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Approximation of functions by singular integrals

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Abstract: In this work questions on approximation of locally summable functions by singular integrals are investigated. Was estimated the rate of approximation in terms of various metric characteristics describing the structural properties of the given function.

Keywords: Approximation, Singular Integrals, Bounded Mean Oscillation, Vanishing Mean Oscillation

1. Introduction

Let $L(R^n)$ be class of function which is integrable on Euclidean space R^n. The function $K \in L(R^n)$ is called kernel if

$$\int_{R^n} K(x)dx = 1.$$

Let $x \in R^n$, $\varepsilon > 0$, $K_\varepsilon(x) = \varepsilon^{-n} K\left(\frac{x}{\varepsilon}\right)$ and $K_\varepsilon f(x) =$

$$(K_\varepsilon * f)(x) = \int_{R^n} K_\varepsilon(x-t) f(t)dt, \tag{1.1}$$

where f is locally integrable function such that for all $\varepsilon > 0$ the integral on the right-hand side is finite.

In this work questions on approximation of locally summable functions by singular integrals of type (1.1) are investigated. Was estimated the rate of approximation in terms of various metric characteristics describing the structural properties of the given function.

Note that various aspects of questions on approximation of function f by singular integrals of a kind (1.1) have been investigated in works of many authors (see, e.g., [1], [2], [3], [5], [8], [9], [10], [15], [16] and the literature quoted there).

2. Some Definitions, Notation and Preliminary Facts

Let f be a locally integrable on R^n function, i.e. $f \in L_{loc}(R^n)$, $B(x,r)$ closed ball in R^n with center $x \in R^n$ and radius $r > 0$, i.e.

$$B(x,r) := \{y \in R^n: \ |x-y| \le r\},$$

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(t)dt,$$

$$\Omega\big(f, B(x,r)\big) = \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(t) - f_{B(x,r)}\right| dt,$$

where $|B(x,r)|$ denotes the volume of a ball $B(x,r)$.

For $f \in L_{loc}(R^n)$ and $x_0 \in R^n$ introduce the following notation ([10], [11]):

$$m_f(x_0;\delta) = \sup\{\Omega\big(f, B(x,r)\big): \ \ r \le \delta\} \ (\delta > 0).$$

It is obvious that, $m_f(x_0;\delta)$ is monotone increasing on interval $(0, +\infty)$ due to the argument δ.

The point $x_0 \in R^n$ is called d-point of $f \in L_{loc}(R^n)$, if there exists a finite limit $\lim_{r\to 0} f_{B(x_0,r)} = s_f(x_0)$. The collection of all d-points of f is denoted by $D(f)$.

The point $x_0 \in R^n$ is called Lebesgue point (or l-point) of function $f \in L_{loc}(R^n)$, if there exists a number $l_f(x_0)$, such that

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} \left|f(t) - l_f(x_0)\right| dt = 0.$$

Set of all l-points of function f is denoted by $L(f)$. It is clear that if $x_0 \in L(f)$, then $x_0 \in D(f)$ and $s_f(x_0) = l_f(x_0)$.

Point $x_0 \in R^n$ we will name a m-point of the function $f \in L_{loc}(R^n)$ (see [10]), if $\lim_{r\to 0} m_f(x_0; r) = 0$. Set of all m-points of function $f \in L_{loc}(R^n)$ is denoted by $M(f)$.

Theorem A [10], [12]. If $f \in L_{loc}(R^n)$, then the following equality is satisfied:

$$L(f) = D(f) \cap M(f).$$

For $f \in L_{loc}(R^n)$ and $x_0 \in D(f)$ we also introduce the following notation

$$\omega_f(x_0; \delta) = \sup_{0<r\le\delta} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |f(t) - s_f(x_0)| dt,$$

$$\delta > 0.$$

It easy to see that $\omega_f(x_0; \delta)$ is monotone increasing on interval $(0, +\infty)$ due to the argument δ. It is also easy to see that the point $x_0 \in R^n$ is l-point of a function f if and only if $\lim_{r\to 0} \omega_f(x_0; r) = 0$.

Now let's note some facts which we will use in the future.

Theorem B [10]. Let $K \in L(R^n)$ be kernel and

$$k(x) = \operatorname{esssup}\{|K(y)|: \ |y| \ge |x|\}, \ k \in L(R^n), \ k_0(|x|) = k(x),$$

$f \in L_{loc}(R^n)$, $x_0 \in R^n$. If right hand side integrals are convergent, then the following inequality holds:

$$|K_\varepsilon f(x_0) - f_{B(x_0,\varepsilon)}| \le$$

$$\le c(n,k_0) \left(m_f(x_0;\varepsilon) + \int_0^\infty x^{n-1} k_0(x) m_f(x_0; 4\varepsilon x) dx + \right.$$

$$+ \int_0^\varepsilon \frac{m_f(x_0;t)}{t} \left(\int_0^{\frac{t}{4\varepsilon}} x^{n-1} k_0(x) dx \right) dt +$$

$$\left. + \int_\varepsilon^\infty \frac{m_f(x_0;t)}{t} \left(\int_{\frac{t}{4\varepsilon}}^\infty x^{n-1} k_0(x) dx \right) dt \right), \ \varepsilon > 0, \qquad (2.1)$$

where $c(n,k_0)$ is constant depending only on the function k_0 and dimension n.

