Controllability of co-operative Neumann parabolic systems

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Abstract: In this communication, we introduce and study the various controllability problems for Neumann co-operative parabolic linear system involving Laplace operator with distributed or boundary controls and with observations belong to different spaces.

Keywords: Optimal Control Problem, Controllability, Solutions of Parabolic System, Co-Operative System

1. Introduction

Controllability is a mathematical problem, which consists in determining the targets to which one can drive the state of some dynamical system, by means of a control parameter present in the equation. Many physical systems such as quantum systems, fluid mechanical systems, wave propagation, diffusion phenomena, etc. are represented by an infinite number of degrees of freedom, and their evolution follows some partial differential equation. Finding active controls in order to properly influence the dynamics of these systems generate highly involved problems. The control theory for PDEs, and among this theory, controllability problems, is a mathematical description of such situations. Any dynamical system represented by a PDE, and on which an external influence can be described, can be the object of a study from this point of view. In 1978, D.L. Russell [1] made a rather complete survey of the most relevant results that were available in the literature at that time. In that paper, the author described a number of different tools that were developed to address controllability problems, often inspired and related to other subjects concerning partial differential equations: multipliers, moment problems, nonharmonic Fourier series, etc.

Various types of controllability of linear abstract dynamical systems defined in a Banach or Hilbert spaces have been recently extensively explored by several authors (see e.g.[2]-[18]). More recently, J.-L. Lions introduced the so called Hilbert Uniqueness Method (H.U.M.; see [19]).

In this work, we will focus our attention on some special aspects of controllability problems for parabolic system involving Laplace operator. In order to explain the results we have in mind, it is convenient to consider the abstract form:

Let V and H be two real Hilbert spaces such that V is a dense subspace of H. Identifying the dual of H with H, we may consider $V \subset H \subset V'$, where the embedding is dense in the following space. Let $A(t)$ ($t \in]0,T[$) be a family of continuous operators associated with a bilinear forms $\pi(t;.,.)$ defined on $V \times V$ which are satisfied Gårding's inequality

$$\pi(t;y,y) + c_0 \|y\|_H^2 \geq c_1 \|y\|_V^2, \quad c_0 \geq 0, c_1 > 0, \quad \text{for } y \in V, t \in [0,T]. \tag{1}$$

Then, from [20] and [21], for given f, y_0 and B be a bounded linear operator the following abstract systems:

$$\left. \begin{array}{l} \dfrac{d}{dt} y(t) + A(t)y(t) = f + Bu, t \in]0,T[, \\ y(0) = y_0 \end{array} \right\} \tag{2}$$

have a unique solution, we denote it by $y(t;u) \in Y$.

We also given an observation equation
$z(u) = Cy(u), \quad C \in L(Y:\mathsf{H}), \quad \mathsf{H}$ being a Hilbert space.

Definition 1. The system whose state is defined by (2) is said to be controllable if the observation $z(u)$ generates a dense (affine) subspace of the space of observations H.

In the above setting, the equation (2) is typically a partial differential equation, where the influence of u can take multiple different forms: typically, u can be an additional

(force) term in the right-hand side of the equation, localized in a part of the domain; it can also appear in the boundary conditions; but other situations can clearly be envisaged (we will describe some of them).

A typical application of a parabolic equation is the heat;

$$\left.\begin{array}{ll} \dfrac{\partial y}{\partial t} = \Delta y + u & \text{in } Q = \Omega \times]0,T[, \\[2mm] y(x,0) = y_0(x) & \text{in } \Omega, \\[2mm] \dfrac{\partial}{\partial \nu} y(x,t) = 0 & \text{on } \Sigma = \Gamma \times]0,T[, \end{array}\right\} \quad (3)$$

where $\Omega \subset R^N$ is a bounded open domain with smooth boundary Γ, $\dfrac{\partial}{\partial \nu}$ is the normal derivative at Γ and $y_0(x)$ is a given function in $L^2(\Omega)$.

The results in [20] partly overlap with results in [22] and they were shown that the system (3) (with $u \in L^2(Q)$) is controllable.

In our papers [23]-[27], the above results have been extended, the controllability questions related to the time optimal control problem of $n \times n$ co-operative parabolic or hyperbolic systems with distributed or boundary controls was considered.

