

# Fractional domination for two classes of graphs

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**Abstract:** Let  $G$  be a graph, a real-valued function  $f$  is said to be a dominating of  $G$  if holds for every vertex  $v$ , the fractional domination function number is defined as, and the fractional total domination function number of  $G$  is analogous. In this paper we research the fractional domination problem for two classes of special graphs, obtained the fractional domination numbers of the generalized star graphs and the fractional total domination function numbers of the generalized wheel graphs.

**Keywords:** Generalized Wheel Graph, Fractional Domination Function Number, Fractional Total Domination Number, Generalized Star Graph

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## 1. Introduction

In this paper, the graphs are undirected simple graphs and for other terminologies we follow [1-3]. Let  $G$  be a graph with vertex  $V(G)$  and edge set  $E(G)$ , for any vertex  $v \in V(G)$ , the open and closed neighborhoods of  $v$  are denoted by  $N(v)$  and  $N[v] = N(v) \cup \{v\}$ , respectively.

For convenience, let  $G = (V, E)$  be a graph, for a real-valued function  $f : V \rightarrow [0, 1]$  and  $S \subseteq V$ , we write  $f(S) = \sum_{v \in S} f(v)$ .

In recent years, the results have been more and more riched with the research of graph theories in growing numbers. G.S.Domke, S.T.Hedetniemi and R.C. Laskar<sup>[4]</sup> first investigated the fractional domination, T.W.Haynes, S.T.Hedetniemi and P.J.Slater<sup>[5]</sup> summarized the main results about the theory of domination in graphs. But so far there have been a few conclusions of the fractional domination in graphs while some special graphs have not been worked out.

**Definition 1[1]** Let  $G = (V, E)$  be a simple graph. A real-valued function  $f : V \rightarrow [0, 1]$  is a fractional dominating function (simply denoting  $F$  - dominating function) of  $G$  if  $f(N[u]) \geq 1$  holds for every vertex  $u \in V$ . The fractional domination number, denoted  $\gamma_f(G)$ , is defined as follows:

$$\gamma_f(G) = \min\{f(V) \mid f \text{ is a fractional dominating function of } G\}.$$

A  $F$  - dominating function  $f$  is minimal if  $\gamma_f(G) = f(V)$ .

**Definition 2[1]** Let  $G = (V, E)$  be a simple graph without isolated vertices. A real-valued function  $f : V \rightarrow [0, 1]$  is a fractional total dominating function of  $G$  if  $f(N(u)) \geq 1$  holds for every vertex  $u \in V$ . The fractional total domination number, denoted  $\gamma_f^0(G)$ , is defined as follows:

$$\gamma_f^0(G) = \min\{f(V) \mid f \text{ is a fractional total dominating function of } G\}.$$

A fractional total dominating function  $f$  is minimal if  $\gamma_f^0(G) = f(V)$ .

**Definition 3[1]** Let  $G = (V, E)$  be a simple graph. A real-valued function  $f : V \rightarrow [0, 1]$  is a packing dominating function of  $G$  if  $f(N[u]) \leq 1$  holds for every vertex  $u \in V$ .

A packing function  $f$  is maximal if for every  $u \in V$  with  $f(u) < 1$ , there exists a vertex  $v \in N[u]$  such that  $f(N[v]) \leq 1$ .

The fractional packing number,  $p_f(G)$ , and the upper fractional packing number,  $P_f(G)$ , are defined as follows:

$$p_f(G) = \min\{f(V) \mid f \text{ is a maximal packing function of } G\}$$

$$P_f(G) = \max\{f(V) \mid f \text{ is a maximal packing function of } G\}$$

**Lemma 1[1]** For every graph  $G$ , then  $P_f(G) = \gamma_f(G)$ .

For a graph  $G = (V, E)$  and a function  $f : V \rightarrow [0, 1]$ , if  $f(N[u]) = 1$  holds if for every  $u \in V$ , then  $f$  are both maximal packing function and minimal  $F$  - dominating

function. By the above Lemma 1, we have the following lemma.

**Lemma 2** For every graph  $G$  if  $f$  is a  $F$  – dominating function of  $G$ , and  $f(N[u])=1$  holds for every  $u \in V$ , then  $\gamma_f(G)=f(V)$ ; Similarly, if  $f$  is a fractional total dominating function of  $G$ , and  $f(N(u))=1$ , then  $\gamma_f^0(G)=f(V)$ .

In this paper, by Lemma 2, we give the fractional domination numbers of the generalized star graphs and the fractional total domination function numbers of the generalized wheel graphs.

## 2. Main Results

### 2.1. $F$ – Domination

Let  $K_{1,t}$  be the star graph of order  $(t+1)$ , the generalized star graph  $P(n, t)$  is the graph obtained from  $K_{1,t}$  by inserting exact  $n-1$  vertices on each edge of  $K_{1,t}$ , the generalized star graph  $P(n, t)$  is shown in Fig. 1.