Note that the function

$$P(x) = \pi^{-(n+1)} \Gamma\left(\frac{n+1}{2}\right) (|x|^2 + 1)^{-\frac{n+1}{2}}, \quad x \in R^n,$$

is called the Poisson kernel. If $K(x) \equiv P(x)$, $x \in R^n$, then it is obvious, that $k(x) \equiv P(x)$, $x \in R^n$, and

$$k_0(t) = \pi^{-(n+1)} \Gamma\left(\frac{n+1}{2}\right) (t^2 + 1)^{-\frac{n+1}{2}}, t \in [0,+\infty).$$

Let $f \in L_{loc}(R^n)$ and

$$M_f(\delta) := \sup\{m_f(x; \delta): x \in R^n\}, \ \delta > 0.$$

A function $M_f(\delta)$ is called modulus of mean oscillation of function f. Note that the function $M_f(\delta)$ was firstly introduced in [14]. If $\varphi(\delta)$ is positive function which is monotone increasing on interval $(0, +\infty)$, then $BMO_\varphi = BMO_\varphi(R^n)$ denotes the set of all functions $f \in L_{loc}(R^n)$, for which the condition (see, for example, [6])

$$\|f\|_{BMO_\varphi} := \sup\left\{\frac{M_f(\delta)}{\varphi(\delta)}: \ \delta > 0\right\} < +\infty$$

satisfied.

If we consider the class BMO_φ as subset in the quotient space $L_{loc}(R^n)/\{constants\}$, then $\|\cdot\|_{BMO_\varphi}$ is the norm in BMO_φ and in this norm BMO_φ is Banach space. If $\varphi(\delta) \equiv 1$, then BMO_φ turns to space BMO, which was firstly introduced in [7].

We also introduce the class [13]

$$VMO = VMO(R^n) := \left\{ f \in BMO: \ \lim_{\delta\to+0} M_f(\delta) = 0 \right\}$$

with norm $\|f\|_{VMO} := \|f\|_{BMO}$.

3. On Approximation in Terms of the Characteristics $\omega_f(x_0; \delta)$

Theorem 3.1. Let $K \in L(R^n)$ be kernel,

$$k(x) = \operatorname{esssup}\{|K(y)|: \ |y| \ge |x|\}, \quad k \in L(R^n), \quad k_0(|x|) = k(x),$$

$f \in L_{loc}(R^n)$, $x_0 \in R^n$ is l-point of function f. If right hand side integrals are convergent, then the following inequality holds:

$$|K_\varepsilon f(x_0) - s_f(x_0)| \le$$

$$\le c \cdot \varepsilon^{-n} \int_0^\infty t^{n-1} k_0\left(\frac{t}{\varepsilon}\right) \omega_f(x_0; 4t) dt, \quad \varepsilon > 0, \qquad (3.1)$$

where c is a positive constant depending only on the dimension n.

Proof. We have

$$|K_\varepsilon f(x_0) - s_f(x_0)| =$$

$$= \left| \int_{R^n} K_\varepsilon(x_0 - t)[f(t) - s_f(x_0)] dt \right| \le$$

$$\le \varepsilon^{-n} \int_{R^n} \left| K\left(\frac{x_0 - t}{\varepsilon}\right) \right| |f(t) - s_f(x_0)| dt \le$$

$$\le \varepsilon^{-n} \int_{R^n} k_0\left(\left|\frac{x_0 - t}{\varepsilon}\right|\right) |f(t) - s_f(x_0)| dt =$$

$$= \sum_{m=-\infty}^{\infty} \varepsilon^{-n} \int_{2^m\varepsilon < |x_0 - t| \le 2^{m+1}\varepsilon} k_0\left(\left|\frac{x_0 - t}{\varepsilon}\right|\right) |f(t) -$$

$$- s_f(x_0)| dt =: \sum_{m=-\infty}^{\infty} \tau_m. \qquad (3.2)$$

We estimate each of the terms $\tau_m, m = 0, \pm 1, \pm 2, \ldots$. Considering that, $k_0(t)$ monotone decreasing function on interval $(0, +\infty)$, we get

$$\tau_m = \varepsilon^{-n} \int_{2^m\varepsilon<|x_0-t|\le 2^{m+1}\varepsilon} k_0\left(\frac{|x_0-t|}{\varepsilon}\right)\left|f(t)-s_f(x_0)\right|dt \le$$

$$\le k_0(2^m)\cdot\frac{1}{\varepsilon^n}\int_{B(x_0,\,2^{m+1}\varepsilon)}\left|f(t)-s_f(x_0)\right|dt =$$

$$= k_0(2^m)\cdot\frac{(2^{m+1})^n\cdot|B(0,1)|}{|B(0,1)|\cdot(2^{m+1}\varepsilon)^n}\times$$

$$\times\int_{B(x_0,\,2^{m+1}\varepsilon)}\left|f(t)-s_f(x_0)\right|dt \le$$

$$= k_0(2^m)\cdot(2^{m+1})^n\cdot|B(0,1)|\cdot\omega_f(x_0;\ 2^{m+1}\varepsilon).$$

