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Rhotrix polynomials and polynomial rhotrices

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Abstract: In this piece of note, polynomials defined over the ring R of rhotrices of $n-dimension$ and rhotrices defined over polynomials in $\Re$ were explored, the aim is to study their nature and present their properties. The hope is that these polynomials (or these rhotrices) will have wider applications than those polynomials defined over the non-commutative ring of $n-square$ matrices (or those matrices defined over polynomials) since R is a commutative ring. The shortcomings of these polynomials and rhotrices were also confirmed as it was proved that the rings $R[x]$ and $R[f]$ are not integral domains.

Keywords: Rhotrix, Group, Ring, Polynomial, Commutative Ring, Integral Domain, Mathematical Modeling

1. Introduction

In mathematical modeling, real life problems are modeled into polynomial equations from where the coefficient matrices are normally formed and used. The study and analysis of the solutions of these polynomials are used to solve these problems. There is no doubt, exploring polynomials enhances substantive application of Mathematics. The purpose of this note is to define polynomials over a relatively new structure termed rhotrix introduced a decade ago and to define rhotrices over polynomials defined in $\Re$. These polynomials (or rhotrices) are expected to have more areas of application than the polynomials defined over the non-commutative ring of $n-square$ matrices (or matrices defined over polynomials) since the ring R of all rhotrices of $n-dimension$ is a commutative ring.

The first work on rhotrices was presented in [1], objects in the set

$$R = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \Re \right\}$$

were defined as rhotrices, as a result of their rhomboid nature. The central entry denoted by $h(R)$, was defined as heart (that is c in the above definition). The addition of two rhotrices R and Q was defined as

$$R + Q = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle$$

and $-A$ was given as the additive inverse of the rhotrix A, for the fact that

$$A + (-A) = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & -a & \\ -b & -h(R) & -d \\ & -e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$$

the additive identity of R. It was shown that $\{R, +\} \cup 0$ where $0 = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$ is a commutative group. Scalar multiplication was defined as follows:

$$\alpha R = \alpha \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & \alpha a & \\ \alpha b & \alpha h(R) & \alpha d \\ & \alpha e & \end{matrix} \right\rangle$$

Multiplication of two Rhotrices R and Q is done as follows:

$$R \bullet Q = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \bullet \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle =$$

$$\left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle$$

It was proved that the set R is a commutative algebra.

$$I = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$$

is the multiplicative identity of R.
If

$$R \bullet Q = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \bullet \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$$

Then

$$Q = R\text{-}1 = -\frac{1}{h(R)^2} \left\langle \begin{matrix} & a & \\ b & -h(R) & d \\ & e & \end{matrix} \right\rangle,\ h(R) \neq 0$$

Definition of R was later generalised to dimension n In Mohammed [2], with the same operations defined above as the set

$$A(n) = \left\{ \left\langle \begin{matrix} & & & a_1 & & & \\ & & a_2 & a_3 & a_4 & & \\ \cdots & \cdots & & \cdots & & \cdots & \cdots \\ a_{\left\{\frac{(t+1)}{2}\right\}-\frac{n}{2}} & \cdots\ \cdots & & a_{\left\{\frac{(t+1)}{2}\right\}} & & \cdots\ \cdots & a_{\left\{\frac{(t+1)}{2}\right\}+\frac{n}{2}} \\ \cdots & \cdots & & \cdots & & \cdots & \cdots \\ & & a_{t-3} & a_{t-2} & a_{t-1} & & \\ & & & a_t & & & \end{matrix} \right\rangle a_i \in \Re \right\}$$

where $t = \frac{(n^2+1)}{2}$, $n \epsilon 2Z^+ + 1$ and $\frac{n}{2}$ is the integer value upon division of n by 2. Hence forth, $\alpha = \left\{\frac{(t+1)}{2}\right\} - \frac{n}{2}$, $\beta = \left\{\frac{(t+1)}{2}\right\}$, $\pi = \left\{\frac{(t+1)}{2}\right\} + \frac{n}{2}$. In [3], R was classified as a ring and as a field under some conditions even though later it was confirmed that R under the operations defined in [1] can never be an integral domain hence can't be a field in [4].

The main contributions of this work were given in sections 2 and 3. Definitions, results and discussions were presented as according to ideas provided in [5, 6-9, 10].

