

The Development of Napoleon's Theorem on the Quadrilateral in Case of Outside Direction

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To cite this article:

Mashadi, Chitra Valentika, Sri Gemawati, Hasriati. The Development of Napoleon's Theorem on the Quadrilateral in Case of Outside Direction. *Pure and Applied Mathematics Journal*. Vol. 6, No. 4, 2017, pp. 108-113. doi: 10.11648/j.pamj.20170604.11

Received: May 6, 2017; **Accepted:** June 14, 2017; **Published:** July 18, 2017

Abstract: In this article we discuss Napoleon's theorem on the rectangles having two pairs of parallel sides for the case of outside direction. The proof of Napoleon's theorem is carried out using a congruence approach. In the last section we discuss the development of Napoleon's theorem on a quadrilateral by drawing a square from the midpoint of a line connecting each of the angle points of each square, where each of the squares is constructed on any quadrilateral and forming a square by using the row line concept.

Keywords: Napoleon's Theorem, Napoleon's Theorem on Rectangles, Outside Direction, Congruence

1. Introductions

Napoleon's theorem was found by Napoleon Bonaparte (1769-1821), an Emperor of French who had a big interest in geometry [8]. Napoleon's theorem on triangle is equilateral triangle constructed on each side of triangle that leads into outside. Furthermore on each equilateral triangle one can obtain a midpoint that is angle from a new equilateral triangle [2]. The new equilateral triangle can be called as Napoleon's triangle [6].

Napoleon's theorem on the triangle can be proved by using congruence [7] and trigonometry algebra [1, 3 and 5]. Many authors have provided a wide range of alternative proofs as well as developed the Napoleon's theorem, for example [6] develops on the inner Napoleon's theorem, whereas [14] describes it if an isosceles (right angle) of mother triangle is given, what properties must the external and internal Napoleon triangles have? What properties must the external and internal Napoleon's triangles of a mother have to ensure that the mother is an isosceles triangle (right angle) triangle? While [2] developed for the case of the pedal triangle. Many other developments associated with the various theorems in geometry are discussed by a

number of authors [3, 4, 5, 10, 11, 12 and 13]. The authors in [5, 6, 7, 8, 9 and 15] developed for the rectangles for the case of inside direction where the proofs are done using a very simple concept.

In this paper we give the development of Napoleon's theorem to the quadrilateral having two pairs of parallel sides and one form of development of Napoleon's theorem for the case of parallelogram. Its development is to construct a quadrilateral formed by the lines connecting the midpoint of each square obtained from the construction. The main content is the process of proofs using a simple geometry concept (the same as in [1, 9] and 17). In the final part we give three corollaries in many cases.

2. Napoleon's Theorem on Quadrilateral

Before discussing Napoleon's theorem on a quadrilateral, we first give an illustration of the Napoleon theorem for triangles for the case of outside direction, the illustration can be seen in Figure 1. Similar conditions will also be applied to the quadrilateral

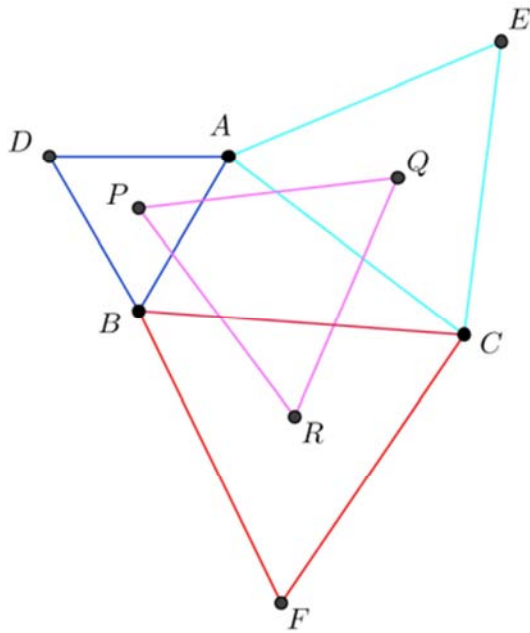


Figure 1. Napoleon's Theorem on triangle for the case of outside direction.