Thus, by (3.2)

$$\left|K_\varepsilon f(x_0)-s_f(x_0)\right| \le |B(0,1)|\times$$

$$\times\sum_{m=-\infty}^{\infty}(2^{m+1})^n\cdot k_0(2^m)\cdot\omega_f(x_0;\ 2^{m+1}\varepsilon). \qquad (3.3)$$

On the other hand, we have

$$\varepsilon^{-n}\int_0^\infty t^{n-1}k_0\left(\frac{t}{\varepsilon}\right)\omega_f(x_0;4t)dt =$$

$$= \sum_{m=-\infty}^{\infty}\varepsilon^{-n}\int_{2^{m-1}\varepsilon}^{2^m\varepsilon} t^{n-1}k_0\left(\frac{t}{\varepsilon}\right)\omega_f(x_0;4t)dt \ge$$

$$\ge \sum_{m=-\infty}^{\infty}\varepsilon^{-n}\cdot k_0(2^m)\omega_f(x_0;4\cdot 2^{m-1}\varepsilon)\int_{2^{m-1}\varepsilon}^{2^m\varepsilon} t^{n-1}dt =$$

$$= \frac{2^n-1}{n\cdot 4^n}\sum_{m=-\infty}^{\infty}k_0(2^m)(2^{m+1})^n\omega_f(x_0;2^{m+1}\varepsilon). \qquad (3.4)$$

Combining inequalities (3.3) and (3.4) we get inequality (3.1) with constant $c = |B(0,1)|\cdot\frac{n\cdot 4^n}{2^n-1}$.

Corollary 3.1. Let kernel $K(x)$ satisfies the conditions of Theorem 3.1, $f\in L_{loc}(R^n)$, $x_0\in R^n$ is Lebesgue point of function f and

$$\int_1^\infty k_0(t)t^{n-1}\omega_f(x_0;4t)dt < +\infty \qquad (3.5)$$

Then $\lim_{\varepsilon\to 0}K_\varepsilon f(x_0) = l_f(x_0)$.

Proof. Let $0<\varepsilon\le 1$. Then we have

$$\varepsilon^{-n}\int_0^\infty t^{n-1}k_0\left(\frac{t}{\varepsilon}\right)\omega_f(x_0;4t)dt =$$

$$= \varepsilon^{-n}\int_0^\infty \varepsilon^{n-1}\cdot y^{n-1}k_0(y)\omega_f(x_0;4\varepsilon y)\varepsilon dy =$$

$$= \int_0^\infty y^{n-1}\cdot k_0(y)\cdot\omega_f(x_0;4\varepsilon y)dy =$$

$$= \int_0^{\frac{1}{\sqrt{\varepsilon}}} y^{n-1}\cdot k_0(y)\cdot\omega_f(x_0;4\varepsilon y)dy +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^\infty y^{n-1}\cdot k_0(y)\cdot\omega_f(x_0;4\varepsilon y)dy \le$$

$$\le \omega_f\left(x_0;4\sqrt{\varepsilon}\right)\cdot\int_0^\infty y^{n-1}k_0(y)dy +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^\infty y^{n-1}k_0(y)\omega_f(x_0;4y)dy. \qquad (3.6)$$

If to consider, that

$$+\infty > \int_{R^n}k(x)dx = \int_0^\infty t^{n-1}\int_{S^{n-1}}k_0(|t\xi|)d\sigma_\xi\,dt =$$

$$= |S^{n-1}|\cdot\int_0^\infty k_0(t)t^{n-1}dt,$$

where $|S^{n-1}|$ is surface area of unit sphere S^{n-1}, $d\sigma_\xi$ is Lebesgue's measure on the sphere S^{n-1}, then from the inequalities (3.1), (3.6) and by condition (3.5) we receive the demanded statement.

Corollary 3.2 [16]. Let kernel $K(x)$ satisfies the conditions of Theorem 3.1 and $f\in L^p(R^n)$, $1\le p\le\infty$, $x_0\in R^n$ is Lebesgue point of f. Then

$$\lim_{\varepsilon\to 0}K_\varepsilon f(x_0) = l_f(x_0).$$

By theorem 3.1 the following statement for Poisson integral is obtained.