2. Rhotrix Polynomials

2.1. Definition

A polynomial in the indeterminate x, over R is an expression of the form

$$f(x) = A_0 + A_1x + \cdots + A_{m-1}x^{m-1} + A_mx^m$$
$$= \sum_{i=0}^{n} A_ix^i$$

where $A_i \in R$, m a positive integer and x is not found in R. f will be termed as rhotrix polynomial.

The set of all polynomials of this form will be denoted by $R[x]$.

2.2. Definition

m is the degree of f, if $A_m \neq 0$ (the zero of R).

2.3. Definition

A_m is called the leading coefficient of f.

2.4. Definition

If

$$A_m = \left\langle \begin{matrix} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ \cdots & \cdots & & \cdots & & \cdots & \cdots \\ 0_\alpha\ \cdots & \cdots & & 1_\beta & & \cdots & \cdots\ 0_\pi \\ \cdots & \cdots & & \cdots & & \cdots & \cdots \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & 0_t & & & \end{matrix} \right\rangle$$

then f monic.

2.5. Definition

Define

$$f = \left\langle \begin{matrix} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ \cdots & \cdots & & \cdots & & \cdots & \cdots \\ 0_\alpha\ \cdots & \cdots & & 0_\beta & & \cdots & \cdots\ 0_\pi \\ \cdots & \cdots & & \cdots & & \cdots & \cdots \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & 0_t & & & \end{matrix} \right\rangle$$

as the zero polynomial.

2.6. Definition

Two polynomials

$$f(x)$$

And

$$g(x) = B_0 + B_1x + \cdots + B_{m-1}x^{m-1} + B_mx^m = \sum_{i=0}^{n} B_ix^i$$

over

$$R$$

are said to be equal only if $A_i = B_i$ for all i.

Now by the usual addition and multiplication of polynomials if

$$f = \sum_{i=0}^{n} A_ix^i$$

And

$$g = \sum_{i=0}^{m} B_ix^i$$

Then

$$f + g = \sum_{i=0}^{\max(n,m)} (A_i + B_i)x^i$$

And

$$fg = \sum_{k=0}^{n+m} C_kx^k$$

Where

$$C_k = \sum\nolimits_{i+j=k} A_i B_j$$

The operation of the coefficients is that defined on the commutative the ring R.

2.7. Definition

If $f(z) = A_0 + A_1 z + \cdots + A_{m-1} z^{m-1} + A_m z^m = 0$, then z is called the root of f.

With these, the following results will follow:

2.8. Theorem

$(R[x], +)$ is a commutative group.

Proof:

The additive identity is the zero polynomial and the additive inverse of

$$f = \sum_{i=0}^{n} A_i x^i \text{ is } -\sum_{i=0}^{n} A_i x^i = -f$$

2.9. Theorem

$(R[x], \bullet)$ is a semi group.

Proof:

This follows from the fact that R is also a semi group under the multiplication of rhotrices.

2.10. Theorem

$(R[x], +, \bullet)$ is a ring.

Proof:

This follows from 2.7, 2.8 and the distributivity of R. The multiplicative inverse of the polynomial f is g such that $fg = 1$.

2.11. Theorem

$(R[x], +, \bullet)$ is a commutative ring.

Proof:

This follows from the commutativity of R.

2.12. Theorem

$(R[x], +, \bullet)$ is not an integral domain.

Proof:

Let $f = \left\langle \begin{array}{c} a \\ b\ 0\ d \\ e \end{array} \right\rangle, g = \left\langle \begin{array}{c} f \\ g\ 0\ j \\ k \end{array} \right\rangle$ such that f may not be equal to $g \neq 0 \in R$. Also let $c = \left\langle \begin{array}{c} l \\ m\ 0\ n \\ p \end{array} \right\rangle \neq 0 \in R$.

But $cf = cg = 0 \in R$ and $f \neq g$.