In [11] the development of the Napoleon theorem is carried out for the case of inside direction. While in this paper we do it for the case that leads out, so the development of Napoleon's theorem in this paper for quadrilateral is also done for the case that leads out. If the Napoleon's triangle theorem applied to any construct the triangle with equilateral triangles outward on each side. However, for any quadrilateral we cannot enforce Napoleon's theorem to construct rectilinear direction out on any quadrilateral proficiency level (as an extreme example only create trapezium), it would not produce such a theorem Napoleon, as intended. But if taken rectangle, it is definitely Napoleon's theorem applies. For the following will be made for cases of a parallelogram.

Napoleon's theorem on rectangle will be discussed is on the quadrilateral that has two pairs parallel line, one of them is parallelogram in case of square that constructed outside direction. On AB side constructed ABHG square, AD side constructed ADEF square, CD side constructed CDKL square, and BC side constructed BCIJ square. Then every squares constructed in outside direction. Furthermore every midpoints are connected to shape the square that can be called as Napoleon's quadrilateral in outside direction.

Theorem 1. Provide rectangle in shaping ABCD parallelogram. On each side is constructed ABHG square, ADEF square, CDKL square, and BCIJ square in outside direction. For example M, N, O, and P are midpoints of square that constructed in outside direction. If the four midpoints are connected they will be shaped MNOP square.

Proof: Alternatif 1. To showing MNOP is square, it will proved by $MN = NO$, and $\angle PMN = 90^\circ$. from $\triangle GAD$ and $\triangle BAF$, are obtained $AG = AB$, $\angle GAD = \angle FAB$, $AD = AF$, so $\triangle GAD \cong \triangle BAF$ [4]. Look at $\triangle GQT$ and $\triangle BAT$ on figure 4, $\angle TGQ = \angle TBA$ and $\angle GTQ = \angle BTA$, so $\angle GQT = \angle BAT = 90^\circ$.

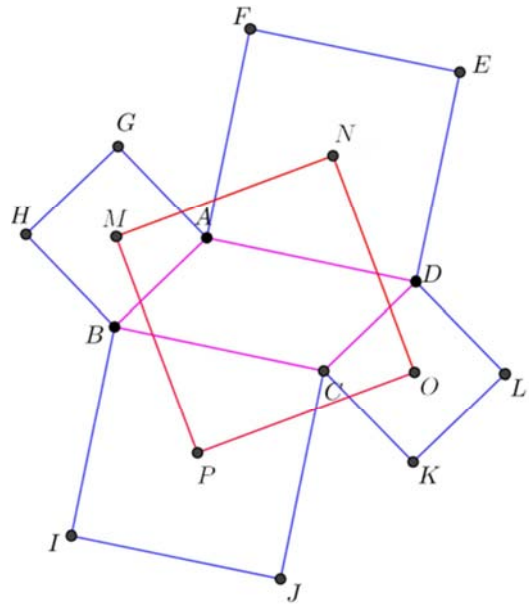


Figure 2. Napoleon's theorem on square.

Furthermore pull line NP and MO then cut it in one point, say point R. For example point S cuts point NP and GD, and U cuts point FB and MO line. It will be shown that BF is parallel with PN. FN and BP is a half diagonal square ADEF and BCIJ, because of ADEF square and BCIJ are parallel then FN is parallel with BP. Then pull FP line then it is obtained opposite angle those are $\angle BFP = \angle FPN$ and $\angle BPF = \angle PFN$ that caused BF also parallel with PN. Because of BF is parallel with PN so $\angle GQT = \angle QSR = \angle MRN = 90^\circ$. To make it clearer look at Figure 3.