Corollary 3.3. Let $P(x)$ be Poisson kernel, $P_\varepsilon(x) := \varepsilon^{-n}P\left(\frac{x}{\varepsilon}\right)$, $\varepsilon>0$, $x\in R^n$, $f\in L_{loc}(R^n)$, $x_0\in R^n$ is Lebesgue point of the function f and

$$\int_1^\infty\frac{\omega_f(x_0;t)}{t^2}dt < +\infty.$$

Then the following inequality is satisfied:

$$\left|(P_\varepsilon * f)(x_0) - l_f(x_0)\right| \le c\cdot\varepsilon\int_\varepsilon^\infty\frac{\omega_f(x_0;t)}{t^2}dt,\ \ \varepsilon>0,$$

where $c>0$ is only depended on n, and hence $\lim_{\varepsilon\to+0}(P_\varepsilon * f)(x_0) = l_f(x_0)$.

It is easy to see that for the Poisson kernel $P(x)$ the conditions of Corollary 3.2 are satisfied. Therefore, if $f\in L^p(R^n)$, $1\le p\le\infty$ and x_0 is Lebesgue point of the function f, we have the equality $\lim_{\varepsilon\to+0}(P_\varepsilon * f)(x_0) =$

$l_f(x_0)$.

Theorem 3.2. Let $x_0 \in R^n$ and

1) $K(x)$ is nonnegative kernel such, that $K(x) \equiv k_0(|x|)$, $x \in R^n$, where $k_0(t)$ is monotone decreases on $[0,+\infty)$;

2) $\omega(\delta)$ is positive monotone increasing function on $(0,+\infty)$ and satisfying $\lim_{\delta\to+0}\omega(\delta)=0$, for which there is a number $c_0>0$ such that $\omega(2\delta)\le c_0\cdot\omega(\delta)$, $\delta\in(0,+\infty)$.

Then there is a function $f_0 \in L_{loc}(R^n)$ such that, $\omega_{f_0}(x_0;\ \delta) \asymp \omega(\delta)$, $\delta\in(0,+\infty)$[1], and

$$\left|K_\varepsilon f_0(x_0) - s_{f_0}(x_0)\right| \ge$$

$$\ge c\cdot\varepsilon^{-n}\int\limits_0^\infty t^{n-1}k_0\left(\frac{t}{\varepsilon}\right)\omega_{f_0}(x_0;4t)dt\,,\qquad \varepsilon>0.$$

Proof. Let $f_0(x)=\omega(|x-x_0|)$, $x\in R^n$. Then

$$\lim_{r\to+0}\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)} f_0(t)dt =$$

$$=\lim_{r\to+0}\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)}\omega(|t-x_0|)dt = 0,$$

i.e. $s_{f_0}(x_0)=0$. Therefore

$$\omega_{f_0}(x_0;\ \delta)=\sup_{0<r\le\delta}\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)}\left|f_0(t)-s_{f_0}(x_0)\right|dt =$$

$$=\sup_{0<r\le\delta}\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)}\omega(|t-x_0|)dt =$$

$$=\sup\nolimits_{0<r\le\delta}\tfrac{1}{|B(0,r)|}\int_{B(0,r)}\omega(|t|)dt. \tag{3.7}$$

From here we get that, $\omega_{f_0}(x_0;\ \delta)\le\omega(\delta)$, $\delta\in(0,+\infty)$. By (3.7) also obtained that,

$$\omega_{f_0}(x_0;\ \delta)\ge\frac{1}{|B(0,\delta)|}\int\limits_{B(0,\delta)}\omega(|t|)dt \ge$$

$$\ge\frac{1}{|B(0,\delta)|}\int\limits_{B(0,\delta)\setminus B\left(0,\frac{\delta}{2}\right)}\omega(|t|)dt\ge$$

$$\ge\omega\left(\frac{\delta}{2}\right)\cdot\frac{1}{|B(0,1)|\cdot\delta^n}\cdot|B(0,1)|\cdot\left(\delta^n-\left(\left(\frac{\delta}{2}\right)\right)^n\right)=$$

$$=\omega\left(\frac{\delta}{2}\right)\cdot\frac{2^n-1}{2^n}\ge$$

$$\ge\frac{1}{c_0}\cdot\omega(\delta)\cdot\frac{2^n-1}{2^n}=c_1\cdot\omega(\delta),\qquad \delta>0.$$

Further we have

$$\left|K_\varepsilon f_0(x_0)-s_{f_0}(x_0)\right|=\varepsilon^{-n}\int\limits_{R^n}k_0\left(\frac{|x_0-t|}{\varepsilon}\right)f_0(t)dt=$$

$$=\varepsilon^{-n}\int\limits_{R^n}k_0\left(\frac{|x_0-t|}{\varepsilon}\right)\omega(|t-x_0|)dt=$$

$$=\varepsilon^{-n}\int\limits_{R^n}k_0\left(\frac{|t|}{\varepsilon}\right)\omega(|t|)dt=$$

$$=\varepsilon^{-n}\int\limits_0^\infty t^{n-1}\left(\int\limits_{S^{n-1}}k_0\left(\frac{|t\xi|}{\varepsilon}\right)\omega(|t\xi|)d\sigma_\xi\right)dt=$$