In this paper, we will consider various time-optimal control problems for the following $n \times n$ Neumann co-operative linear parabolic system (here and everywhere below the vectors are denoted by bold letters and the index $i = 1,2,...,n$):

$$\left.\begin{array}{ll} \dfrac{\partial y_i}{\partial t} = (A(t)\mathbf{y})_i + u_i(x,t) & \text{in } Q, \\[2mm] y_i(x,0) = y_{i,0}(x) & \text{in } \Omega, \\[2mm] \dfrac{\partial}{\partial \nu} y_i(x,t) = v_i(x,t), & \text{on } \Sigma \end{array}\right\} \quad (4)$$

where $y_{i,0}$, is a given functions, u_i represents either a distributed control or a given function defined in Q, v_i represents either a Neumann boundary control or a given function defined in Σ and $A(t)$ ($t \in]0,T[$) are a family of $n \times n$ continuous matrix operators,

$$A(t)y = \begin{pmatrix} \Delta + a_1 & a_{12} & & a_{1n} \\ a_{21} & \Delta + a_2 & & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & & \Delta + a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

with co-operative coefficient functions a_i, a_{ij} satisfying the following conditions:

$$\left.\begin{array}{l} a_i, a_{ij} \text{ are positive functions in } L^\infty(Q), \\[2mm] a_{ij}(x,t) \le \sqrt{a_i(x,t)a_j(x,t)}. \end{array}\right\} \quad (5)$$

A classical controllability problems consists in steering an initial vector state $(y_1(0), y_2(0),..., y_n(0))^T$ for system (4), with a distributed control $\mathbf{u} = (u_1, u_2,..., u_n)^T$ belonging to $(L^2(Q))^n$ or with a Dirichlet boundary control $\mathbf{v} = (v_1, v_2,..., v_n)^T$, belonging to $(L^2(\Sigma))^n$ so that $R_u(T)$ or $R_v(T)$ generates a dense (affine) subspace of H. where

$$R_u(T) = \{y(T;\mathbf{u}) : \mathbf{u} \in (L^2(Q))^n\},$$
$$R_v(T) = \{y(T;\mathbf{v}) : \mathbf{v} \in (L^2(\Sigma))^n\}$$

2. Solutions of Neumann Co-Operative Parabolic Systems

This section is devoted to the analysis of the existence and uniqueness of solutions of system (4). We distinguish three classes of solutions: strong, weak and ultra weak solutions defined by transposition.

Let $H^\ell(\Omega)$, be the usual Sobolev space (see [28]) of order ℓ which consists of all $\varphi \in L^2(\Omega)$ whose distributional derivatives $D^q \varphi \in L^2(\Omega)$, $|q| \le \ell$, with the scalar product

$$< y, \varphi >_{H^\ell(\Omega)} = \sum_{|q| \le \ell} < D^q y, D^q \varphi >_{L^2(\Omega)},$$
$$q = \{q_1,...,q_N\}, \quad |q| = q_1 + ... + q_N,$$
$$D^q = D_1^{q_1}...D_N^{q_N}, \quad D_i = \frac{\partial}{\partial x_i}.$$

Definition 2.

1. A real function $y = y(x,t)$ defined in Q is said to be a weak solution for system (4) if

$$y \in C([0,T]; (L^2(\Omega))^n) \cap L^2(0,T; (H^1(\Omega))^n)$$

and

$$\int_Q y_i \left[-\frac{\partial \varphi}{\partial t} + A^* \varphi_i \right] dxdt = \int_\Omega y_{i,0}\varphi_i(x,0)dx + \int_Q u_i\varphi_i dxdt + \int_\Sigma v_i\varphi_i d\Gamma dt$$

for all $\varphi \in C^1([0,T]; (L^2(\Omega))^n) \cap L^2(0,T; (H^1(\Omega))^n)$, $\varphi_i(x,T) = 0$, $\dfrac{\partial \varphi_i}{\partial \nu}\bigg|_\Sigma = 0$

2. We say that the function y is a strong solution of (4) if

$$y \in C([0,T];(H^1(\Omega))^n) \cap L^2(0,T;(H^2(\Omega))^n \cap (H^1(\Omega))^n) \cap H^1(0,T;(L^2(\Omega))^n)$$

and the three conditions in (4) are satisfied almost everywhere in their corresponding domains.

