2 be the topology induced on Y by d_1. To show that $\tau_1 = \tau_2$.

Firstly, by the definition of d_1, $f : (X, \tau) \to (X, \tau_2)$ is certainly continuous.

Observe that for a subset C of Y,

C is a closed subset of (Y, τ_1)

$\Rightarrow f^{-1}(C)$ is a closed subset of (X, τ)

$\Rightarrow f^{-1}(C)$ is a Q*-compact subset of (X, τ)

$\Rightarrow f\left(f^{-1}(C)\right)$ is a Q*-compact subset of (X, τ_2)

SO $\tau_1 \subseteq \tau_2$

Similarly, we have $\tau_2 \subseteq \tau_1$, and thus $\tau_1 = \tau_2$

Theorem: Let (X,τ) be a Q*-compact space and $f:(X,\tau) \to \mathbb{R}$ a continuous mapping. Then $f(X)$ has a greatest element and a least element.

Theorem: If $(X_1,\tau_1),(X_2,\tau_2),...,(X_n,\tau_n)$ are Q*- compact spaces, then $\prod_{i=1}^{n}(X_i,\tau_i)$ is a Q*- compact space.

Proof: The first part of this proof is to show that the product of any two Q*- compact topological spaces is Q*-compact.

Suppose (X_1,τ_1) and (X_1,τ_1) are Q*- compact then $(X_1,\tau_1)\times(X_2,\tau_2)$ is also Q*- compact Padma (2015).

Then by induction, we can say that:

$(X_1,\tau_1)\times(X_2,\tau_2)\times(X_3,\tau_3) = [(X_1,\tau_1)\times(X_2,\tau_2)]\times(X_3,\tau_3)$ is also Q*- compact since it is also a product of two Q*-compact spaces.

Conclusively, suppose that the product of any two N Q*-compact spaces is Q*- compact. Consider the product $(X_1,\tau_1)\times(X_2,\tau_2)\times...\times(X_{N+1},\tau_{N+1})$ of Q*- compact spaces (X_i,τ_i), i = 1,..., N+1. Then,

$$(X_1,\tau_1)\times...\times(X_N,\tau_N)\times(X_{N+1},\tau_{N+1}) \cong [(X_1,\tau_1)\times...\times(X_N,\tau_N)]\times(X_{N+1},\tau_{N+1})$$

By our inductive hypothesis $(X_1,\tau_1)\times...\times(X_N,\tau_N)$ is Q*-compact, so the right-hand side is the product of two Q*-compact spaces and thus is Q*- compact. Therefore, the left-hand side is also Q*- compact.

Theorem: Let $\{(X_i,\tau_i): i \in I\}$ be any family of topological spaces. Then $\prod_{i\in I}(X_i,\tau_i)$ is Q*-compact if and only if each (X_i,τ_i) is Q*-compact.

Theorem: If X is not Q*-compact, then X is homeomorphic to an open dense set in χ. (Where χ is not too larger than X).

Proof: Suppose we ensure that χ is not "too large", that is, not "too much larger" than X.

First show that X is homeomorphic to the set $\{X\} \subset \chi$.

Construct a function that sends each point of X to the corresponding point in $\{X\}$. This function is obviously one-to-one and onto, and it is continuous (and so is its inverse) because the open sets in $\{X\}$ are exactly the open sets in X.

The set $\{X\}$ is open in χ, because it does not contain ∞ and it is open in X. To show that $\{X\}$ is dense, we can simply show that it is not closed, or that ∞ is not open. (If that's the case, then $\{X\}$ is not its own closure, and the only other option is that its closure is χ). If ∞ is open, then its complement, $\{X\}$, must be compact. But this would imply that X is Q*-compact, contradicting our earlier assumption. So ∞ cannot be open, meaning $\{X\}$ must be dense.

Theorem: If none of the components of X is Q*-compact, then χ is connected.

Proof: Assume that χ is not connected, i.e. there is some set U in χ that is open and closed, but is not ϕ or χ. Its complement, V, is also open and closed without being ϕ or χ. Either U or V contains ∞; take the one that does not, and call it W. W is Q*-compact because its complement is open and contains ∞.

First let us consider the case that X is connected. We have already established that W is not ϕ. It cannot be all of X either, because W is Q*-compact and X is not. W is open in X because it is open in χ and does not contain ∞. It is closed in X because its complement (either $U \cap X$ or $V \cap X$) is open in X. So, W is open, closed, not ϕ, and not X, which implies that X is not connected. This contradicts our assumption, so χ must be connected.

4. Conclusion

But what if X is not connected? In this case, we look at the connected components of X. Any open set including ∞ must also contain points in each of the components of X (because the complement of the open set is Q*-compact, and if the complement included an entire connected component, then that component would need to be Q*-compact, but it is not). So W contains some points in each of the components. But this would imply that the connected components are not connected, which is our contradiction. So again, χ must be connected.

It is also true that every Q*O compact space is a Q* - Lindelof space. Every Q*O –compact topological space is Q* - countably compact. Since the space is Q*O –compact, every τ- Q* open covering of X has a finite subcover. Hence, every countable τ - Q* open covering of X has a finite subcover and therefore it is countably compact.

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