



Utility of Correlation Measures for Weighted Hesitant Fuzzy Sets in Medical Diagnosis Problems

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Abstract: Due to importance of correlation measure in data analysis, some researchers have shown great interest in the concept of correlation measure for extensions of fuzzy sets, in particular, for a new extension known as hesitant fuzzy set (HFS). Recently, an extension of HFS called the weighted hesitant fuzzy set (WHFS) has been developed by Zhang and Wu [1] to allow the membership of a given element is defined in terms of several possible values together with their importance weight. But, Zhang and Wu's definition of WHFS gives rise to a number of disadvantages which violate the well-known axioms for mathematical operations. To circumvent this issue, we refine the definition of WHFS and then we put forward some correlation measures for WHFSs. Finally, we give a practical example to illustrate the application of proposed correlation measures for WHFSs in medical diagnosis.

Keywords: Weighted Hesitant Fuzzy Set, Correlation Measure, Medical Diagnosis Problem

1. Introduction

Correlation measure is one of the most broadly applied indices in a variety of fields, such as pattern recognition and fuzzy multiple attribute decision making problems. Recently, some researchers have shown great interest in the concept of correlation measure for extensions of fuzzy sets as well as that for fuzzy sets, and applied it to the field of fuzzy decision making. Murthy and Pal [2], and Chiang and Lin [3] studied the correlation between two fuzzy sets. Gerstenkorn and Manko [4], and Mitchell [5] derived the correlation coefficient of intuitionistic fuzzy sets from different viewpoints.

A new generalization of fuzzy set called hesitant fuzzy set (HFS) [6] has received great attention in handling decision making problems where the decision makers have some hesitations among several possible memberships for an element to a set. Later, a number of other extensions of the HFS have been developed such as dual hesitant fuzzy sets (DHFSs) [7], generalized hesitant fuzzy sets (G-HFSs) [8], hesitant fuzzy linguistic term sets (HFLTSS) [9] and higher order hesitant fuzzy sets (HOHFSs) [10].

However, HFS [6] has its inherent drawbacks, because it expresses the membership degrees of an element to a given set only by possible values without emphasizing on the

importance of each possible value. In many practical decision making problems, the information provided by decision makers who are familiar with the area might often be described by the same preferences. In such situations, the value repeated several times is more important than that appeared only one time. Thus, the importance of possible membership degrees (i.e., their repetition rate) should be considered in improving the definition of HFS. To consider this fact, Zhang and Wu [1] introduced the concept of a weighted hesitant fuzzy set, denoted hereafter by (Z-WHFS). To clarify the necessity of introducing WHFS, consider a situation in which L experts are asked to evaluate the membership degree of the element x in the set w_H . l_1 experts provide $h^{\sigma(1)}(x)$, l_2 experts provide $h^{\sigma(2)}(x)$, ..., and l_m experts provide $h^{\sigma(m)}(x)$ such that $\sum_{k=1}^m l_k = 1$. Keeping in the mind that these L experts cannot persuade each other to change their opinions. In such a situation, the membership degree of the element x in the set w_H has m possible values $h^{\sigma(1)}(x)$, $h^{\sigma(2)}(x)$, ..., and $h^{\sigma(m)}(x)$ associated respectively with the weights of $w^{\sigma(1)}(x) = \frac{l_1}{1}$, $w^{\sigma(2)}(x) = \frac{l_2}{1}$, ..., and $w^{\sigma(m)}(x) = \frac{l_m}{1}$. In this regard, the membership degree of the element x in the set w_H should be represented by a weighted hesitant fuzzy element (WHFE) ${}^w h = \cup_{1 \leq j \leq m} \{ \langle h^{\sigma(j)}(x), w^{\sigma(j)}(x) \rangle \}$.

In this contribution, we will show that Zhang and Wu's

definition of union, intersection, addition and multiplication operations for Z-WHFS have not been correctly set up. This motivates us to modify and emend a fault of WHFS definition proposed by Zhang and Wu [1] so as not only the modified definition of WHFS is acceptable in accordance with the well-known axioms for mathematical operations, but also it allows that all information measures are to be defined reasonably as well as those defined for HFSs [10]–[17]. Due to the potential applications of correlation measures in HFS theory, Chen et al. [18], and Xu and Xia [19] have further studied them for HFSs. Farhadinia [7] proposed an approach for deriving the correlation measures of dual HFSs, and then extended the approach to the dual interval-valued HFS (IVHFS) theory. In this paper, we develop some correlation measures for WHFSs and then, the proposed correlation measures are applied to a medical diagnosis problem.

The present paper is organized as follows: Section 2 introduces some correlation measures for WHFSs. Section 3 is shown the application of correlation measures of WHFSs in medical diagnosis problems. This paper is concluded in Section 4.