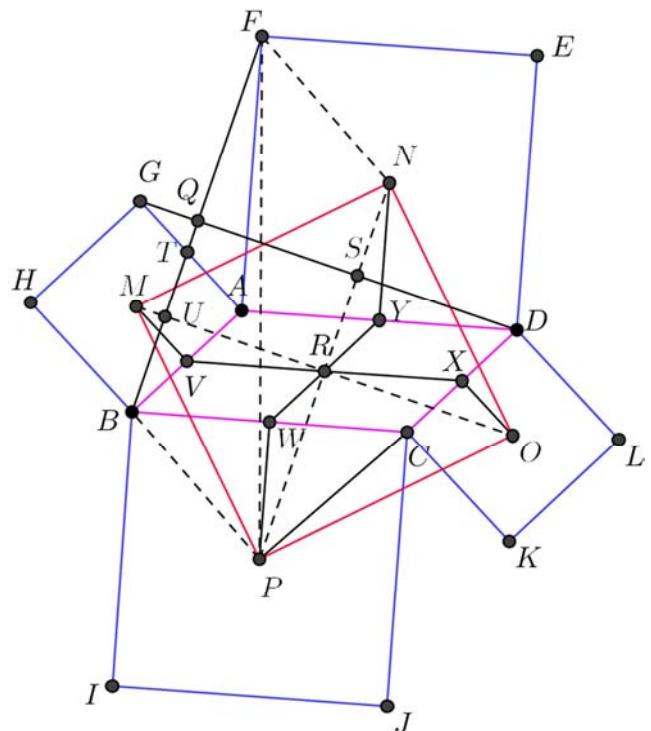


Figure 3. Proof of Napoleon's Theorem on a quadrilateral using congruence.

For example, T , U , and V is the midpoint of the line AD , CD , BC and AB . From $\triangle MVR$ and $\triangle NSR$, $MV = SR$, $\angle MVR = \angle NSR$, $VR = NS$, then $\triangle MVR \cong \triangle NSR$, causing $MR = RN$. Because $MR = RN$ and $\angle MRN = 90^\circ$, so $\triangle MRN$ is an isosceles triangle, as well as $\triangle RNO$ that causes $\angle MNR = \angle RNO = 45^\circ$. Then it is obtained that $\angle MNO = 90^\circ$, in the same way $\angle OPM = \angle NOP = \angle PMN = 90^\circ$. From $\triangle MRP$ and $\triangle NRO$ on, $MR = RO$, $\angle MRN = \angle NRO$, $NR = NR$, then $\triangle MRP \cong \triangle NRO$. So that led to $MN = ON$, then in the same way then $MP = OP$. $MNOP$ it is clear that the quadrilateral is a square.

Alternative 2. $MNOP$ is a square will be proven by showing $MN = NO$, and $\angle PMN = \angle MNO = 90^\circ$ with trigonometric approach is by using the cosine rule and sine rule [6].

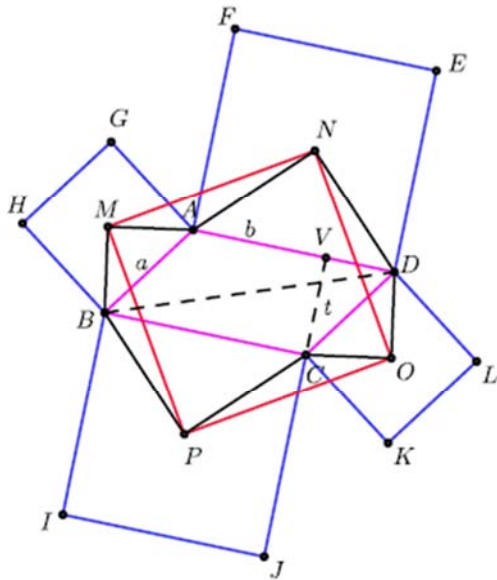


Figure 4. Proof of Napoleon's Theorem on a quadrilateral using trigonometry.

