The Ito Formula for the Ito Processes Driven by the Wiener Processes in a Banach Space

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Abstract: Using traditional methods it is possible to prove the Ito formula in a Hilbert space and some Banach spaces with special geometrical properties. The class of such Banach spaces is very narrow-they are subclass of reflexive Banach spaces. Using the definition of a generalized stochastic integral, early we proved the Ito formula in an arbitrary Banach space for the case, when as initial Ito process was the Wiener process. For an arbitrary Banach space and an arbitrary Ito process it is impossible to find the sequence of corresponding step functions with the desired convergence. We consider the space of generalized random processes, introduce general Ito process there and prove in it the Ito formula. Afterward, from the main Ito process in a Banach space we receive the generalized Ito process in the space of generalized random processes and we get the Ito formula in this space. Then we check decompasibililty of the members of the received equality and as they turn out Banach space valued, we get the Ito formula in an arbitrary Banach space. We implemented this approach when the stochastic integral in the Ito process was taken from a Banach space valued non-anticipating random process by the one dimensional Wiener process. In this paper we consider the case, when the stochastic integral is taken from an operator- valued non-anticipating random process by the Wiener process with values in a Banach space.

Keywords: Wiener Process in a Banach Space, Covariance Operators, Ito Stochastic Integrals and Ito Processes, the Ito Formula, Stochastic Differential Equations in a Banach Space

1. Introduction and Preliminaries

As in the finite dimensional case, the Ito formula plays an important role in the infinite dimensional stochastic analysis. For the cases when the Banach spaces have special geometrical properties, the Ito formula was proved in [1] and [2]. For the Wiener process in an arbitrary separable Banach space, the Ito formula was proved in [3]. The Ito formula for the case, when the stochastic integral that appears in the Ito process is taken from Banach space valued non-anticipating process by the one dimensional Wiener process was proved in [4]. In this paper we prove the Ito formula for the Ito processes when the stochastic integral is taken from the operator valued non-anticipating random processes by the Wiener processes in a Banach space. The main unsolved problem to prove this formula in an arbitrary Banach space is to find such a sequence of step functions converging to the integrand function that their stochastic integrals converge to the stochastic integral from the integrand function. We use the concept of the generalized random element; we consider the space of generalized random processes and introduce the

generalized Ito process there. Firstly we prove the Ito formula for the generalized random processes. Then from the initial Ito process in a Banach space we receive a generalized Ito process and write the Ito formula there. Afterward, In the obtained equality we check decomposability of the members of the equality; we found that all of them are Banach space valued. Therefore, we get the Ito formula for the Banach space valued random process. Now we give, some definitions and preliminary results.

Let X be a real separable Banach space, X^* - its conjugate, $\mathrm{B}(X)$ – the Borel σ-algebra of X, $(\Omega, \mathrm{B}, \mathrm{P})$ - a probability space. The continuous linear operator $T: X^* \to \mathrm{L}_2(\Omega, B, \mathrm{P})$ is called a generalized random element (GRE) (sometimes it is used the terms: random linear function or cylindrical random element). Denote by $M_1 := \mathrm{L}(X^*, \mathrm{L}_2(\Omega, \mathrm{B},\mathrm{P}))$ the Banach space of GRE with the norm $\|T\|_{M_1} := \sup_{\|x^*\|\leq 1} \|Tx^*\|_{L_2}$. A random element (measurable map)

$\xi : \Omega \to X$ is said to have a weak second order, if, for all $x^* \in X^*$, $E\langle \xi, x^* \rangle^2 < \infty$. We can realize the random element ξ as an element of M_1: $T_\xi x^* = \langle \xi, x^* \rangle$, but not conversely: in infinite dimensional Banach space for all $T : X^* \to L_2$ (Ω, B, P), there does not always exists the random element $\xi : \Omega \to X$ such that $T x^* = \langle \xi, x^* \rangle$ for all $x^* \in X^*$. The problem of existence such random element is the well known problem of decompasibility of the GRE. Denote by M_2 the linear normed space of all random elements of the weak second order with the norm $\|\xi\| = \|T_\xi\|$. Thus, we can assume $M_2 \subset M_1$.

Let $(W_t)_{t \in [0,1]}$ be the Wiener process in a Banach space. That is 1) $W_0 = 0$ almost surely (a.s.); 2. for all $0 \le t_0 < t_1 < \cdots < t_n \le 1$, the random elements in X, $W_{t_{i+1}} - W_{t_i}$, $i = 1, 2, \cdots, n-1$ are independent; 3. for all $t > s$ $W_t - W_s$ is a Gaussian random element with a mean 0 and the covariance operator $(t-s)R$, where $R : X^* \to X$ is a fixed Gaussian covariance; 4. $(W_t)_{t \in [0,1]}$ has continuous sample pats.

Let $(F_t)_{t \in [0,1]}$-be an increasing family of σ – algebras such that a) W_t is F_t-measurable for all $t \in [0,1]$; b) $W_t - W_s$ -is independent of the σ -algebra F_s for all $s < t$. F_0 contains all P-null sets in B. In this case we say that $(W_t)_{t \in [0,1]}$ is F_t-adapted.

