Pure and Applied Mathematics Journal
2013; 2(2) : 94-97
Published online April 2, 2013 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.20130202.16

Analysis of m/g/1 queue model with priority

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To cite this article:
Pan Quanru. Analysis of M/G/1 Queue Model with Priority, *Pure and Applied Mathematics Journal*. Vol. 2, No. 2, 2013, pp. 94-97.
doi: 10.11648/j.pamj.20130202.16

Abstract: Due to the queue phenomenon different customers needing different service quality, a model is established as follows: there are two types of customers in the system and their arrival rates are different; first-class customers have no preemptive priority, the different service time for the different customers and all the service time obeys the general distribution. The following conclusions are drawn: the Laplace - Steele Kyrgyz transform of the low-priority customers' waiting time stationary distribution; the average waiting time in the system of low priority customers; the Laplace - Steele Kyrgyz transform of the low-priority customers' staying time stationary distribution; At last, this paper points out the problems to be solved.

Keywords: Non-Preemptive, Priority, General Distribution

1. Introduction

Differences in quality of service requirements due to the different types of business, single service standards are often unable to meet certain business requirements, which require that the customers are divided into different priority, and according to the priority to get the services and quality. Actually, model with priority has been widely used [1−2], and have come to some better results [3−4]. However, most of the literatures are based on service time with exponentially distributed [5−6].In this paper, a queuing model with no preemptive and the service time obeyed general distribution is established.

2. Model Assumes

1. The system contains two types of customers, each type customer reaches the system at Poisson flow, and the arrival rate of the first class customer is λ_1, the arrival rate of the second class customer is λ_2. The arrival processes of different type customers are independent.
2. First-come, first-served for the same priority customers. The first type customer owns a higher priority than the second type customers.
3. There is only one service window in the system, and its capacity is unlimited.
4. General distribution is applied to the service time "T" for each type customers, and the service time distribution for the first type customer is denoted by $B_1(x)$, the service time distribution for the second type customer is denoted by $B_2(x)$
5. The arrival time of the second type customer is less than the service time of the first type customers.
6. The service processes of the different type customers are independent.
7. If the second type customer is being served when the first type customer arrives, the system does not interrupt existing services. Let

$$\beta_i = \int_0^\infty x dB_i(x),\ \beta_i^{(2)} = \int_0^\infty x^2 dB_i(x),\ B_i^*(s) = \int_{0-}^\infty e^{-sx} dB_i(x),\ i = 1,2$$

3. Model Analysis

Assumed $\lambda_1\beta_1 + \lambda_2\beta_2 < 1$ and stationary distribution of the first type customers' queue length exists.

Let $q_n(i)$ is the queue length of the i-th $(i=1,2)$ type customers when the customer leaves the system after n-th time.

Let generating function of the stationary distribution of $q_n(1)$:

$$\widetilde{Q}_1(z) = E\left\{z^{q_n(1)}\right\} \tag{1}$$

Let P_0 is the probability that there is no first type customers in the system or the second type customers in the system after the moment the second type customer leaves the system.

Easy to know

$$P_0 = 1 - \lambda_1\beta_1 - \lambda_2\beta_2 \qquad (2)$$

Let $\widetilde{P}_0$ is the probability that there is no customer in the system after the moment one customer leaves the system, the one who leaves the system can be the first type customer or the second type customer.

Let

$$x_n = \begin{cases} 1 & \text{the n-th customer leaves the system and he is the first type customer} \\ 0 & \text{the n-th customer leaves the system and he is the second type customer} \end{cases} \qquad (3)$$

Due to the arrival interval time of the second type customers is less than service time of the first class customer, then there must have second type customer in the system after the leaving of the first type customer, so

$$\widetilde{P}_0 = \mathrm{P}\{x_n = 0\} P_0 \qquad (4)$$

Theorem

After the moment that the first type customer leaves the system, the probability generating function of the queue length stationary distribution of the first type customers

$$Q_1(z) = \frac{(2-\lambda_1\beta_1-\lambda_2\beta_2)\left[1-B_2^*(\lambda_1(1-z))\right]+(1-\lambda_1\beta_1)(1-\lambda_1\beta_1-\lambda_2\beta_2)\left[B_2^*(\lambda_1(1-z))-z\right]}{\left(B_1^*(\lambda_1(1-z))-z\right)\left(1-\lambda_1\beta_1+(\lambda_1-\lambda_2)\beta_2\right)} \times\left[B_1^*(\lambda_1(1-z))\right] \qquad (5)$$