$$=|S^{n-1}|\cdot\varepsilon^{-n}\int\limits_0^\infty k_0\left(\frac{t}{\varepsilon}\right)\cdot t^{n-1}\omega(t)dt\ge$$

$$\ge\frac{1}{c_0^2}\cdot|S^{n-1}|\cdot\varepsilon^{-n}\int\limits_0^\infty k_0\left(\frac{t}{\varepsilon}\right)\cdot t^{n-1}\omega(4t)dt\ge$$

$$\ge\frac{1}{c_0^2}\cdot|S^{n-1}|\cdot\varepsilon^{-n}\int\limits_0^\infty k_0\left(\frac{t}{\varepsilon}\right)\cdot t^{n-1}\omega_{f_0}(x_0;\ 4t)dt,$$

$$\varepsilon>0.$$

This theorem shows unimprovability of estimations (3.1) in a certain class of kernels.

4. Approximation in Terms of the Mean Oscillation

Theorem 4.1. Let the kernel $K(x)$ satisfies conditions of the theorem B, $f\in L_{loc}(R^n)$, x_0 is Lebesgue point of f and let satisfies the following conditions also:

1) $\int_1^\infty t^{n-1}k_0(t)m_f(x_0;4t)dt<+\infty$,

2) $\int_0^1\left(\frac{1}{t}\int_0^t x^{n-1}k_0(x)dx\right)m_f(x_0;4t)dt<+\infty$,

3) $\int_1^\infty\left(\frac{1}{t}\int_t^\infty x^{n-1}k_0(x)dx\right)m_f(x_0;4t)dt<+\infty$.

Then $\lim_{\varepsilon\to+0}K_\varepsilon f(x_0)=l_f(x_0)$.

Proof. Since $x_0\in R^n$ is Lebesgue point of function f, then by the Theorem A the equality $\lim_{\delta\to+0}m_f(x_0;\ \delta)=0$ is satisfied and the finite limit $s_f(x_0)\coloneqq\lim_{\varepsilon\to+0}f_{B(x_0,\varepsilon)}=l_f(x_0)$ exists.

Let's show that, if the conditions 1), 2) and 3) are satisfied, then all terms on the right hand side of inequality (2.1) approaches to zero as $\varepsilon\to+0$. We can assume that $0<\varepsilon\le1$. Then we get

$$\int\limits_0^\infty t^{n-1}k_0(t)m_f(x_0;4\varepsilon t)dt=$$

[1] i.e. $\exists c_1,c_2>0$ $\forall\delta\in(0,+\infty)$: $c_1\cdot\omega_{f_0}(x_0;\ \delta)\le\omega(\delta)\le c_2\cdot\omega_{f_0}(x_0;\ \delta)$.

$$= \int_0^{\frac{1}{\sqrt{\varepsilon}}} t^{n-1} k_0(t) m_f(x_0; 4\varepsilon t)dt +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} t^{n-1} k_0(t) m_f(x_0; 4\varepsilon t)dt \le$$

$$\le m_f\left(x_0; 4\sqrt{\varepsilon}\right) \int_0^{\infty} t^{n-1} k_0(t)dt +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} t^{n-1} k_0(t) m_f(x_0; 4t)dt\,; \qquad (4.1)$$

$$\int_0^{\varepsilon} \frac{m_f(x_0;t)}{t} \left(\int_0^{\frac{t}{4\varepsilon}} x^{n-1} k_0(x)dx \right) dt =$$

$$= \int_0^{\frac{1}{4}} \frac{m_f(x_0; 4\varepsilon y)}{4\varepsilon y} \left(\int_0^{y} x^{n-1} k_0(x)dx \right) 4\varepsilon dy =$$

$$= \int_0^{\frac{1}{4}} \left(\frac{1}{y} \int_0^{y} x^{n-1} k_0(x)dx \right) m_f(x_0; 4\varepsilon y)dy \le$$

$$\le \int_0^{m_f(x_0;\,\varepsilon)} \left(\frac{1}{y} \int_0^{y} x^{n-1} k_0(x)dx \right) m_f(x_0; 4y)dy +$$

$$+ m_f(x_0;\,\varepsilon) \int_{m_f(x_0;\,\varepsilon)}^{\frac{1}{4}} \left(\frac{1}{y} \int_0^{y} x^{n-1} k_0(x)dx \right) dy, \qquad (4.2)$$

where ε is positive number such that $0 < \varepsilon \le 1$ and $m_f(x_0;\,\varepsilon) \le \frac{1}{4}$;

$$\int_{\varepsilon}^{\infty} \frac{m_f(x_0;t)}{t} \left(\int_{\frac{t}{4\varepsilon}}^{\infty} x^{n-1} k_0(x)dx \right) dt =$$

$$= \int_{\frac{1}{4}}^{\infty} \frac{m_f(x_0; 4\varepsilon y)}{4\varepsilon y} \left(\int_y^{\infty} x^{n-1} k_0(x)dx \right) 4\varepsilon dy =$$

$$= \int_{\frac{1}{4}}^{\infty} \left(\frac{1}{y} \int_y^{\infty} x^{n-1} k_0(x)dx \right) m_f(x_0; 4\varepsilon y)dy \le$$