If $R : X^* \to X$ is a covariance operator of the random element W_1, and $Rx^* = \sum_{k=1}^\infty \langle a_k, x^* \rangle a_k$, $(a_k)_{k \in N} \subset X$, is a representation of the operator R (see [5], Lemma 3.1.1), then there exists the sequence of independent, standard, real valued Wiener processes $(w_k(t))_{k \in N}$ such that $W_t = \sum_{k=1}^\infty a_k w_k(t)$ and the convergence is a. s. uniformly for t in X (see [6], Th. 1.4, [7], Th.1.4). We may choose $(a_k)_{k \in N} \subset X$ and $(w_k(t))_{k \in N}$ such that $(w_k(t))_{k \in N}$ will be F_t-adapted for all $k \in N$ (see [7], prop. 2.1).

Let $(T_t)_{t \in [0,1]}$ be a family of GRE. We call it a generalized random process (GRP). If we have a weak second order random process $(\xi_t)_{t \in [0,1]}$, $\xi_t : \Omega \to X$, it will be realized as a GRP: $T_{\xi_t} x^* = \langle \xi_t, x^* \rangle$.

Denote by $G_R(X^*)$ the linear space of random functions $\phi : \Omega \to X^*$ such that for all $x \in X$ $\langle \phi(\omega), x \rangle$ is measurable and $\tau^2_R(\phi) \equiv \int_\Omega \langle R\phi(\omega), \phi(\omega) \rangle dP < \infty$. $\tau_R(\phi)$ is a pseudonorm in $G_R(X^*)$.

Definition 1. A function $\phi : [0,1] \times \Omega \to X^*$ is called non-anticipating with respect to $(F_t)_{t \in [0,1]}$ if the function

$(t, \omega) \to \langle \phi(t, \omega), x \rangle$ from $([0,1] \times \Omega, B[0,1] \times B)$ into $(R^1, B(R^1))$ is measurable for all $x \in X$, and the function $\omega \to \langle \phi(t, \omega), x \rangle$ is F_t-measurable for all $t \in [0,1]$.

By $TG_R(X^*)$ we denote the class of non-anticipating random functions $\phi : [0,1] \times \Omega \to X^*$, for which $P^2_R(\phi) \equiv (\int_0^1 \int_\Omega \langle R\phi(t, \omega), \phi(t, \omega) \rangle dt dP < \infty$. $TG_R(X^*)$ is a linear space and P_R is a pseudonorm in it.

If $\phi \in TG_R(X^*)$ is a step-function $\phi(t, \omega) = \sum_{i=0}^{n-1} \phi_{t_i}(\omega) \chi_{[t_i, t_{i+1}]}(t)$, $0 = t_0 < t_1 < \cdots < t_n = 1$, $i = 0, \cdots, n-1$, then the stochastic integral of ϕ with respect to $(W_t)_{t \in [0,1]}$, is naturally defined by the equality

$$\int_0^1 \phi(t, \omega) dW_t = \sum_{i=0}^{n-1} \langle \phi_{t_i}(\omega), (W_{t_{i+1}} - W_{t_i}) \rangle .$$

The following lemma is true:

Lemma 1 ([7]). For an arbitrary $\phi \in TG_R(X^*)$ there exists a sequence of step-functions $(\phi_n)_{n \in N} \subset TG_R(X^*)$ such that $\phi_n \xrightarrow{P_R} \phi$ and $\int_0^1 \phi_n dW_t$ converges in $L_2(\Omega, B, P)$.

Definition 2 ([7]). Let $\phi \in TG_R(X^*)$ and $(\phi_n)_{n \in N} \subset TG_R(X^*)$ be step-functions such that $\phi_n \xrightarrow{P_R} \phi$ and $\int_0^1 \phi_n dW_t$ converges in $L_2(\Omega, B, P)$. The limit of the sequence $\int_0^1 \phi_n dW_t$ is called the stochastic integral of a random function $\phi \in TG_R(X^*)$ with respect to the Wiener process $(W_t)_{t \in [0,1]}$ and is denoted by $\int_0^1 \phi dW_t$.

Proposition 1. Let $W_t = \sum_{k=1}^\infty a_k w_k(t)$ be such a representation of $(W_t)_{t \in [0,1]}$ that $(w_k(t))_{t \in [0,1]}$ is F_t-adapted for all $k \in N$, then $\int_0^t \phi dW_t = \sum_{k=1}^\infty \int_0^t \langle \phi, a_k \rangle dw_k(t)$ a. s. for all $t \in [0,1]$.

Proof. As $(w_k(t))_{t \in [0,1]}$ is F_t-adapted for all $k \in N$, the real valued stochastic integrals $\int_0^t \langle \phi, a_k \rangle dw_k(t)$ exist. The sum $\sum_{k=1}^\infty \int_0^t \langle \phi, a_k \rangle dw_k(t)$ converges in $L_2(\Omega, B, P)$ and $E(\sum_{k=1}^\infty \int_0^t \langle \phi, a_k \rangle dw_k(t))^2 = \sum_{k=1}^\infty \langle \phi, a_k \rangle^2 = \langle R\phi, \phi \rangle$. Let $(\phi_n)_{n \in N} \subset TG_R(X^*)$ be step-functions such that $\phi_n \xrightarrow{P_R} \phi$ and

$\int_0^1 \phi_n dW_t$ converges in $L_2(\Omega, B, P)$. It is easy to see, that the above equality holds for step-functions ϕ_n, $n \in N$. That is $\int_0^t \phi_n dW_t = \sum_{k=1}^{\infty} \int_0^t \langle \phi_n, a_k \rangle dw_k(t)$ a. s. for all $t \in [0,1]$. $\sum_{k=1}^{m} \int_0^t \langle \phi_n, a_k \rangle dw_k(t)$ converges also to $\sum_{k=1}^{m} \int_0^t \langle \phi, a_k \rangle dw_k(t)$ for all fixed $m \in N$. Therefore, $\sum_{k=1}^{m} \int_0^t \langle \phi_n, a_k \rangle dw_k(t) \rightarrow \sum_{k=1}^{m} \int_0^t \langle \phi, a_k \rangle dw_k(t)$, when $m \to \infty$ and $\sum_{k=1}^{m} \int_0^t \langle \phi_n, a_k \rangle dw_k(t) \rightarrow \int_0^t \phi dW_t$ when $n \to \infty$.