3. We say that the function y is an ultra solution of (4) or solution by transposition if

$$y \in C([0,T];(H^{-1}(\Omega))^n) \cap L^2(0,T;(L^2(\Omega))^n)$$

and

$$\int_Q y_i f_i \, dxdt = <y_{i,0}, \varphi_i(x,0)> + \int_\Sigma v_i \varphi_i \, d\Gamma dt \quad \forall f_i \in L^2(Q)$$

where φ is the unique solution of the adjoint system

$$\left.\begin{array}{l} -\dfrac{\partial \varphi_i}{\partial t} - (A^*(t)\varphi)_i = f_i \quad \text{in} Q, \\[2mm] \varphi_i(x,T) = 0 \quad \text{in} \Omega, \\[2mm] \dfrac{\partial \varphi_i}{\partial \nu} = 0. \quad \text{on} \Sigma \end{array}\right\}$$

and $<.,.>$ denotes the duality paring between $H^{-1}(\Omega)$ and $H^1(\Omega)$.

For $y = (y_1, y_2, ..., y_n)^T$, $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)^T \in (H^1(\Omega))^n$ and $t \in]0,T[$, let us define a family of continuous bilinear forms

$$\pi(t;.,.): (H^1(\Omega))^n \times (H^1(\Omega))^n \to \Re \text{ by}$$

$$\begin{aligned} \pi(t;y,\varphi) &= \sum_{i=1}^n \int_\Omega \left[(\nabla y_i)(\nabla \varphi_i) - a_i(x,t) y_i \varphi_i \right] dx - \sum_{i,j=1}^n \int_\Omega a_{ij}(x,t) y_j \varphi_i \, dx \\ &= \sum_{i=1}^n \int_\Omega \left[(-\Delta y_i) - a_i(x,t) y_i \right] \varphi_i \, dx - \sum_{i,j=1}^n \int_\Omega a_{ij}(x,t) y_j \varphi_i \, dx \qquad (6) \\ &= \sum_{i=1}^n < -(A(t)y)_i, \varphi >_{L^2(\Omega)} . \end{aligned}$$

Lemma 1. If Ω is a regular bounded domain in R^N, with boundary Γ, and if m is positive on Ω and smooth enough (in particular $m \in L^\infty(\Omega)$) then the eigenvalue problem:

$$\left.\begin{array}{ll} -\Delta y = \lambda m(x) y & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \end{array}\right\}$$

possesses an infinite sequence of positive eigenvalues:

$$0 < \lambda_1(m) < \lambda_2(m) \leq \dots \lambda_k(m) \dots; \lambda_k(m) \to \infty, \text{as} k \to \infty.$$

Moreover $\lambda_1(m)$ is simple, its associate eigenfunction e_m is positive, and $\lambda_1(m)$ is characterized by:

$$\lambda_1(m) \int_\Omega m y^2 \, dx \leq \int_\Omega |\nabla y|^2 \, dx \qquad (7)$$

Proof. See[29].

Now, let

$$\lambda_1(a_i) \geq n-1, \quad i = 1, 2, \dots, n. \qquad (8)$$

Lemma 2. If (5) and (8) hold then, the bilinear form (6) satisfy the Gårding inequality

$$\pi(t;y,y) + c_0 \|y\|^2_{(L^2(\Omega))^n} \geq c_1 \|y\|^2_{(H^1(\Omega))^n}, \quad c_0, c_1 > 0. \qquad (9)$$

Proof. In fact

$$\begin{aligned} \pi(t;y,y) &= \sum_{i=1}^n \int_\Omega \left[|\nabla y_i|^2 - a_i(x,t) y_i^2 \right] dx - \sum_{i,j=1}^n \int_\Omega a_{ij}(x,t) y_i y_j \, dx \\ &\geq \sum_{i=1}^n \int_\Omega \left[|\nabla y_i|^2 - a_i(x,t) y_i^2 \right] dx - 2\sum_{i>j}^n \int_\Omega \sqrt{a_i(x,t) a_j(x,t)} \, y_i y_j \, dx. \end{aligned}$$