3. Polynomial Rhotrices

Observe that by scalar multiplication of rhotrices,

$$\left\langle \begin{array}{c} a_1 \\ a_2 \quad a_3 \quad a_4 \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ a_\alpha \cdots\ \cdots \quad a_\beta \quad \cdots\ \cdots\ a_\pi \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ a_{t-3} \quad a_{t-2} \quad a_{t-1} \\ a_t \end{array} \right\rangle x^n = \left\langle \begin{array}{c} a_1 x^n \\ a_2 x^n \quad a_3 x^n \quad a_4 x^n \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ a_\alpha x^n \cdots\ \cdots \quad a_\beta x^n \quad \cdots\ \cdots\ a_\pi x^n \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ a_{t-3} x^n \quad a_{t-2} x^n \quad a_{t-1} x^n \\ a_t x^n \end{array} \right\rangle$$

This made the entries of the above rhotrix to be either terms of polynomials or polynomials themselves.

In general, the rhotrix above is better be represented as follows:

$$R[f] = \left\langle \begin{array}{c} f_1(x) \\ f_2(x) \quad f_3(x) \quad f_4(x) \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ f_\alpha(x) \cdots\ \cdots \quad f_\beta(x) \quad \cdots\ \cdots\ f_\pi(x) \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ f_{t-3}(x) \quad f_{t-2}(x) \quad f_{t-1}(x) \\ f_t(x) \end{array} \right\rangle$$

3.1. Definition

$R[f]$ will be called polynomial rhotrix.

3.2. Theorem

$(R[f], +)$ is a commutative group.

Proof:

The additive identity is the zero of R and the additive inverse of

$$R[f] = \left\langle \begin{array}{c} f_1(x) \\ f_2(x) \quad f_3(x) \quad f_4(x) \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ f_\alpha(x) \cdots\ \cdots \quad f_\beta(x) \quad \cdots\ \cdots\ f_\pi(x) \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ f_{t-3}(x) \quad f_{t-2}(x) \quad f_{t-1}(x) \\ f_t(x) \end{array} \right\rangle$$

is

$$-R[f] = \left\langle \begin{array}{c} -f_1(x) \\ -f_2(x) \quad -f_3(x) \quad -f_4(x) \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ -f_\alpha(x) \cdots\ \cdots \quad -f_\beta(x) \quad \cdots\ \cdots\ -f_\pi(x) \\ \cdots\ \cdots \quad \cdots \quad \cdots\ \cdots \\ -f_{t-3}(x) \quad -f_{t-2}(x) \quad -f_{t-1}(x) \\ -f_t(x) \end{array} \right\rangle$$

3.3. Theorem

$(R[f], \bullet)$ is a semi group.

Proof:

This also follows from the fact R is also a semi group under the multiplication of rhotrices.

3.4. Theorem

($R[f]$, +,•) is a ring.
Proof:
This also follows from the fact that R is distributive. Here, the multiplicative inverse of

$$R[f]=\left\langle\begin{array}{ccccccc} & & & f_1(x) & & & \\ & & f_2(x) & f_3(x) & f_4(x) & & \\ & \cdots & \dots & \dots & \dots & \cdots & \\ f_\alpha(x)\cdots & & \dots & f_\beta(x) & \dots & \cdots & f_\pi(x) \\ & \cdots & \dots & \dots & \dots & \cdots & \\ & & f_{t-3}(x) & f_{t-2}(x) & f_{t-1}(x) & & \\ & & & f_t(x) & & & \end{array}\right\rangle$$

is

$$R[f]^{-1}=-\frac{1}{(f_\beta(x))^2}\left\langle\begin{array}{ccccccc} & & & f_1(x) & & & \\ & & f_2(x) & f_3(x) & f_4(x) & & \\ & \cdots & \dots & \dots & \dots & \cdots & \\ f_\alpha(x)\cdots & & \dots & -f_\beta(x) & \dots & \cdots & f_\pi(x) \\ & \cdots & \dots & \dots & \dots & \cdots & \\ & & f_{t-3}(x) & f_{t-2}(x) & f_{t-1}(x) & & \\ & & & f_t(x) & & & \end{array}\right\rangle$$

Such that $-f_\beta(x)\neq 0.$

3.5. Theorem

($R[f]$, +,•) is a commutative ring.
Proof:
This follows from the commutativity of R.

3.6. Theorem

($R[x]$, +,•) is not an integral domain.
Proof:
To prove this theorem, see 2.12 above.

4. Conclusion

In this paper, properties of polynomials defined over rhotrices and those of rhotrices defined over polynomials were discussed. The hope is that the presentation of these new structures will pave more ways for further researches and applications of Mathematics in finding solutions to real life problems.

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