Now consider the linear bounded operator $\phi : X^* \to G_R(X^*)$, for all fixed $x^* \in X^*$, $\phi x^* : \Omega \to X^*$. Denote by $M_1^G \equiv L(X^*, G_R(X^*))$ the space of such operators with the property: $\tau_R^2(\phi) \equiv \sup_{\|x^*\| \leq 1} E \langle R \phi x^*, \phi x^* \rangle < \infty$. $\tau_R(\phi)$ is a pseudonorm in M_1^G. Consider now the family of linear bounded operators $(T_t)_{t \in [0,1]}$, $T_t : X^* \to G_R(X^*)$, such that for all $x^* \in X^*$, the random process $T_t x^*$ is non -

anticipating   and $\tau_R^2((T_t)_{t \in [0,1]}) \equiv$ $\sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \langle R T_t x^*, T_t x^* \rangle dt dP < \infty$. Denote by TM_1^G the space of such family of operators.

Afterward, we will consider the family $(T_t)_{t \in [0,1]}$, $T_t : X^* \to G_R(X^*)$ with the property $\sup_{\|x^*\| \leq 1} \sup_{\|x\| \leq 1}$ $\int_0^1 \int_\Omega \langle T_t x^\bullet, x \rangle^2 dt dP < \infty$.

Proposition 2. If the family of linear bounded operators $(T_t)_{t \in [0,1]}$, $T_t : X^* \to G_R(X^*)$ are such that $\sup_{\|x^*\| \leq 1} \sup_{\|x\| \leq 1}$ $\int_0^1 \int_\Omega \langle T_t x^\bullet, x \rangle^2 dt dP < \infty$, then $\sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \langle R T_t x^*, T_t x^* \rangle dt dP < \infty$, that is $(T_t)_{t \in [0,1]} \in TM_1^G$.

Proof. . As $R : X^* \to X$ is a Gaussian covariance operator, by the Kwapien-Szymanski's theorem (see [5] p. 262, [8]) there exist the sequences $(x_k)_{k \in N} \subset X$ and $(x_k^*)_{k \in n} \subset X^*$ such that $\langle x_k, x_j^* \rangle = \delta_{k,j}$, $R x_k^* = x_k$, $k = 1, 2, \cdots$, for all $x^* \in X^*$, $R x^* = \sum_{k=1}^{\infty} \langle x_k, x^* \rangle x_k$ and $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$. Then

$$\sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \langle R T_t x^*, T_t x^* \rangle dt dP = \sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \sum_{k=1}^{\infty} \langle x_k, T_t x^* \rangle^2 dt dP = \sum_{k=1}^{\infty} \|x_k\|^2 \sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \langle x_k / \|x_k\|, T_t x^* \rangle^2 dt dP \leq$$

$$\sum_{k=1}^{\infty} \|x_k\|^2 \sup_{\|x\| \leq 1} \sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \langle T_t x^*, x \rangle^2 dt dP < \infty.$$

We can naturally define the stochastic integral from $(T_t)_{t \in [0,1]} \in TM_1^G$ which is the GRE, defined by the equality $I(T_t)_{t \in [0,1]} x^* = \int_0^1 T_t x^* dW_t$. Accordingly, we have the isometrical operator $I : \quad TM_1^G \quad \to \quad M_1$, $I(T_t)_{t \in [0,1]} x^* = \int_0^1 T_t x^* dW_t$.

Lemma 2. For an arbitrary separable-valued $\phi : [0,1] \to M_1^G$ there exists a sequence of step-functions $(\phi_n)_{n \in N} \subset TM_1^G$ such that $\phi_n \xrightarrow{\tau_R} \phi$ in TM_1^G and $\int_0^1 \phi_n dW_t$

converges in M_1.

Proof. Analogous to the case of proposition 2, consider the sequences $(x_k)_{k \in N} \subset X$ and $(x_k^*)_{k \in n} \subset X^*$ such that $\langle x_k, x_j^* \rangle = \delta_{k,j}$, $R x_k^* = x_k$, $k = 1, 2, \cdots$, for all $x^* \in X^*$, $R x^* = \sum_{k=1}^{\infty} \langle x_k, x^* \rangle x_k$ and $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$. Let $\phi \in TM_1^G$ and denote by $f_n(t) x^* = \sum_{k=1}^{n} \langle \phi(t) x^*, x_k \rangle x_k^*$, $n = 1, 2, \cdots$. the functions from TM_1^G.

$$\sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \sum_{k=1}^{\infty} \langle (\phi x^* - \sum_{j=1}^{n} \langle \phi x^*, x_j \rangle x_j^*), x_k \rangle^2 dt dP = \sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \sum_{k=n+1}^{\infty} \langle \phi x^*, x_k \rangle^2 dt dP \leq \sum_{k=n+1}^{\infty} \|x_k\|^2 \sup_{\|x^*\| \leq 1} \int_0^1 \int_\Omega \langle \phi x^*, \frac{x_k}{\|x_k\|} \rangle^2 dt dP \leq$$

$$\sum_{k=n+1}^{\infty} \|x_k\|^2 \sup_{\|x^*\| \leq 1} \sup_{\|x\| \leq 1} \int_0^1 \int_\Omega \langle \phi x^\bullet, x \rangle^2 dt dP \to 0.$$