By Cauchy Schwarz inequality and (7),we obtain

$$\begin{aligned} \pi(t;y,y) \geq & \sum_{i=1}^n \left(1 - \frac{1}{\lambda_1(a_i)} \right) \int_\Omega |\nabla y_i|^2 \, dx - \sum_{i=1}^n \int_\Omega a_i y_i^2 \, dx \\ & -2\sum_{i>j}^n \int_\Omega \left(\frac{1}{\sqrt{\lambda_1(a_i)\lambda_1(a_j)}} \right) \left(\int_\Omega |\nabla y_i|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla y_j|^2 \, dx \right)^{\frac{1}{2}} \\ & \geq \sum_{i=1}^n \left(\frac{\lambda_1(a_i) - n + 1}{\lambda_1(a_i)} \right) \int_\Omega |\nabla y_i|^2 \, dx - \sum_{i=1}^n \int_\Omega a_i y_i^2 \, dx. \end{aligned}$$

Finally, from (8) we have (9).

Under the above lemma (Lemma 2) and using the results of Lions [20] and Lions and Magenes [21] we can prove the following theorems:

Theorem 1. Assume that (5) and (8) hold. Then, problem (4)

1. has a unique weak solution if $u_i \in L^2(0,T;H^{-1}(\Omega)), v_i \in H^{\frac{-1}{2},\frac{-1}{4}}(\Sigma)$ and $y_{i,0} \in L^2(\Omega)$. Moreover, the mapping $t \to y(t;u)$ is continuous from $[0,T] \to (L^2(\Omega))^n$.

2. has a unique strong solution if $u_i \in L^2(0,T;L^2(\Omega)), v_i \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and $y_{i,0} \in H^1(\Omega)$. Moreover, the mapping $t \to y(t;u)$ is continuous from $[0,T] \to (H^1(\Omega))^n$.

3. has a unique ultra solution if $u_i \in L^2(0,T;L^2(\Omega)), v_i \in L^2(0,T;L^2(\Gamma))$ and $y_{i,0} \in H^{-1}(\Omega)$. Moreover, the mapping $t \to y(t;u)$ is continuous from $[0,T] \to (H^{-1}(\Omega))^n$.

where the sobolev space $H^{r,s}(Q), r,s \ge 0$ (see [21])is defined by

$$H^{r,s}(Q) = H^0\left(0,T;H^r(\Omega)\right) \bigcap H^s\left(0,T;H^0(\Omega)\right)$$

$H^s(0,T;X)$ denotes the sobolev space of order s of functions defined on $[0,T]$ and taking values in X.

Based on the above theorem, we may consider the following problems:

1. distributed control ($u \in (L^2(Q))^n$)) problem with observation in $C([0,T];(L^2(\Omega))^n)$

2. distributed control ($u \in (L^2(Q))^n$)) problem with observation in $C([0,T];(H^1(\Omega))^n)$

3. boundary control ($v \in (L^2(\Sigma))^n$)) problem with observation in $C([0,T];(L^2(\Omega))^n)$

4. boundary control ($v \in (L^2(\Sigma))^n$)) problem with observation in $C([0,T];(H^{-1}(\Omega))^n)$

3. Controllability with Deferent Observations

In this section, we take the deferent cases of observations:

3.1. Distributed Control with Observation in $C([0,T];(L^2(\Omega))^n)$

Let $y(t;u)$ denote to the unique weak solution of (4), at time t corresponding to a given control $u \in (L^2(Q))^n$ and a given functions v_i, $y_{i,0}$, satisfying the hypothesis of Theorem 1, a). Occasionally, we write $y(x,t;u)$ when the

explicit dependence on x is required.

Let the observations be given by

$$y_i(T;u) \in L^2(\Omega) \tag{10}$$

Theorem 2. Assume that (5) and (8) hold, then the system (4) with control $u \in (L^2(Q))^n$ and observation (10) is controllable.

Proof. let us first remark that by translation we may always reduce the problem of controllability to the case were the system (4) with $y_{i,0} = v_i = 0$. We can show quit easily that (4) is controllable in $(L^2(\Omega))^n$ if and only if the set of reachable states $R_u(T)$ at any finite time $T > 0$ is dense in $(L^2(\Omega))^n$. By the Hahn-Banach theorem, this will be the case if

$$\int_\Omega \psi_i y_i(T;u)dx = 0, \quad \psi_i \in L^2(\Omega), \tag{11}$$

for all $u \in (L^2(Q))^n$, implies that $\psi_i = 0, i = 1,2,...,n.$

We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the following system

$$\left. \begin{aligned} -\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i &= 0 & \text{in} Q, \\ \xi_i(T) = \psi_i \in L^2(\Omega) & & \text{in} \Omega, \\ \frac{\partial}{\partial \nu}\xi_i = 0. & & \text{on} \Sigma. \end{aligned} \right\} \tag{12}$$

The existence of a unique solution for system (12) can be proved using Theorem1,a), with an obvious change of variables.