$$\le \left\{ \int_{\frac{1}{4}}^{\frac{1}{\sqrt{\varepsilon}}} \left(\frac{1}{y} \int_y^{\infty} x^{n-1} k_0(x)dx \right) dy \right\} \cdot m_f\left(x_0; 4\sqrt{\varepsilon}\right) +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} \left(\frac{1}{y} \int_y^{\infty} x^{n-1} k_0(x)dx \right) m_f(x_0; 4y)dy \le$$

$$\le \frac{m_f\left(x_0; 4\sqrt{\varepsilon}\right)}{m_f(x_0; 1)} \int_{\frac{1}{4}}^{\infty} \left(\frac{1}{y} \int_y^{\infty} x^{n-1} k_0(x)dx \right) m_f(x_0; 4y)dy +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} \left(\frac{1}{y} \int_y^{\infty} x^{n-1} k_0(x)dx \right) m_f(x_0; 4y)dy. \qquad (4.3)$$

By considering that in the case of satisfying conditions of the theorem the following integral

$$\int_0^{\infty} x^{n-1} k_0(x)dx$$

converges, from inequality (4.1), (4.2) and (4.3) the required assertion is obtained.

Corollary 4.1. Let the kernel $K(x)$ satisfies conditions of the theorem B, $f \in BMO$, x_0 is Lebesgue point of function f, the condition 2) is satisfied and

$$\int_1^{\infty} \left(\frac{1}{t} \int_t^{\infty} x^{n-1} k_0(x)dx \right) dt < +\infty.$$

Then $\lim_{\varepsilon \to +0} K_\varepsilon f(x_0) = l_f(x_0)$.

Corollary 4.2. Let $f \in L_{loc}(R^n)$, x_0 is Lebesgue point of function f and let the following condition is satisfied:

$$\exists \gamma > 0 \;\; sup\{(1+|t|)^{n+\gamma}|K(t)|:\; t \in R^n\} < +\infty, \qquad (4.4)$$

$$\int_1^{\infty} t^{-1-\gamma} \cdot m_f(x_0;t)dt < +\infty. \qquad (4.5)$$

Then $\lim_{\varepsilon \to +0} K_\varepsilon f(x_0) = l_f(x_0)$.

Proof. By the condition (4.4) follows that,

$$\exists C > 0 \qquad \forall t \ge 0: \qquad k_0(t) \le C \cdot \frac{1}{(1+t)^{n+\gamma}}.$$

Now let's show that conditions of theorem 4.1 are satisfied. We have that

$$\int_1^{\infty} t^{n-1} k_0(t) m_f(x_0; 4t)dt \le$$

$$\le C \cdot \int_1^{\infty} t^{n-1} \cdot \frac{1}{(1+t)^{n+\gamma}} \cdot m_f(x_0; 4t)dt \le$$

$$\le C_1 \cdot \int_1^{\infty} \frac{m_f(x_0;t)}{t^{1+\gamma}} dt < +\infty, \qquad C_1 = const;$$

$$\int_0^1 \left(\frac{1}{t} \int_0^t x^{n-1} k_0(x)dx \right) m_f(x_0; 4t)dt \le$$

$$\leq C\cdot\int_0^1\left(\frac{1}{t}\int_0^t\frac{x^{n-1}dx}{(1+x)^{n+\gamma}}\right)m_f(x_0;4t)dt\leq$$

$$\leq C\cdot\int_0^1\left(\frac{1}{t}\int_0^t x^{n-1}dx\right)m_f(x_0;4t)dt=$$

$$=\frac{C}{n}\cdot\int_0^1 t^{n-1}\cdot m_f(x_0;4t)dt<+\infty;$$

$$\int_1^\infty\left(\frac{1}{t}\int_t^\infty x^{n-1}k_0(x)dx\right)m_f(x_0;4t)dt\leq$$

$$\leq C\cdot\int_1^\infty\left(\frac{1}{t}\int_t^\infty\frac{x^{n-1}}{x^{n+\gamma}}dx\right)m_f(x_0;4t)dt=$$

$$=C\cdot\int_1^\infty\left(\frac{1}{t}\int_t^\infty x^{-\gamma-1}dx\right)m_f(x_0;4t)dt\leq$$

$$\leq C_2\cdot\int_1^\infty\frac{m_f(x_0;t)}{t^{1+\gamma}}dt<+\infty,\quad C_2=const.$$

Thus, all conditions of the theorem 4.1 are satisfied and because of this $\lim_{\varepsilon\to+0}K_\varepsilon f(x_0)=l_f(x_0)$.

Note that, particularly, any function $f\in BMO$ satisfy condition (4.5) for all points $x_0\in R^n$. The kernel $K(x)$, which satisfies conditions (4.4) is called kernel of Fejer type.