For any fixed k, consider now the GRP $(\varphi_k(t))_{t \in [0,1]} \equiv \langle \phi(t) x^\bullet, x_k \rangle$, $\varphi_k(t) : X^\bullet \to L_2(\Omega, B, P)$, so we have the map $\varphi_k : [0,1] \to M_1$, $\int_0^1 \|\varphi_k\|_{M_1}^2 dt < \infty$ and φ_k is

separable-valued, therefore, by the lemma 1 from [9], there exists the sequence of non-anticipating step functions $\varphi_{km} : [0,1] \to M_1$ such that $\int_0^1 \|\varphi_k(t) - \varphi_{km}(t)\|_{M_1}^2 dt \to 0$.

Afterward for $f_n(t)x^* = \sum\limits_{k=1}^{n} \langle \phi(t)x^*, x_k \rangle x_k^*$, the sequence $f_{nm} \equiv \sum\limits_{k=1}^{n} \varphi_{km}(t)x_k^\bullet$ is such that $\tau_R^2(f_{nm} - f_n) \to 0$, when $m \to \infty$. Therefore, we may choose a sequence of step-functions $(\phi_n)_{n \in N} \subset TM_1^G$ such that $\phi_n \xrightarrow{\tau_R} \phi$. It is easy to see, that $\int\limits_0^1 \phi_n dW_t$ converges in M_1 to the $\int\limits_0^1 \phi dW_t$.

If we have $F(t, \omega) : [0,1] \times \Omega \to L(M_1, M_1)$ non-anticipating operator valued random process and non-anticipating $(T_t)_{t \in [0,1]} \in TM_1^G$, then we can define the generalized stochastic integral $I : X^* \to L_2(\Omega, B, P)$,

$$Ix^* = \int\limits_0^1 F(t, \omega)T_t x^* dW_t \equiv \sum\limits_{m=1}^{\infty} \int\limits_0^1 F(t, \omega)\langle T_t x^*, x_k \rangle dw_m(t),$$

as $\langle T_t(\cdot), x \rangle : X^* \to L_2(\Omega, B, P)$ is a generalized random process. Here we use the representation of the Wiener process in a Banach space by the sum of one dimensional, independent, non-anticipating Wiener processes, and $(x_k)_{k \in N}$ is such that $Rx^* = \sum\limits_{k=1}^{\infty} \langle x^*, x_k \rangle x_k$ is the representation of the covariance operator of W_1. Let now X be a separable Banach space and $L(X, X)$ be the space of bounded linear operators from X to X. We will consider the random processes $\xi_t : \Omega \to L(X, X)$ such that $\int\limits_0^1 \int\limits_\Omega \langle \xi_t(\omega)x, x^\bullet \rangle^2 dtdP < \infty$ for all $x \in X$ and $x^\bullet \in X^\bullet$.

Proposition 2. If the random process $\xi_t : \Omega \to L(X, X)$ is

such, that $\int\limits_0^1 \int\limits_\Omega \langle \xi_t(\omega)x, x^\bullet \rangle^2 dtdP < \infty$ for all $x \in X$ and $x^\bullet \in X^\bullet$, then $\sup_{\|x^\bullet\| \leq 1} \int\limits_0^1 \int\limits_\Omega \langle R\xi_t^\bullet(\omega)x^\bullet, \xi_t^\bullet(\omega)x^\bullet \rangle^2 dtdP < \infty$, where the operator $\xi_t^\bullet(\omega)$ is conjugate of the operator $\xi_t(\omega)$.

Proof. Firstly we prove that $\sup_{\|x^\bullet\| \leq 1} \sup_{\|x\| \leq 1} \int\limits_0^1 \int\limits_\Omega \langle \xi_t(\omega)x, x^\bullet \rangle^2 dtdP < \infty$. Consider the family of linear operators $T_x : X^\bullet \to L_2([0,1] \times \Omega, B[0,1] \times B, \lambda \times P)$, $T_x x^\bullet = \langle \xi_t(\omega)x, x^\bullet \rangle$, $x \in B(X) \equiv \{x : \|x\| \leq 1\}$. From the closed graph theorem it follows that T_x is a continuous operator for all fixed $x \in X$. That is $\{T_x, x \in B(X)\}$ is a collection of continuous linear operators from $X^\bullet$ to $L_2([0,1], \Omega)$. For all fix $x^\bullet \in X^\bullet$, if we consider the linear operator $S : X \to L_2([0,1], \Omega)$, $Sx = \langle \xi_t(\omega)x, x^\bullet \rangle$, by the closed graph theorem, we can proof boundedness of the operator S. That is, for all fixed $x^\bullet \in X^\bullet$, $\sup_{x \in B(X)} \|T_x(x^\bullet)\|^2 = \sup_{\|x\| \leq 1} \|S(x)\|^2$ $= \sup_{\|x\| \leq 1} \int\limits_0^1 \int\limits_\Omega \langle \xi_t(\omega)x, x^\bullet \rangle^2 dtdP < \infty$. Then, by the uniform boundedness principle,

$$\sup_{\|x^\bullet\| \leq 1} \sup_{\|x\| \leq 1} \int\limits_0^1 \int\limits_\Omega \langle \xi_t(\omega)x, x^* \rangle^2 dtdP < \infty..$$