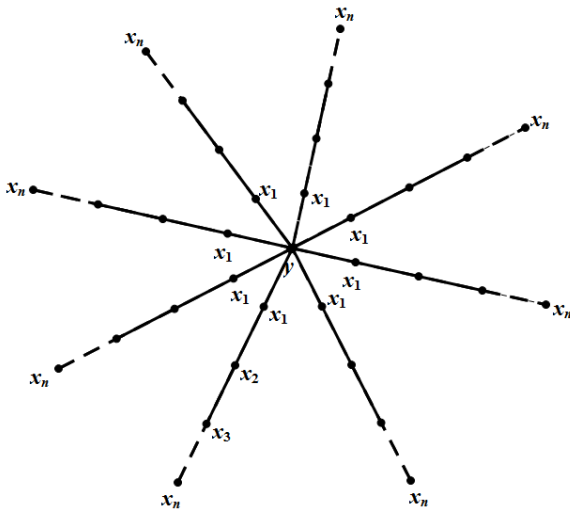


Fig. 1. The generalized star graph

**Theorem 1** For the generalized star graph  $P(n, t)$  with  $n \geq 1$  and  $t \geq 3$ , then

$$\text{When } n \equiv 0(\text{mod } 3), \gamma_f(P_{n,t}) = 1 + \frac{nt}{3} \quad (1)$$

$$\text{When } n \equiv 1(\text{mod } 3), \gamma_f(P_{n,t}) = 1 + \left\lfloor \frac{n}{3} \right\rfloor t \quad (2)$$

$$\text{When } n \equiv 2(\text{mod } 3), \gamma_f(P_{n,t}) = \frac{(n+1)t}{3} \quad (3)$$

**Proof** Let  $V = V(P_{n,t})$ . See in Fig. 1, the labels of the vertices are shown in Fig. 1.

Define a function  $f: V \rightarrow [0, 1]$  as follows: let  $f(v_0) = y$ ,

$$f(v_{ij}) = x_i \quad (j=1, 2, \dots, t).$$

Next we give the exact value of  $y$  and  $x_i$  ( $i=1, 2, \dots, n$ ) as four cases.

**Case 1** When  $n \equiv 1$ ; From the equation  $\begin{cases} y + tx = 1 \\ y + x = 1 \end{cases}$  We can obtain  $\begin{cases} x = 0 \\ y = 1 \end{cases}$ , then we have  $\gamma_f(P_{1,t}) = 1$ .

When  $n \equiv 2$ ; Similarly, we can obtain  $\gamma_f(P_{2,t}) = t$ .

**Case 2** When  $n \equiv 0(\text{mod } 3)$ ; let  $y = 1$ , and for each integer  $k$   $\left(0 \leq k \leq \frac{n}{3} - 1\right)$ ,

$$\text{let } x_{3k+1} = 0, x_{3k+2} = 0, x_{3(k+1)} = 1.$$

It is easy to see that  $f(N[v_{ij}])=1$  holds for any vertex  $v_{i,j}$ , and  $0 \leq x_i \leq 1$  for every integer  $i$  ( $i=1, 2, \dots, n$ ) and  $j$  ( $j=1, 2, \dots, t$ ). By Lemma 2,  $f$  is a minimal  $F$  – dominating function, and hence, we have

$$\gamma_f(P_{n,t}) = f(V) = y + t \sum_{i=1}^n x_i = 1 + \frac{nt}{3}.$$

**Case 3** When  $n \equiv 1(\text{mod } 3)$ ; let  $y = 1$ , and for each integer  $k$   $\left(0 \leq k \leq \frac{n}{3} - 1\right)$ ,

$$\text{let } x_{3k+1} = 0, x_{3k+2} = 0, x_{3(k+1)} = 1, x_n = 0.$$

Similar to Case 2, it is easy to see that  $f(N[v_{ij}])=1$  holds for any vertex  $v_{i,j}$ , and  $0 \leq x_i \leq 1$  for every integer  $i$  ( $i=1, 2, \dots, n$ ) and  $j$  ( $j=1, 2, \dots, t$ ). By Lemma 2,  $f$  is a minimal  $F$  – dominating function, and hence, we have

$$\gamma_f(P_{n,t}) = f(V) = y + t \sum_{i=1}^n x_i = 1 + \left\lfloor \frac{n}{3} \right\rfloor t.$$

**Case 4** When  $n \equiv 2(\text{mod } 3)$ ; let  $y = 0$ , and for each integer  $k$   $\left(0 \leq k \leq \frac{n}{3} - 1\right)$ ,

let

$$x_{3k+1} = \frac{1}{t}, x_{3k+2} = 1 - \frac{1}{t}, x_{3(k+1)} = 0, x_{n-1} = \frac{1}{t}, x_n = 1 - \frac{1}{t}.$$