In parallelogram $ABCD$, Let $AB = CD = a$, side $AC = BD = b$, use basic trigonometry to obtain

$$AM = MB = OC = OD = \frac{1}{2}a\sqrt{2}$$

$$AM = MB = OC = OD = \frac{1}{2}b\sqrt{2}$$

So that at $\triangle MAN$ using the cosine rule [5] apply

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - 2 \cdot \frac{1}{2}a\sqrt{2} \cdot \frac{1}{2}b\sqrt{2} \cdot \cos \angle MAN$$

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot \cos(270^\circ - \angle BAD)$$

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot (\cos 270^\circ \cdot \cos \angle BAD + \sin 270^\circ \cdot \sin \angle BAD)$$

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \sin \angle BAD \quad (1)$$

and on $\triangle ABD$ apply

$$\sin \angle BAD = \frac{2 \cdot L_{\triangle ABD}}{a \cdot b} \quad (2)$$

Substitute the equation (2) to (1) in order to obtain

$$\begin{aligned} MN^2 &= \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \frac{2 \cdot L_{\triangle ABD}}{a \cdot b} \\ MN &= \sqrt{\frac{1}{2}a^2 + \frac{1}{2}b^2 + 2 \cdot L_{\triangle ABD}} \\ MN &= \sqrt{\frac{1}{2}a^2 + \frac{1}{2}b^2 + L \cdot ABCD} \end{aligned} \quad (3)$$

Subsequently the $\triangle NDO$ we obtained

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - 2 \cdot \frac{1}{2}a\sqrt{2} \cdot \frac{1}{2}b\sqrt{2} \cdot \cos \angle NDO$$

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot \cos(90^\circ + \angle ADC)$$

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot (\cos 90^\circ \cdot \cos \angle ADC - \sin 90^\circ \cdot \sin \angle ADC)$$

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \sin \angle ADC \quad (4)$$

If

$$t = \frac{L \cdot ABCD}{b} \text{ and } \sin \angle ADC = \frac{t}{a}$$

Then we have

$$\sin \angle ADC = \frac{L \cdot ABCD}{a \cdot b} \quad (5)$$

Substitute the equation (5) to (4) we obtained

$$\begin{aligned} NO^2 &= \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \frac{L \cdot ABCD}{a \cdot b} \\ NO &= \sqrt{\frac{1}{2}a^2 + \frac{1}{2}b^2 + L \cdot ABCD} \end{aligned} \quad (6)$$

Based on the equation (5) and equation (6) it is clear that $MN=NO$. Furthermore, to show $\angle MNO = 90^\circ$ consider Figure 5.

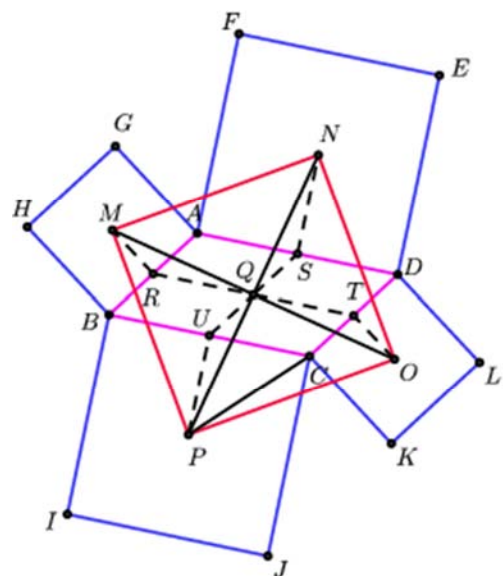


Figure 5. Proof of Napoleon's Theorem on a quadrilateral using trigonometry.

Then to show $\angle MNO = 90^\circ$, must first be proven $OQ = NQ$ with the cosine rule. Furthermore, by using the cosine rule on ΔNOQ obtained $\angle NQO = 90^\circ$, so ΔNOQ is an isosceles triangle that causes $\angle ONQ = \angle NOQ = 45^\circ$, as well as on ΔMNQ the same way in order to obtain $\angle MNQ = \angle NMQ = 45^\circ$. Thus obtained $\angle MNO = 90^\circ$, in the same way also obtained $\angle NOP = \angle OPM = \angle NMP = 90^\circ$. Because $MN = NO$ and $\angle NOP = \angle OPM = \angle MNQ = \angle NMP = 90^\circ$, so it is evident that $MNOP$ quadrilateral is a square.