Let now the sequences $(x_k)_{k \in N} \subset X$ and $(x_k^*)_{k \in n} \subset X^*$ be such that $\langle x_k, x_j^* \rangle = \delta_{k,j}$, $Rx_k^* = x_k$, $k = 1, 2, \cdots$, for all $x^* \in X^*$, $Rx^* = \sum\limits_{k=1}^{\infty} \langle x_k, x^* \rangle x_k$ and $\sum\limits_{k=1}^{\infty} \|x_k\|^2 < \infty$. Then we have

$$\sup_{\|x^\bullet\| \leq 1} \int\limits_0^1 \int\limits_\Omega \langle R\xi_t^\bullet(\omega)x^\bullet, \xi^\bullet x^\bullet \rangle^2 dtdP = \sup_{\|x^\bullet\| \leq 1} \int\limits_0^1 \int\limits_\Omega \sum\limits_{k=1}^{\infty} \langle \xi_t^\bullet(\omega)x^\bullet, x_k \rangle^2 dtdP =$$

$$\sum\limits_{k=1}^{\infty} \|x_k\|^2 \sup_{\|x^\bullet\| \leq 1} \int\limits_0^1 \int\limits_\Omega \langle \xi^\bullet(\omega)x^\bullet, \frac{x_k}{\|x_k\|} \rangle^2 dtdP \leq \sum\limits_{k=1}^{\infty} \|x_k\|^2 \sup_{\|x^\bullet\| \leq 1} \sup_{\|x\| \leq 1} \int\limits_0^1 \int\limits_\Omega \langle \xi_t(\omega)x, x^\bullet \rangle^2 dtdP < \infty$$

Definition 3. The random process $\xi_t : \Omega \to L(X, X)$ is non-anticipating with respect to the family of the σ-algebra $(F_t)_{t \in [0,1]}$ if, for all $x \in X$, $\xi_t(\omega)x : [0,1] \times \Omega \to X$ is measurable and, for all $t \in [0,1]$, the random element $\xi_t x : \Omega \to X$ is F_t-measurable.

Definition 4. We say that the non-anticipating random process $(\xi_t)_{t \in [0,1]}$, $\xi_t : \Omega \to L(X, X)$ belongs to the class $TG_R(L(X, X))$ if

$$\tau^2(\xi) \equiv \sup_{\|x^*\| \leq 1} \int\limits_0^1 \int\limits_\omega \langle R\xi_t^*(\omega)x^*, \xi_t^*(\omega)x^* \rangle dtdP < \infty,$$

where $\xi_t^*(\omega)$ is the linear operator, conjugate to the operator

$\xi_t(\omega)$. $TG_R(L(X, X))$ is a linear space with the pseudonorm $\tau(\xi)$.

Let $\xi \in TG_R(L(X, X))$ and $x^* \in X^*$. Then $\xi^* x^* : [0,1] \times \Omega \to X^*$ is non-anticipating and $\int\limits_0^1 \int\limits_\omega \langle R\xi_t^*(\omega)x^*, \xi_t^*(\omega)x^* \rangle dtdP < \infty$. We can define the stochastic integral $\int\limits_0^1 \xi_t^*(\omega)x^* dW_t$, which is the random variable with a mean 0 and variance $\int\limits_0^1 \int\limits_\omega \langle R\xi_t^*(\omega)x^*, \xi_t^*(\omega)x^* \rangle dtdP$. Therefore, we can consider

the GRE $I_\xi : X^* \to L_2(\Omega, B, P)$, $I_\xi x^* = \int_0^1 \xi^*_t(\omega)x^* dW_t$.

Definition 5. The generalized random element $I_\xi x^* = \int_0^1 \xi^*_t(\omega)x^* dW_t$ is called the generalized stochastic integral from the random process $\xi \in TG_R(L(X,X))$. If there exists the random element $\eta : \Omega \to X$ such that $\langle \eta, x^* \rangle = I_\xi x^* = \int_0^1 \xi^*_t(\omega)x^* dW_t$ for all $x^* \in X^*$, then we say that there exists the stochastic integral from the operator-valued non-anticipating random process $(\xi_t)_{t \in [0,1]}$, $\xi_t : \Omega \to L(X,X)$ by the Wiener process in a Banach space X and then we write $\eta = \int_0^1 \xi_t(\omega)dW_t$.

2. The Ito Formula

We will prove the Ito formula for the generalized Ito processes and, as a consequence, we receive the Ito formula for the Banach space valued Ito processes, where the stochastic integral is taken from the operator-valued random process by the Wiener processes in a Banach space.