the average queue length

$$\begin{aligned}\overline{Q}_1 = \Big\{&\left[\lambda_1^2(2-\lambda_1\beta_1-\lambda_2\beta_2)\right]\left(\beta_2^2+2\beta_1\beta_2\lambda_1+\beta_1\beta_2\right) \times(\lambda_1\beta_1-1) + \\ &\lambda_1(\lambda_1\beta_1-1)\begin{bmatrix}(1-\lambda_1\beta_1-\lambda_2\beta_2)(\lambda_1\beta_2^{(2)}-1) \\ +(\beta_1\lambda_1-1)(\beta_2-1)\end{bmatrix} \\ &+\frac{\lambda_1(\lambda_1\beta_1-1)(1-\lambda_1\beta_1-\lambda_2\beta_2)}{\left[\beta_1(\lambda_1-1)+\beta_1(\beta_1\lambda_1-1)(\lambda_2\beta_2-1)\right]}\Big\} / 2(\lambda_1\beta_1-1)^2\end{aligned} \qquad (6)$$

Proof

$$\begin{aligned}\widetilde{Q}_1(z) &= E\{z^{q_{n+1}(1)}\} \\ &= E\{z^{q_{n+1}(1)} \mid q_n(1)>0\}P\{q_n(1)>0\} \\ &+E\{z^{q_{n+1}(1)} \mid q_n(1)=0,q_n(2)=0\}P\{q_n(1)=0,q_n(2)=0\} \\ &+E\{z^{q_{n+1}(1)} \mid q_n(1)=0,q_n(2)>0\}\times P\{q_n(1)=0,q_n(2)>0\}\end{aligned} \qquad (7)$$

$$E_1 = E\{z^{q_{n+1}(1)} \mid q_n(1)>0\}P\{q_n(1)>0\},$$

$$E_2 = E\{z^{q_{n+1}(1)} \mid q_n(1)=0,q_n(2)=0\} P\{q_n(1)=0,q_n(2)=0\},$$

$$E_3 = E\{z^{q_{n+1}(1)} \mid q_n(1)=0,q_n(2)>0\} P\{q_n(1)=0,q_n(2)>0\}$$

In E_1, when $q_n(1)>0$, $q_{n+1}(1)=q_n(1)-1+v_{n+1}(1)$, among which $v_{n+1}(1)$ is the number of first type customers who arrive at system during the service time of one first type customer, and its probability generating function

$$\begin{aligned}E\left(z^{v_{n+1}(1)}\right) &= \sum_{j=0}^{\infty} z^j P\{v_{n+1}(1)=j\} \\ &= \sum_{j=0}^{\infty} z^j \int_{0-}^{\infty} e^{-\lambda_1 x}\frac{(\lambda_1 x)^j}{j!}dB_1(x) \\ &= \int_{0-}^{\infty} e^{-\lambda_1 x(1-z)}dB_1(x) = B_1^*(\lambda_1(1-z))\end{aligned} \qquad (8)$$

So

$$\begin{aligned}E_1 &= E\{z^{q_n(1)-1+v_{n+1}(1)} \mid q_n(1)>0\}P\{q_n(1)>0\} \\ &= E\{z^{q_n(1)} \mid q_n(1)>0\}z^{-1}E\{z^{v_{n+1}(1)}\}P\{q_n(1)>0\} \\ &= \sum_{j=0}^{\infty} z^j P\{q_n(1)=j \mid q_n(1)>0\}P\{q_n(1)>0\}z^{-1}B_1^*(\lambda_1(1-z)) \\ &= \sum_{j=0}^{\infty} z^j P\{q_n(1)=j, q_n(1)>0\}z^{-1}B_1^*(\lambda_1(1-z)) \\ &= \sum_{j=1}^{\infty} z^j P\{q_n(1)=j\}z^{-1}B_1^*(\lambda_1(1-z)) \\ &= [\widetilde{Q}_1(z)-\widetilde{Q}_1(0)]z^{-1}B_1^*(\lambda_1(1-z))\end{aligned} \qquad (9)$$

In E_2, when $q_n(1)=0$, $q_n(2)=0$, then

$$q_{n+1}(1) = v_{n+1}(1),$$

and $\mathrm{P}\{q_n(1)=0,q_n(2)=0\}=\widetilde{P}_0$, so,

$$\begin{aligned}E_2 &= \widetilde{P}_0 E\{z^{v_{n+1}(1)} \mid q_n(1)=0,q_n(2)=0\} \\ &= \widetilde{P}_0 E\{z^{v_{n+1}(1)}\} = \widetilde{P}_0 B_1^*(\lambda_1(1-z))\end{aligned} \qquad (10)$$