Multiply the first equation in (12) by $y_i(t;u)$ and using Green formula, we obtain the following identity:

$$\begin{aligned} 0 &= \int_Q \left[-\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i \right] y_i(t;u)dxdt \\ &= -\int_\Omega \xi_i y_i(t;u)\big|_0^T dx + \int_Q \xi_i \left[\frac{\partial}{\partial t}y_i(t;u) + (A(t)y(t;u))_i \right] dxdt \\ &= -\int_\Omega \psi_i y_i(T;u)dx + \int_Q \xi_i u_i dxdt. \end{aligned}$$

and so, if (11) holds, then

$$\int_Q \xi_i u_i dxdt = 0 \quad \forall u_i \in L^2(Q)$$

hence $\xi_i = 0$, and hence $\psi_i = 0$.

3.2. Distributed Control with Observation in $C([0,T];(H^1(\Omega))^n)$

In this section, let $y(t;u)$ denote to the unique strong solution of (4), at time t corresponding to a given control $u \in (L^2(Q))^n$ and a given functions v_i $y_{i,0}$, satisfying the

hypothesis of Theorem 1,b). Let the observation be given by

$$y_i(T;u) \in H^1(\Omega) \qquad (13)$$

Theorem 3. Assume that (5) and (8) hold, then the system (4) with control $u \in (L^2(Q))^n$ and observation (13) is controllable.

Proof. We can reduce the problem of controllability to the case were the system (4) with $y_{i,0} = v_i = 0$. Here $y(T;u) \in (H^1(\Omega))^n$. To show the system is controllable let $\psi_i \in H^{-1}(\Omega)$ such that

$$< \psi_i, y_i(T;u) >= 0 \quad \forall u \in (L^2(Q))^n.$$

We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the following system

$$\left. \begin{aligned} -\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i &= 0 && \text{in} Q, \\ \xi_i(T) = \psi_i &\in H^{-1}(\Omega) && \text{in} \Omega, \\ \frac{\partial}{\partial v}\xi_i &= 0. && \text{on} \Sigma. \end{aligned} \right\} \qquad (14)$$

since $\psi_i \in H^{-1}(\Omega)$ (it is sufficient to apply Theorem 1,c), after reversing sense of time) then there exist a unique ultra weak solution ξ for (14) and

$$\int_Q \xi_i u_i dxdt =< \psi_i, y_i(T;u) >= 0$$

then $\xi_i = 0$, and hence $\psi_i = 0$.

3.3. Boundary Control with Observation in $C([0,T]; (L^2(\Omega))^n)$

$$\begin{aligned} 0 \;\; &= \int_Q \left[-\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i \right] y_i(t;v) dxdt \\ &= -\int_\Omega \xi_i y_i(t;v)\big|_0^T \, dx + \int_Q \xi_i \left[\frac{\partial}{\partial t} y_i(t;v) + \left(A(t)y(t;v)\right)_i \right] dxdt + \int_\Sigma \xi_i \frac{\partial y_i}{\partial v} d\Sigma \\ &= -\int_\Omega y_i(T;v), \psi_i dx \int_\Sigma \frac{\partial \xi}{\partial v} y_i d\Sigma. \end{aligned}$$

implies

$$\int_\Sigma \xi v_i d\Sigma = 0;$$

hence ξ on Σ. The Cauchy data of ξ on Σ being zero, we conclude (see [30]) $\xi = 0$ and hence $\psi = 0$.