Definition 4. A non-anticipating GRP is called the Generalized Ito process, if there exist non-anticipating GRP $(a(t))_{t \in [0,1]}$, $a(t) : X^* \to L_2(\Omega, B, P)$, $\int_0^1 \|a(t)\|_{M_1}^2 dt < \infty$, and non-anticipating $(B(t))_{t \in [0,1]}$ $\in TM_1^G$, $\sup_{\|x^*\| \le 1} \int_0^1 \int_\Omega \langle RB(t)x^*, B(t)x^* \rangle dt d P < \infty$, such that, for all $x^* \in X^*$,

$$\text{T}_t x^* = T_0 x^* + \int_0^t a(s)x^* ds + \int_0^t B(s)x^* dW_s \text{ a.s.}$$

Lemma 3. Let the generalized Ito process $T_t x^* = T_0 x^* + \int_0^t a(s)x^* ds + \int_0^t B(s)x^* dW_s$ be such that $a : [0,1] \to M_1$ and $B : [0,1] \to M_1^G$ are separable-valued, then there exist the sequences of non-anticipating step functions $(a_n)_{n \in N}$ and $(B_n)_{n \in N}$ such that $\int_0^1 \|a_n(t) - a(t)\|_{M_1}^2 dt \to 0$, $\int_0^1 \|B_n(t) - B(t)\|_{M_1^G}^2 dt \to 0$ and $\|T^{(n)}_t - T_t\|_{M_1}^2 \to 0$ uniformly for t , where

$$\text{T}^{(n)}_t x^* = T_0 x^* + \int_0^t a_n(s)x^* ds + \int_0^t B_n(s)x^* dW_s \ .$$

Proof. The existence of the sequences of non-anticipating step functions $(a_n)_{n \in N}$ is proved in [3] (lemma 1) and existence of the sequence of non-anticipating step functions

$(B_n)_{n \in N}$ follows from the lemma 2. Uniformness of the convergence for t of $\|T^{(n)}_t - T_t\|_{M_1}^2 \to 0$ follows from the inequality

$$\|T^{(n)}_t - T_t\|_{M_1}^2 \le \|T^{(n)}_1 - T_1\|_{M_1}^2 \to 0 \, .$$

Theorem 1(Formula Ito). Let $\text{T}_t x^* = T_0 x^* + \int_0^t a(s)x^* ds + \int_0^t B(s)x^* dW_s$ be a generalized Ito process, where a and B are separable-valued non-anticipating GRP such that $\int_0^1 \|a(t)\|_{M_1}^2 dt < \infty$, $\int_0^1 \|B(t)\|_{M_1^G}^2 dt < \infty$. Let $f : [0,1] \times M_1 \to M_1$ be a continuous function such that the derivatives $f'_t : [0,1] \times M_1 \to M_1$, $f'_T : [0,1] \times M_1 \to L(M_1, M_1)$ and $f''_{T,T} : [0,1] \times M_1 \to L(M_1, L(M_1, M_1))$ are continuous. Then

$$f(t, T_t) = f(0, T_0) + \int_0^t f'_t(s, T_s)ds + \int_0^t f'_T(s, T_s)a(s)ds +$$

$$\frac{1}{2}\int_0^t (\sum_{k=1}^\infty f''_{T,T}(s, T_s)B(s)a_k B(s)a_k)ds + \int_0^t f'_T(s, T_s)B(s)dW(s) \ .$$

Proof. As in the finite dimensional case, we show that it is enough to prove this theorem for the step functions a and B : let $(a_n)_{n \in N}$ and $(B_n)_{n \in N}$ be the sequences of step functions such that $\int_0^1 \|a_n(t) - a(t)\|_{M_1}^2 dt \to 0$ and $\int_0^1 \|B_n(t) - B(t)\|_{M_1^G}^2 dt \to 0$, then , $\|T^{(n)}_t - T_t\|_{M_1}^2 \to 0$ uniformly for t, where

$$T^{(n)}_t x^* = T_0 x^* + \int_0^t a_n(s)x^* ds + \int_0^t B_n(s)x^* dW_s \ .$$

Let the Ito formula be true for the step functions:

$$f(t, T^{(n)}_t) = f(0, T_0) + \int_0^t f'_t(s, T^{(n)}_s)ds + \int_0^t f'_T(s, T^{(n)}_s)a_n(s)ds +$$

$$\frac{1}{2}\int_0^t (\sum_{k=1}^\infty f''_{T,T}(s, T^{(n)}_s)B(s)a_k B(s)a_k)ds +$$

$$\int_0^t f'_T(s, T^{(n)}_s)B(s)dW(s) \ ..$$

As $f'_t(s, T^{(n)}_s)$ are continuous functions on $[0,1]$ converging to the continuous function $f'_t(s, T_s)$, then they are bounded. Thereby, by the Lebesgue theorem, we have the

convergence $\int_0^t f_t'(s,T_s^{(n)})ds \rightarrow \int_0^t f_t'(s,T_s)ds$. Furthermore, we have

$$\int_0^t \left\| f_T'(s,T^{(n)}{}_s)a_n(s)-f_T'(s,T_s)a_s \right\| ds \le \int_0^1 \left\| f_T'(s,T_s^{(n)}) \right\| \left\| a_n(s)-a(s) \right\| ds + + \int_0^1 \left\| a(s) \right\| \left\| f_T'(s,T_s^{(n)})-f_T'(s,T_s) \right\| ds \le C$$

$$\int_1^1 \left\| a_n(s)-a(s) \right\| ds + + (\int_0^1 \left\| a(s) \right\|^2 ds)^{\frac{1}{2}} (\int_0^1 \left\| f_T'(s,T_s^{(n)})-f_T'(s,T_s) \right\|^2 ds)^{\frac{1}{2}} \rightarrow 0 ;$$

In principle, we can similarly prove

$$\frac{1}{2}\int_0^t (\sum_{k=1}^\infty f''_{T,T}(s,T^{(n)}{}_s)B(s)a_k B(s)a_k)ds \rightarrow \frac{1}{2}\int_0^t (\sum_{k=1}^\infty f''_{T,T}(s,T_s)B(s)a_k B(s)a_k)ds$$

and

$$\int_0^t f_T'(s,T^{(n)}{}_s)B(s)dW(s) \rightarrow \int_0^t f_T'(s,T_s)B(s)dW(s) .$$