In E_3, when $q_n(1)=0$, $q_n(2)>0$, then $q_{n+1}(1)=v'_{n+1}(1)$, among which $v'_{n+1}(1)$ is the number of first type customers who arrive at system during the service time of one second type customer, and its probability generating function(Similar to the certificate in E_1)

$$E\{z^{v'_{n+1}(1)}\} = B_2^*(\lambda_1(1-z))$$

so

$$\begin{aligned} E_3 &= E\left\{z^{v'_{n+1}(1)} \mid q_n(1)=0, q_n(2)>0\right\}\times \\ &P\{q_n(1)=0, q_n(2)>0\} \\ &= E\left\{z^{v'_{n+1}(1)}\right\}[P\{q_n(1)=0\}-P\{q_n(1)=0, q_n(2)=0\}] \\ &= B_2^*(\lambda_1(1-z))\left[\widetilde{Q}_1(0)-\widetilde{P}_0\right] \end{aligned} \tag{11}$$

Put E_1, E_2, E_3 into $\widetilde{Q}_1(z)$,we get:

$$\begin{aligned} \widetilde{Q}_1(z) &= \frac{\widetilde{Q}_1(0)\left[B_1^*(\lambda_1(1-z))-zB_2^*(\lambda_1(1-z))\right]}{B_1^*(\lambda_1(1-z))-z} \\ &+\frac{\widetilde{Q}_1(0)+\widetilde{P}_0 z\left[B_2^*(\lambda_1(1-z))-B_1^*(\lambda_1(1-z))\right]}{B_1^*(\lambda_1(1-z))-z} \end{aligned} \tag{12}$$

$\lim\limits_{z\to 1}\widetilde{Q}_1(z)=1$, so

$$\begin{aligned} \lim_{z\to 1}\widetilde{Q}_1(z) = \lim_{z\to 1}\Bigg\{ &\frac{\widetilde{Q}_1(0)\left[B_1^*(\lambda_1(1-z))-zB_2^*(\lambda_1(1-z))\right]}{B_1^*(\lambda_1(1-z))-z} \\ &+\frac{\widetilde{P}_0 z\left[B_2^*(\lambda_1(1-z))-B_1^*(\lambda_1(1-z))\right]}{B_1^*(\lambda_1(1-z))-z}\Bigg\} \end{aligned}$$

So,

$$\frac{\left[\widetilde{Q}_1(0)[\lambda_1\beta_1-1-\lambda_1\beta_2]+\widetilde{P}_0[\lambda_1\beta_2-\lambda_1\beta_1]\right]}{\lambda_1\beta_1-1}=1$$

further,

$$\widetilde{Q}_1(0)=\frac{\lambda_1\beta_1-1-\widetilde{P}_0[\lambda_1\beta_2-\lambda_1\beta_1]}{\lambda_1\beta_1-1-\lambda_1\beta_2} \tag{13}$$

among which $\widetilde{P}_0=\mathrm{P}\{x_n=0\}P_0$.

Let

$$G_1(z)=P\{x_{n+1}=1\}E\left\{z^{q_{n+1}(1)} \mid x_{n+1}=1\right\} \tag{14}$$

Then

$$\begin{aligned} G_1(z) &= E\left\{z^{q_{n+1}(1)}\right\}-P\{x_{n+1}=0\}E\left\{z^{q_{n+1}(1)} \mid x_{n+1}=0\right\} \\ &= \widetilde{Q}_1(z)-P\{q_n(1)=0, q_n(2)>0\} \\ &\times E\left\{z^{q_{n+1}(1)} \mid q_n(1)=0, q_n(2)>0\right\} \end{aligned}$$

put E_3 into it:

$$G_1(z)=\widetilde{Q}_1(z)-B_2^*(\lambda_1(1-z))\left[\widetilde{Q}_1(0)-\widetilde{P}_0\right] \tag{15}$$

put(12)into(15):

$$\begin{aligned} G_1(z) &= \frac{\widetilde{Q}_1(0)\left[1-B_2^*(\lambda_1(1-z))\right]}{B_1^*(\lambda_1(1-z))-z} \\ &+\frac{\widetilde{P}_0\left[B_2^*(\lambda_1(1-z))-z\right]B_1^*(\lambda_1(1-z))}{B_1^*(\lambda_1(1-z))-z} \end{aligned} \tag{16}$$