Theorem 4.2. Let kernel $K(x)$ satisfies conditions of theorem B and $f\in L_{loc}(R^n)$. Then for a finite value of right hand side we have the following inequality

$$\|f-K_\varepsilon f\|_{BMO}\leq$$

$$\leq C\cdot\left(M_f(\varepsilon)+\int_0^\infty x^{n-1}k_0(x)M_f(4\varepsilon x)dx+\right.$$

$$+\int_0^\varepsilon\frac{M_f(t)}{t}\left(\int_0^{\frac{t}{4\varepsilon}}x^{n-1}k_0(x)dx\right)dt+$$

$$\left.+\int_\varepsilon^\infty\frac{M_f(t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^\infty x^{n-1}k_0(x)dx\right)dt\right),\quad\varepsilon>0,\qquad(4.6)$$

where the constant $C>0$ depends only on n and k_0.

Proof. From inequality (2.1) we have

$$\left\|K_\varepsilon f-f_{B(\cdot\,;\,\varepsilon)}\right\|_{BMO}\leq 2\left\|K_\varepsilon f-f_{B(\cdot\,;\,\varepsilon)}\right\|_{L^\infty(R^n)}\leq$$

$$\leq C\left(M_f(\varepsilon)+\int_0^\infty x^{n-1}k_0(x)M_f(4\varepsilon x)dx+\right.$$

$$+\int_0^\varepsilon\frac{M_f(t)}{t}\left(\int_0^{\frac{t}{4\varepsilon}}x^{n-1}k_0(x)dx\right)dt+$$

$$\left.+\int_\varepsilon^\infty\frac{M_f(t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^\infty x^{n-1}k_0(x)dx\right)dt\right),\quad\varepsilon>0.\qquad(4.7)$$

It is known that (see [6])

$$\left\|f-f_{B(\cdot\,;\,\varepsilon)}\right\|_{BMO}\leq C\cdot M_f(\varepsilon),\qquad\varepsilon>0,\qquad(4.8)$$

where $C>0$ is independent of f and ε. From inequalities (4.7) and (4.8) the inequality (4.6) turns out.

Corollary 4.3. Let kernel $K(x)$ satisfies conditions of theorem B, $f\in VMO$ and also satisfies the following conditions:

1^0) $\int_0^1\left(\frac{1}{t}\int_0^t x^{n-1}k_0(x)dx\right)M_f(4t)dt<+\infty$,

2^0) $\int_1^\infty\left(\frac{1}{t}\int_t^\infty x^{n-1}k_0(x)dx\right)dt<+\infty$.

Then $\lim_{\varepsilon\to+0}\|K_\varepsilon f-f\|_{BMO}=0$.

Proof. It is enough to check up that under our assumptions all terms on the right hand side of the inequality (4.6) approaches to zero, with $\varepsilon\to+0$. If $f\in VMO$, then $\lim_{\varepsilon\to+0}M_f(\varepsilon)=0$. In addition, if $0<\varepsilon\leq 1$, then

$$\int_0^\infty x^{n-1}k_0(x)M_f(4\varepsilon x)dx\leq M_f\left(4\sqrt{\varepsilon}\right)\int_0^\infty x^{n-1}k_0(x)dx+$$

$$+\|f\|_{BMO}\cdot\int_{\frac{1}{\sqrt{\varepsilon}}}^\infty x^{n-1}k_0(x)dx\,;$$

$$\int_0^\varepsilon\frac{M_f(t)}{t}\left(\int_0^{\frac{t}{4\varepsilon}}x^{n-1}k_0(x)dx\right)dt=$$

$$=\int_0^{\frac{1}{4}}\left(\frac{1}{y}\int_0^y x^{n-1}k_0(x)dx\right)M_f(4\varepsilon y)dy\leq$$

$$\leq\int_0^{M_f(\varepsilon)}\left(\frac{1}{y}\int_0^y x^{n-1}k_0(x)dx\right)M_f(4y)dy+$$

$$+M_f(\varepsilon)\int_{M_f(\varepsilon)}^{\frac{1}{4}}\left(\frac{1}{y}\int_0^y x^{n-1}k_0(x)dx\right)dy,$$

where ε is positive number such that $\varepsilon\leq 1$ and $M_f(\varepsilon)\leq\frac{1}{4}$;

$$\int_{\varepsilon}^{\infty} \frac{M_f(t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^{\infty} x^{n-1}k_0(x)dx\right)dt =$$

$$= \int_{\frac{1}{4}}^{\infty}\left(\frac{1}{y}\int_{y}^{\infty} x^{n-1}k_0(x)dx\right)M_f(4\varepsilon y)dy \le$$

$$\le \left(\int_{\frac{1}{4}}^{\frac{1}{\sqrt{\varepsilon}}}\left(\frac{1}{y}\int_{y}^{\infty} x^{n-1}k_0(x)dx\right)dy\right)\cdot M_f\left(4\sqrt{\varepsilon}\right) +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty}\left(\frac{1}{y}\int_{y}^{\infty} x^{n-1}k_0(x)dx\right)M_f(4y)dy \le$$

$$\le \frac{1}{M_f(1)}\cdot M_f\left(4\sqrt{\varepsilon}\right)\cdot\int_{\frac{1}{4}}^{\infty}\left(\frac{1}{y}\int_{y}^{\infty} x^{n-1}k_0(x)dx\right)M_f(4y)dy +$$

$$+ \int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty}\left(\frac{1}{y}\int_{y}^{\infty} x^{n-1}k_0(x)dx\right)M_f(4y)dy.$$

It is easy to check that if conditions 1^0) and 2^0) are satisfied, then the integral $\int_0^{\infty} x^{n-1}k_0(x)dx$ converges. Therefore the obtained relations show the validity of the required statement by virtue of inequality (4.6).