Therefore, it is enough to prove the Ito lemma for the step functions and, by additivity of integrals, we need to prove it when $T_t = T_0 + at + BW_t$, where a and B are the elements of M_1 and M_1^G correspondingly. For simplicity, we can assume that $T_0 = 0$. Then the function $u(t,BW_t) = f(t,at+BW_t)$ has the same smoothness as f and so, it is enough to prove the Ito formula for function $u(t,BW_t)$. Let $l=[2^n t]$, $\Delta W = W_{\frac{k}{2^n}} - W_{\frac{(k-1)}{2^n}}$, $\Delta = \frac{1}{2^n}$, $n=1,2,\dots$. Then, by the Taylor's formula, we have

$$u(t,BW_t)-u(0,0) = \sum_{k\le l} u(\frac{k}{2^n},BW_{\frac{k}{2^n}})-u(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})+u(t,BW_t)-u(\frac{l}{2^n},BW_{\frac{l}{2^n}}) = \sum_{k\le l}\int_0^1 u_t'(\frac{k-1}{2^n}+\frac{s}{2^n},BW_{\frac{k-1}{2^n}})\cdot\frac{1}{2^n}ds +$$

$$\sum_{k\le l} u_T'(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})B\Delta W + + \sum_{k\le l}\int_0^1 (1-s)u_{TT}''(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}}+sB\Delta W)B\Delta WB\Delta Wds + u(t,BW_t)-u(\frac{l}{2^n},BW_{\frac{l}{2^n}}) =$$

$$\sum_{k\le l} u_t'(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})\cdot\frac{1}{2^n} + \sum_{k\le l}\int_0^1 [u_t'(\frac{k-1}{2^n}+\frac{s}{2^n},BW_{\frac{k-1}{2^n}})-u_t'(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})]\cdot\frac{1}{2^n}ds + \sum_{k\le l} u_T'(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})B\Delta W +$$

$$\frac{1}{2}\sum_{k\le l} u_{TT}''(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})]\cdot B\Delta WB\Delta W + \sum_{k\le l}\int_0^1 (1-s)[u_{TT}''(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}}+sB\Delta W)-u_{TT}''(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})]B\Delta WB\Delta Wds +$$

$$u(t,BW_t)-u(\frac{l}{2^n},BW_{\frac{l}{2^n}}) = \sum_{k\le l} u_t'(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})\cdot\frac{1}{2^n} + \sum_{k\le l} u_T'(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})B\Delta W +$$

$$\frac{1}{2}\sum_{k\le l}\frac{1}{2^n}\sum_{m=1}^\infty u''_{TT}(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})Bx_m Bx_m + A_n + B_n + C_n + \delta_n ,$$

where

$$A_n = \sum_{k\le l}\int_0^1 [u_t'(\frac{k-1}{2^n}+\frac{s}{2^n},BW_{\frac{k-1}{2^n}})-u_t'(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})]\cdot\frac{1}{2^n}ds ,$$

$$B_n = \sum_{k\le l}\int_0^1 (1-s)u_{TT}''(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}}+sB\Delta W)B\Delta WB\Delta Wds - \frac{1}{2}u_{TT}''(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})B\Delta WB\Delta W ,$$

$$C_n = \frac{1}{2}\sum_{k\le l} u''_{TT}(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})B\Delta WB\Delta W - \frac{1}{2}\sum_{k\le l}\frac{1}{2^n}\sum_{m=1}^\infty u''_{TT}(\frac{k-1}{2^n},BW_{\frac{k-1}{2^n}})Bx_m Bx_m ,$$

$$\delta_n = u(t,BW_t)-u(\frac{l}{2^n},BW_{\frac{l}{2^n}})$$

and $(x_k)_{k\in N} \subset X$ is such, that there exists $(x_k^*)_{k\in n} \subset X^*$, $\langle x_k, x_j^* \rangle = \delta_{k,j}$, $Rx_k^* = x_k$, $k=1,2,\cdots$, for all $x^* \in X^*$, $Rx^* = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k$ and $\sum_{k=1}^\infty \left\| x_k \right\|^2 < \infty$.

Using the technique developed in [3], (theorem 1) it is not difficult to prove, that $\|A_n\|_{M_1} \to 0$, $B_n x^* \to 0$, $C_n x^* \to 0$ and $\delta_n x^* \to 0$ for all $x^* \in X^*$ in $L_2(\Omega, \mathbf{B}, \mathbf{P})$. Therefore, we have

$$u(t, BW_t)x^* = u(0,0)x^* + \int_0^t u'_t(s, BW_s)x^* ds + \int_0^t \sum_{m=1}^\infty \langle u'_T(s, BW_s)Bx^*, x_m \rangle dw_m(s) + \frac{1}{2}\sum_{m=1}^\infty \int_0^t \langle u''_{TT}(s, BW_s)Bx_m Bx^*, x_m \rangle ds$$

for all $x^* \in X^*$. That is, we have the equality in M_1:

$$u(t, BW_t) = u(0,0) + \int_0^t u'_t(s, BW_s)ds + \int_0^t u'_T(s, B_s)BdW(s) + \frac{1}{2}\sum_{m=1}^\infty \int_0^t u''_{TT}(s, BW_s)Bx_m Bx_m ds .$$