It is also easy to see that  $f(N[v_{ij}])=1$  for any vertex  $v_{i,j}$ , and note that  $0 \leq x_i \leq 1$  for every integer  $i$  ( $i=1, 2, \dots, n$ ) and  $j$  ( $j=1, 2, \dots, t$ ). Similarly,  $f(N[u])=1$  holds for every  $u \in V$ ,  $f$  is the minimal  $F$  – dominating function, thus we have

$$\gamma_f(P_{n,t}) = f(V) = y + t \sum_{i=1}^n x_i = \frac{(n+1)t}{3}.$$

Combining the above four cases, we have completed the proof of Theorem 1. #

## 2.2. Fractional Total Domination

Let  $W_{n,t}$  be a wheel graph of order  $(t+1)$ . A generalized wheel graph  $W(n,t)$  is shown in Fig.2.

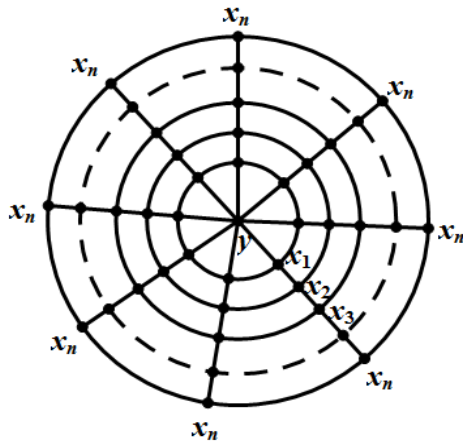


Fig 2. the generalized wheel graph

Theorem 2 For the generalized wheel graph  $W(n,t)$  with  $n \geq 2$  and  $t \geq 3$ , then if  $n$  is odd,

$$\gamma_f^0(W_{n,t}) = \frac{2(1+2+\dots+\left\lfloor \frac{n}{2} \right\rfloor) t^2 + (n+1)t - (n+1)}{nt}$$

if  $n$  is even,

$$\gamma_f^0(W_{n,t}) = \frac{\frac{n^2 t^2}{4} + nt - (n+1)}{nt}.$$

Proof Let  $V = V(W_{n,t})$ . Similar to the proof of Theorem 1, and the labels  $y$  and  $x_i (1 \leq i \leq n)$  of all vertices are shown in Fig. 2, define a function  $f: V \rightarrow [0, 1]$  as follows:

$$\text{let } f(v_0) = y, f(v_{ij}) = x_i \quad (j = 1, 2, \dots, t).$$

Case 1 When  $n = 2r + 1$  is odd ;

$$\text{let } y = \frac{n+1}{2n} - \frac{n+1}{nt}, x_1 = \frac{1}{t}, \text{ and for integer } i \quad (1 \leq i \leq r),$$

$$\text{let } x_{2i} = \frac{n+1-2i}{2n} - \frac{n+1-2i}{nt} \text{ and } x_{2i+1} = \frac{i}{n} + \frac{n-2i}{nt}.$$

It is easy to see that  $f(N(v_{ij})) = 1$  holds for any vertex  $v_{i,j} \quad (1 \leq i \leq n, 1 \leq j \leq t)$ . Note that  $0 \leq x_i \leq 1$  holds for each integer  $i \quad (1 \leq i \leq n)$ . By Lemma 2,  $f$  is the minimal fractional total domination function. Thus we have

$$\gamma_f^0(W_{n,t}) = f(V) = x_0 + t \sum_{i=1}^n x_i = \frac{2(1+2+\dots+\left\lfloor \frac{n}{2} \right\rfloor) t^2 + (n+1)t - (n+1)}{nt}.$$

Case 2 When  $n = 2r$  is even;

$$\text{let } y = \frac{1}{2} - \frac{n+1}{nt} \text{ and } x_1 = \frac{1}{t}, \text{ and for integer } i \quad (1 \leq i \leq r)$$

$$\text{let } x_{2i} = \frac{1}{2} - \frac{n-2i+1}{nt} \text{ and } x_{2i+1} = \frac{n-2i}{nt}$$

Similarly to case 1, it is also easy to see that  $f(N(v_{ij})) = 1$  for any vertex  $v_{i,j} \quad (1 \leq i \leq n, 1 \leq j \leq t)$ , and  $0 \leq x_i \leq 1$  holds for each integer  $i \quad (1 \leq i \leq n)$ . By Lemma 2,  $f$  is the minimal fractional total domination function, so we have

$$\gamma_f^0(W_{n,t}) = f(V) = x_0 + t \sum_{i=1}^n x_i = \frac{\frac{n^2 t^2}{4} + nt - (n+1)}{nt}$$

Combining the above three cases, we have completed the proof of Theorem 2. #

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