3. The Development of Napoleon's Theorem on Quadrilateral

Basically Napoleon's theorem on the quadrilateral can only be developed in a quadrilateral that has two pairs of parallel sides. The following will be given the development of Napoleon's theorem on a parallelogram, theorem developed based on a rectangle to a square case built leading to the outside which is a square formation by connecting endpoints to construct a square formed on the sides of the initial parallelogram.

Theorem 2 Given a parallelogram quadrilateral $ABCD$, and on each side of the square was built leading out. Then draw a line FG , EL , KJ , and HI . Suppose point Q , R , S , and T is the midpoint of the fourth line. If the four points are connected, the square formed $QRST$.

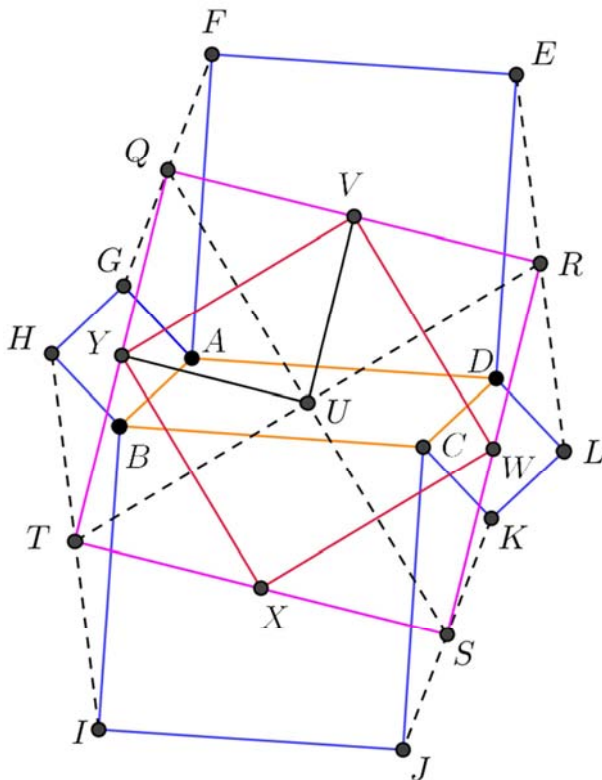


Figure 6. Development of Napoleon's theorem on the quadrilateral.

Proof. Suppose point Q , R , S , and T is the midpoint of the line FG , EL , KJ , and HI . To show $QRST$ is square it will be proved $TQ = QR$, and $\angle TQR = 90^\circ$. Consider Figure 6, draw a line from point Q to point S and point R to the point Q . So the lines QS and RQ lines intersect at one point, said point U . Before the show $TQ = QR$ will at first show $UT = UR$. Note Figure 4, $UY = UT$, $YT = VR$ so $UT = UR$. Then Note figure 6, $UY = UT$, $YT = VR$ so $UT = UR$. Then look at ΔQUT and ΔQUR , $UT = UR$, $\angle TUQ = \angle RUQ$ and $UQ = UQ$ then obtained $TQ = QR$. From theorem 1, $VU = UY$, and $\angle VUY = 90^\circ$, therefore $\angle QTR = 90^\circ$, so it proved $QRST$ quadrilateral is a square.

Based on the Theorem 1 and Theorem 2, and by way of connecting the vertex and the lines are connected are three types of effect, that the evidence is not given here because it always can easily be shown, however, given the geometry illustration

Corollary 1. On the square due $MNOP$ and $TQRS$ formed parallel lines $PQ \parallel SN$ and $MR \parallel TO$, then the square formed $VWZU$, and if formed parallel lines $MS \parallel QO$ and $TN \parallel PR$, then the square formed $A_1B_1C_1D_1$. The illustration is in Figure 7.