Now let us return to the function f and remember that $f(t, at+BW_t) = u(t, BW_t)$, then $u'_t(t, BW_t) = f'_t(t, at+BW_t) + f'_T(t, at+BW_t)a$. Therefore, we have

$$f(t, T_t) = f(0, T_0) + \int_0^t f'_t(s, T_s)ds + \int_0^t f'_T(s, T_s)a(s)ds + \frac{1}{2}\sum_{m=1}^\infty \int_0^t f''_{TT}(s, T_s)B(s)x_m B(s)x_m ds + \int_0^t f'_T(s, T_s)B(s)dW_s ,$$

where $(x_k)_{k \in N} \subset X$ is such, that there exists $(x_k^*)_{k \in n} \subset X^*$, $\langle x_k, x_j^* \rangle = \delta_{k,j}$, $Rx_k^* = x_k$, $k = 1, 2, \cdots$, for all

$$x^* \in X^*, \quad Rx^* = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k \quad \text{and} \quad \sum_{k=1}^\infty \|x_k\|^2 < \infty .$$

Let the generalized Ito process T $_t x^* = T_0 x^* + \int_0^t a(s)x^* ds + \int_0^t B(s)x^* dW_s$ be such that there exists the X-valued random process $(\xi_t)_{t \in [0,1]}$ with property $\langle \xi_t, x^* \rangle = T_t x^*$ for all $x^* \in X^*$ and

$\xi_t = \xi_0 + \int_0^t a(t, \omega)dt + \int_0^t B(t, \omega)dW_t$, where $a : [0,1] \times \Omega \to X$, $B : [0,1] \times \Omega \to L(X, X)$ are (t, ω)-measurable, F $_t$-adapted and $\int_\Omega \int_0^1 \|a(t, \omega)\|_{M_1}^2 dPdt < \infty$, $B \in TG_R(L(X, X))$. Let also $f : [0,1] \times X \to X$, be such that $f : [0,1] \times M_2 \to M_2$, f'_t, f'_T and f''_{TT} are continuous by the norm of M_1 ($X \subset \overline{M_2} \subset M_1$). Then, taking into consideration that the step functions for $T_t \in \overline{M}_2$ we can take X-valued, we have

$$\langle f(t, \xi_t), x^* \rangle = \langle f(0, \xi_0), x^* \rangle + \left\langle \int_0^t f'_t(s, \xi_s)ds, x^* \right\rangle + \left\langle \int_0^t f'_T(s, \xi_s)a(s)ds, x^* \right\rangle + \left\langle \frac{1}{2}\int_0^t \sum_{m=1}^\infty \langle f''_{TT}(s, \xi_s)B(s)x_m B(s)x^*, x_m \rangle ds, \right\rangle +$$

$$\int_0^t \left\langle (f'_T(s, \xi_s)B(s))^* x^* \right\rangle dW_s .$$

The first five members of the aforementioned equality are functionals from the X-valued processes. Therefore, the stochastic integral $\int_0^1 f'_T(s, \xi_s)B(s)dW_s$ as the X-valued random process exists. Consequently, we have received the Ito formula for the Banach space-valued Ito process.

Theorem 2. Let $\xi_t = \xi_0 + \int_0^t a(s, \omega)ds + \int_0^t B(s, \omega)dW_s$, where

$a : [0,1] \times \Omega \to X$, $B : [0,1] \times \Omega \to L(X, X)$ be (t, ω) measurable, F $_t$-adapted and $\int_\Omega \int_0^t \|a(t, \omega)\|_{M_1}^2 dPdt < \infty$, $B \in TG_R(L(X, X))$. Let $f : [0,1] \times X \to X$ be such that $f : [0,1] \times M_2 \to M_2$, f'_t, f'_T and f''_{TT} are, continuous then

$$f(t, \xi_t) = f(0, \xi_0) + \int_0^t f'_t(s, \xi_s)ds + \int_0^t f'_T(s, \xi_s)a(s)ds + \frac{1}{2}\int_0^t \sum_{m=1}^\infty f''_{TT}(s, \xi_s)B(s, \omega)x_m B(s, \omega)x_m ds + \int_0^t f'_T(s, \xi_s)B(s, \omega)dW_s .$$

We have chosen $(x_n)_{n \in N}$ such that $Rx^* = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k$, $x^* \in X^*$, and $\sum_{k=1}^\infty \|x_k\|^2 < \infty$. Analogously of proposition 3 from [3], we can prove, that the expression $\frac{1}{2}\int_0^t \sum_{m=1}^\infty f''_{TT}(s, \xi_s)B(s, \omega)x_m B(s, \omega)x_m ds$ is the same for all $(x_n)_{n \in N}$ such that $Rx^* = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k$

3. Conclusion

The theory of stochastic differential equations in Banach space is develops in three directions. The first direction is the case, when the stochastic integral of the equation is taken from Banach space valued non-anticipating random function by the real-valued Wiener process; the second direction is the case, when the stochastic integral of the equation is taken from operator-valued non-anticipating random function by the Wiener process in a Banach space; the third direction is the case, when the stochastic integral of the equation is taken from operator-valued (from Hilbert space to Banach space) non-anticipating random function by the canonical generalized Wiener process in Hilbert space. Analogous to the finite dimensional case, the Ito formula is one of the main tools in stochastic analysis in Banach space. In this paper we consider the second case and prove the Ito Formula in this case. Existence and uniqueness of solutions is considered in [12] for this case. As we mentioned above the first case is considered in [4] and the third case, when the Banach space has special geometry are considered in [1] and [2] . in [10] is considered the case, when the function f maps from Banach space to real line.

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