Note that, if $K(x)$ is Poisson kernel and $f \in BMO$, then the conditions 1^0) and 2^0) are satisfied. Thus, if $f \in VMO$ and K is Poisson kernel, then $\lim_{\varepsilon\to+0}\|K_\varepsilon f - f\|_{BMO} = 0$. It is known that, if $f \in BMO(R)$, K is Poisson kernel (for $n = 1$) and $\lim_{\varepsilon\to+0}\|K_\varepsilon f - f\|_{BMO} = 0$, then $f \in VMO$ (see [4]).

Corollary 4.4. Let kernel $K(x)$ satisfies condition (4.4), $f \in L_{loc}(R^n)$ and

$$\int_1^{\infty} \frac{M_f(t)}{t^{1+\gamma}}dt < +\infty. \tag{4.9}$$

Then the following inequality is true

$$\|f - K_\varepsilon f\|_{BMO} \le$$
$$\le C\cdot\varepsilon^\gamma \int_\varepsilon^{\infty} \frac{M_f(t)}{t^{1+\gamma}}dt,\ \ \varepsilon > 0, \tag{4.10}$$

where the constant $C > 0$ is independent of f and ε.

Proof. Estimate terms on the right hand side of inequality (4.6). We have

$$\int_0^{\infty} x^{n-1}k_0(x)M_f(4\varepsilon x)dx =$$

$$= \int_0^{\frac{1}{4}} x^{n-1}k_0(x)M_f(4\varepsilon x)dx + \int_{\frac{1}{4}}^{\infty} x^{n-1}k_0(x)M_f(4\varepsilon x)dx \le$$

$$\le c_1\cdot\int_0^{\frac{1}{4}} x^{n-1}\cdot M_f(4\varepsilon x)dx +$$

$$+c_2\cdot\int_{\frac{1}{4}}^{\infty} x^{n-1}\cdot\frac{1}{x^{n+\gamma}}\cdot M_f(4\varepsilon x)dx \le$$

$$\le c_3\cdot\left(M_f(\varepsilon) + \varepsilon^\gamma\int_\varepsilon^{\infty}\frac{M_f(x)}{x^{1+\gamma}}dx\right) \le c_4\cdot\varepsilon^\gamma\int_\varepsilon^{\infty}\frac{M_f(t)}{t^{1+\gamma}}dt;$$

$$\int_0^{\varepsilon}\frac{M_f(t)}{t}\left(\int_0^{\frac{t}{4\varepsilon}} x^{n-1}k_0(x)dx\right)dt \le c_5\cdot\int_0^{\varepsilon}\frac{M_f(t)}{t}\left(\frac{t}{\varepsilon}\right)^n dt =$$

$$= c_5\cdot\frac{1}{\varepsilon^n}\int_0^{\varepsilon} M_f(t)t^{n-1}dt \le$$

$$\le c_6\cdot M_f(\varepsilon) \le c_7\cdot\varepsilon^\gamma\int_\varepsilon^{\infty}\frac{M_f(t)}{t^{1+\gamma}}dt;$$

$$\int_\varepsilon^{\infty}\frac{M_f(t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^{\infty} x^{n-1}k_0(x)dx\right)dt \le$$

$$\le c_8\cdot\int_\varepsilon^{\infty}\frac{M_f(t)}{t}\left(\int_{\frac{t}{4\varepsilon}}^{\infty} x^{n-1}\cdot\frac{1}{x^{n+\gamma}}dx\right)dt \le$$

$$\le c_9\cdot\varepsilon^\gamma\int_\varepsilon^{\infty}\frac{M_f(t)}{t^{1+\gamma}}dt,$$

where c_i $(i = 1,2,\dots,9)$ are positive constants, not depending on f and ε. By obtained inequalities from inequality (4.6) we get estimation (4.10).

Corollary 4.5. Let kernel $K(x)$ satisfies condition (4.4), $f \in BMO_\varphi$,

$$\varepsilon^\gamma\int_\varepsilon^{\infty}\frac{\varphi(t)}{t^{1+\gamma}}dt = O\big(\varphi(\varepsilon)\big), \qquad \varepsilon > 0.$$

Then the following relation is true

$$\|f - K_\varepsilon f\|_{BMO} = O\big(\varphi(\varepsilon)\big), \qquad \varepsilon > 0.$$

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