3.4. Boundary Control with Observation in $C([0,T]; (H^{-1}(\Omega))^n)$

Let $y(t;v)$ denote to the unique ultra weak solution of (4), at time t corresponding to a given control $v \in (L^2(\Sigma))^n$ and a given functions u_i, $y_{i,0}$, satisfying the hypothesis of

Let $y(t;v)$ denote to the unique weak solution of (4), at time t corresponding to a given control $v \in (L^2(\Sigma))^n$ and a given functions u_i, $y_{i,0}$, satisfying the hypothesis of Theorem 1,a). Let the observation be given by

$$y_i(T;v) \in L^2(\Omega) \qquad (15)$$

Theorem 4. Assume that (5) and (8) hold, then the system (4) with control $v \in (L^2(\Sigma))^n$ and observation (15) is controllable.

Proof. Here $y(T;v) \in (L^2(\Omega))^n$. To show the system is controllable let $\psi_i \in L^2(\Omega)$ such that

$$\int_\Omega y_i(T;v)\psi_i dx = 0 \quad \forall v \in (L^2(\Sigma))^n.$$

We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the following system

$$\left. \begin{aligned} -\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i &= 0 && \text{in} Q, \\ \xi_i(T) = \psi_i &\in L^2(\Omega) && \text{in} \Omega, \\ \frac{\partial}{\partial v}\xi_i &= 0. && \text{on} \Sigma. \end{aligned} \right\} \qquad (16)$$

The existence of a unique solution for system (18) can be proved using Theorem1,a), with an obvious change of variables.

Multiply the first equation in (18) by $y_i(t;v)$ and using Green's formula, we obtain the following identity:

Theorem 1,c). Let the observation be given by

$$y_i(T;v) \in H^{-1}(\Omega) \qquad (17)$$

Theorem 5. Assume that (5) and (8) hold, then the system (4) with control $v \in (L^2(\Sigma))^n$ and observation (17) is controllable.

Proof. Here $y(T;v) \in (H^{-1}(\Omega))^n$. To show the system is controllable let $\psi_i \in H^1(\Omega)$ such that

$$< y_i(T;v), \psi_i >= 0 \quad \forall v \in (L^2(\Sigma))^n.$$

We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the

following system

$$-\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i = 0 \qquad \text{in } Q,$$
$$\xi_i(T) = \psi_i \in H^1(\Omega) \qquad \text{in } \Omega,$$
$$\frac{\partial}{\partial \nu}\xi_i = 0. \qquad \text{on } \Sigma. \qquad (18)$$

Since $\psi_i \in H^1(\Omega)$, then according to Theorem 1,b), system (18) admits an unique strong solution ξ. (after

$$0 = \int_Q \left[-\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i \right] y_i(t;\mathrm{v}) dx dt$$

$$= -\int_\Omega \xi_i y_i(t;\mathrm{v})\big|_0^T \, dx + \int_Q \xi_i \left[\frac{\partial}{\partial t} y_i(t;\mathrm{v}) + \left(A(t)y(t;\mathrm{v})\right)_i \right] dx dt + \int_\Sigma \xi_i \frac{\partial y_i}{\partial \nu} d\Sigma$$

$$= - < y_i(T;\mathrm{v}), \psi_i > + \int_\Sigma \frac{\partial \xi}{\partial \nu} y_i d\Sigma.$$

implies

$$\int_\Sigma \frac{\partial \xi}{\partial \nu} y_i d\Sigma = 0;$$

hence $\frac{\partial \xi}{\partial \nu} = 0$ on Σ. The Cauchy data of ξ on Σ being zero, we conclude (see [30]) $\xi = 0$ and hence $\psi = 0$.

4. Conclusion

In this study, we have proved the controllability to a special co-operative parabolic systems with Neumann conditions, with deferent cases of observations. Most of the results we described in this paper apply, without any change on the results, to more general parabolic systems involving the following second order operator :

$$L(x,.) = \sum_{i,j=1}^n b_{ij}(x,.) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x,.) \frac{\partial}{\partial x_j} + b_0(x,.)$$

with sufficiently smooth coefficients (in particular, $b_{ij}, b_j, b_0 \in L^\infty(Q), b_j, b_0 > 0$) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^n \eta_i \eta_j \geq \sigma \sum_{i=1}^n \eta_i \qquad \forall (x,t) \in Q,$$

for all $\eta_i \in \Re$ and some constant $\sigma > 0$.

In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator L (see [29]).

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reversing sense of time)

Multiply the first equation in (18) by $y_i(t;\mathrm{v})$ and using Green's formula, we obtain the following identity:

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