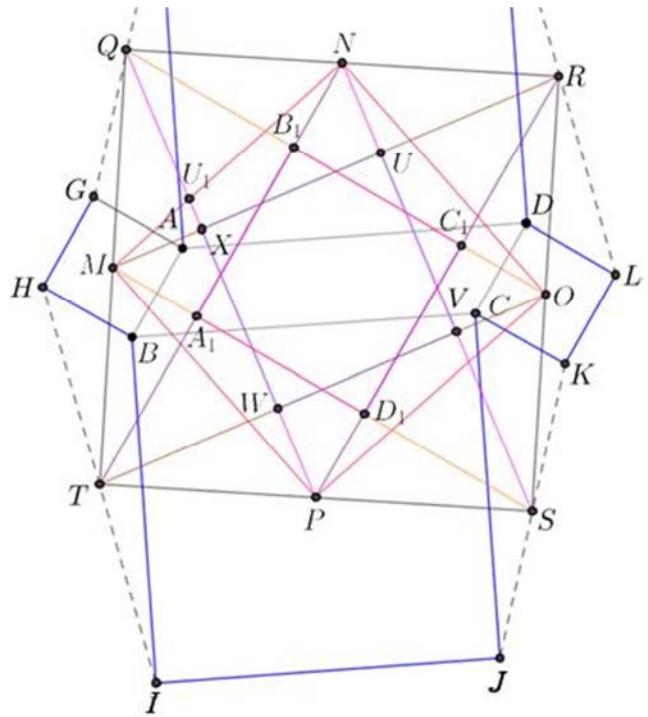


Figure 7. Corollary 1 of Theorem Napoleon in the quadrilateral.

Corollary 2. On the square $VWZU$ and $A_1B_1C_1D_1$ parallel lines $UA_1 \parallel VC_1$ and $ZD_1 \parallel WB_1$ are formed, then a square $K_1N_1M_1H_1$ is formed, and if parallel lines $VB_1 \parallel UD_1$ and $ZD_1 \parallel WB_1$ are formed, then a square $O_1P_1Q_1R_1$ is formed. The illustration is in Figure 8.