In(15),let $z=1$,we get $G_1(1)=1-\widetilde{Q}_1(0)+\widetilde{P}_0$
noticed(14):

$$P\{x_{n+1}=1\}=G_1(1)=1-\widetilde{Q}_1(0)+\widetilde{P}_0 \tag{17}$$

so, $P\{x_{n+1}=0\}=1-P\{x_{n+1}=1\}=\widetilde{Q}_1(0)-\widetilde{P}_0$,put it into(4):

$\widetilde{P}_0=\left[\widetilde{Q}_1(0)-\widetilde{P}_0\right]P_0$,so,

$$\widetilde{Q}_1(0)=\frac{(1+P_0)\widetilde{P}_0}{P_0} \tag{18}$$

put (18) into (13), and noticed (2):

$$\widetilde{P}_0=\frac{1-\lambda_1\beta_1}{2-2\lambda_1\beta_1+\lambda_1\beta_2-\lambda_2\beta_2}P_0 \tag{19}$$

compared it with the formula (4), we get:

$$P\{x_n=0\}=\frac{1-\lambda_1\beta_1}{2-2\lambda_1\beta_1+\lambda_1\beta_2-\lambda_2\beta_2} \tag{20}$$

thus

$$\begin{aligned} P\{x_n=1\} &= 1-P\{x_n=0\} \\ &= \frac{1-\lambda_1\beta_1+(\lambda_1-\lambda_2)\beta_2}{2-2\lambda_1\beta_1+\lambda_1\beta_2-\lambda_2\beta_2} \end{aligned} \tag{21}$$

put(19)into(18):

$$\widetilde{Q}_1(0)=\frac{(1+P_0)(1-\lambda_1\beta_1)}{2-2\lambda_1\beta_1+(\lambda_1-\lambda_2)\beta_2} \tag{22}$$

put(2)into(19):

$$\widetilde{P}_0=\frac{(1-\lambda_1\beta_1)(1-\lambda_1\beta_1-\lambda_2\beta_2)}{2-2\lambda_1\beta_1+\lambda_1\beta_2-\lambda_2\beta_2} \tag{23}$$

put(2)into(22):

$$\widetilde{Q}_1(0)=\frac{(2-\lambda_1\beta_1-\lambda_2\beta_2)(1-\lambda_1\beta_1)}{2-2\lambda_1\beta_1+(\lambda_1-\lambda_2)\beta_2} \tag{24}$$

put(23)(24)into(16):

$$\begin{aligned} G_1(z) &= (2-\lambda_1\beta_1-\lambda_2\beta_2)\left[B_1^*(\lambda_1(1-z))\right]\times \\ &\frac{\left[1-B_2^*(\lambda_1(1-z))\right]+(1-\lambda_1\beta_1)(1-\lambda_1\beta_1-\lambda_2\beta_2)\left[B_2^*(\lambda_1(1-z))-z\right]}{\left[B_1^*(\lambda_1(1-z))-z\right](2-2\lambda_1\beta_1+(\lambda_1-\lambda_2)\beta_2)} \end{aligned} \tag{25}$$

So by the definition of $Q_1(z)$, and (14) we have:

$$Q_1(z)=E\left\{z^{q_{n+1}(1)} \mid x_{n+1}=1\right\}=\frac{G_1(z)}{P\{x_{n+1}=1\}} \tag{26}$$

put(21)(25)into(26),we get

$$Q_1(z)=$$
$$\frac{(2-\lambda_1\beta_1-\lambda_2\beta_2)\left[1-B_2^*(\lambda_1(1-z))\right]+(1-\lambda_1\beta_1)(1-\lambda_1\beta_1-\lambda_2\beta_2)\left[B_2^*(\lambda_1(1-z))-z\right]}{\left(B_1^*(\lambda_1(1-z))-z\right)\left(1-\lambda_1\beta_1+(\lambda_1-\lambda_2)\beta_2\right)}$$
$$\times\left[B_1^*(\lambda_1(1-z))\right]$$

namely(5),the average queue length is determined by

$$\overline{Q}_1(z)=\frac{dQ_1(z)}{dz}\,|\,z=1.$$

4. Conclusion

Queuing theory with priority has important applications, this article made a quantitative analysis of some indicators of the low priority customers. About priority theory, there are many issues worthy of further study, such as the utilization of the system, the system with more than two types of customers, and so on. In the application process, high priority should be given to the customers with a shorter average service time, which makes it possible to shorten the average waiting time of the entire customers.

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