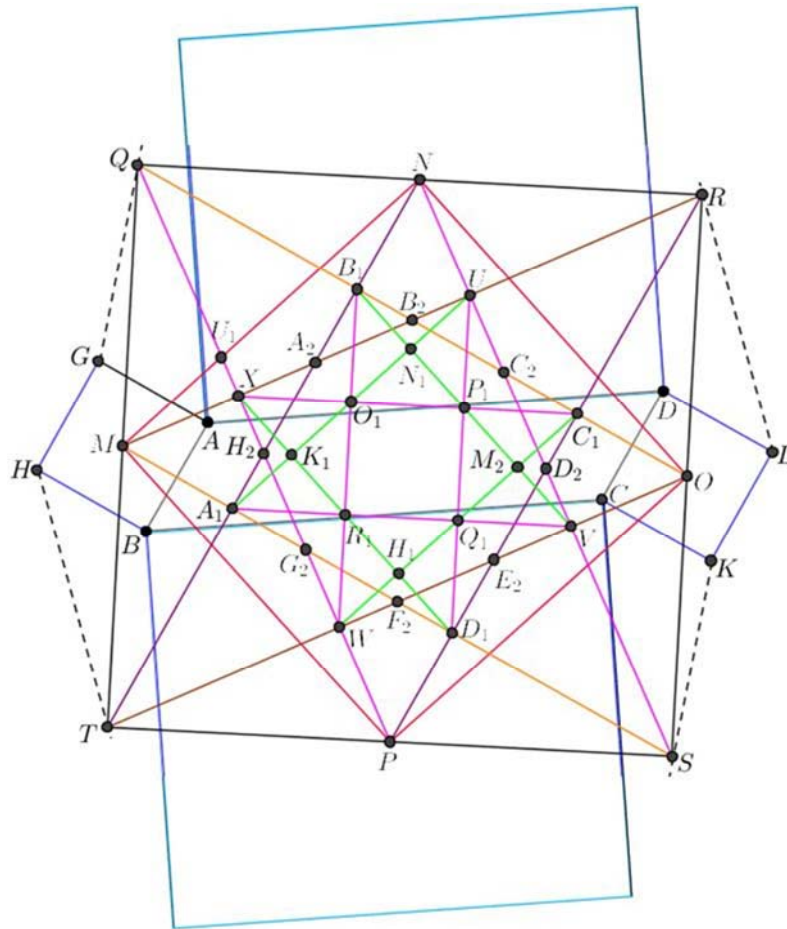


Figure 8. Corollary 2 of Theorem Napoleon in the quadrilateral.

Corollary 3. On the Picture $K_1N_1M_1H_1$ and $O_1P_1Q_1R_1$ form parallel lines $K_1P_1 \parallel M_1R_1$ and $K_1Q_1 \parallel M_1O_1$, then they form a square $E_1F_1I_1G_1$, and if parallel lines $N_1R_1 \parallel M_1O_1$ and $K_1Q_1 \parallel M_1O_1$ are formed, then a square $J_1L_1T_1S_1$ is formed. Illustration is in Figure 9.

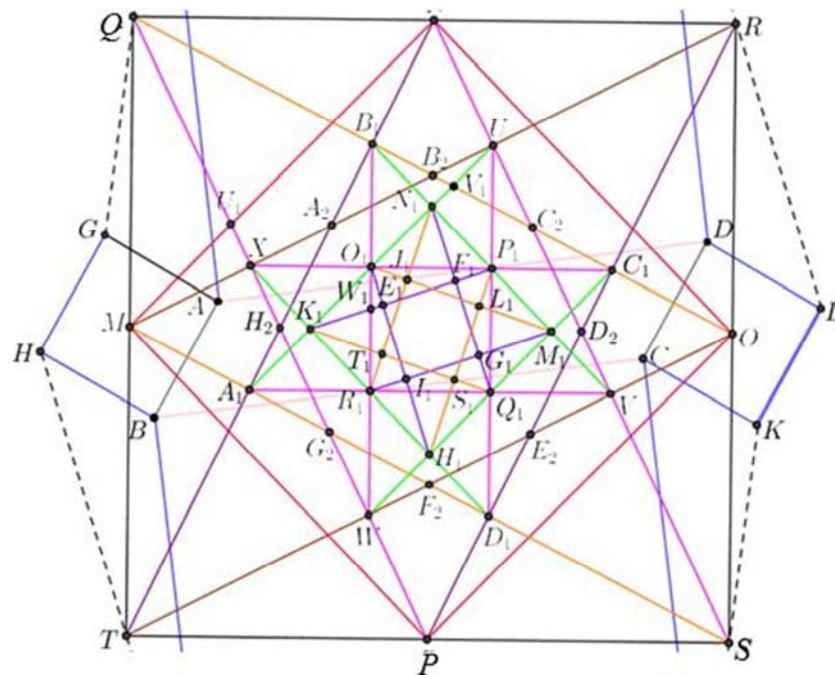


Figure 9. Corollary 3 of Theorem Napoleon in the quadrilateral.

4. Conclusion

After several experiments on Napoleon's Theorem in the quadrilateral we thus observe that Napoleon's theorem applied only to the quadrilateral which possess two pairs of parallel sides like a square, rhombus, rectangle, and parallelogram. Napoleon's Theorem in the ranks of the cases led to the outside that is if the square was built on each side of the square center of the fourth point would be to the square Napoleon called quadrilateral outside. Proof of that is done by using the concept of congruence. Development of Napoleon on a quadrilateral theorem can be developed to form a square on the midpoint of the line so as to form a new square, and forming a square with the concept of parallel lines that intersect.

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