2. Correlation Measures for Weighted Hesitant Fuzzy Sets

This section starts with the definition of hesitant fuzzy sets (HFSs) which were first introduced by Torra [6] as an extension of fuzzy sets.

Definition 2.1 [6] Let X be a reference set, a HFS A on X is defined in terms of a function $h_A(x)$ when applied to X returns a subset of $[0, 1]$, i.e.,

$$A = \{ \langle x, h_A(x) \rangle : x \in X \},$$

where $h_A(x)$ is a set of some different values in $[0, 1]$, representing the possible membership degrees of the element $x \in X$ to A .

For convenience, we call $h_A(x)$ a hesitant fuzzy element (HFE) [20] and denoted briefly by h_A .

Example 2.2 Let $x = \{x_1, x_2, x_3\}$ be a reference set, $h_A(x_1) = \{0.2, 0.4, 0.5\}$, $h_A(x_2) = \{0.3, 0.4\}$ and $h_A(x_3) =$

$\{0.3, 0.2, 0.5, 0.6\}$ be the HFEs of x_i ($i = 1, 2, 3$) to a set A , respectively. Then A can be considered as a HFS, i.e.,

$$A = \{ \langle x_1, \{0.2, 0.4, 0.5\} \rangle, \langle x_2, \{0.3, 0.4\} \rangle, \langle x_3, \{0.3, 0.2, 0.5, 0.6\} \rangle \}.$$

From a mathematical point of view, a HFS A can be seen as a fuzzy set if there is only one element in $h_A(x)$ which indicates that fuzzy sets are a special type of HFSs. That is, the theory for HFSs can also be applied to fuzzy sets.

Assumption 2.3 (See e.g. [11, 20]) Notice that the number of values in different HFEs may be different. Suppose that $l(h)$ stands for the number of values in the HFE h . Hereafter, the following assumptions are made: (A1) All the elements in each HFE h are arranged in increasing/decreasing order, and then $h^{\sigma(j)}$ is referred to as the j th largest/smallest value in the HFE h . (A2) If, for two HFEs h_1, h_2 , $l(h_1) \neq l(h_2)$, then $l = \max\{l(h_1), l(h_2)\}$. To have a correct comparison, the two HFEs h_1 and h_2 should have the same length l . If there are fewer elements in h_1 than in h_2 , an extension of h_1 should be considered optimistically/pessimistically by repeating its maximum/minimum element until it has the same length with h_2 .

Hereafter, we assume that all HFEs have the same length N , and let $h = \cup_{1 \leq j \leq N} \{h^{\sigma(j)}\}$ throughout the paper.

As can be seen from Definition 2.1, HFS expresses the membership degrees of an element to a given set only by several real numbers between 0 and 1 of equal importance, while in many real-world situations assigning exact values without importance weight to the membership degrees does not describe properly the imprecise or uncertain decision information. Thus, it seems to be difficult for the decision makers to rely on the present form of HFSs for expressing uncertainty of an element. To overcome the difficulty associated with the present form of HFSs, Zhang and Wu [1] have attempted to introduce the concept of weighted hesitant fuzzy set (Z-WHFS) in which the membership degrees of an element to a given set can be expressed by several possible values together with their importance weight.

Definition 2.4 [1] Let X be the universe of discourse. A Zhang and Wu's representation of weighted hesitant fuzzy set (Z-WHFS) on X is defined as

$${}^wA = \{ \langle x, w_{h_A}(x) \rangle : x \in X \} = \left\{ \langle x, \gamma_A \in \cup_{h_A(x)} \{(\gamma_A, w_{x\gamma_A})\} : x \in X \right\}, \quad (1)$$

Where $w_{h_A}(x)$ is a set of some different values in $[0, 1]$, denoting all possible membership degrees of the element $x \in X$ to the set wA , $w_{x\gamma_A} \in [0, 1]$ is the weight of γ_A such that $\sum_{\gamma_A \in w_{h_A}(x)} w_{x\gamma_A} = 1$ for any $x \in X$.

Zhang and Wu [1] called $w_{h_A}(x) = \cup_{\gamma_A \in w_{h_A}(x)} \{(\gamma_A, w_{x\gamma_A})\}$ a weighted hesitant fuzzy element (Z-WHFE). A Z-WHFE is conveniently denoted by $w_{h_A} = \cup_{\gamma_A \in w_{h_A}} \{(\gamma_A, w_{\gamma_A})\}$.

Zhang and Wu [1] defined for three Z-WHFEs $w_h = \cup_{\gamma \in w_h} \{(\gamma, w_\gamma)\}$, $w_{h_1} = \cup_{\gamma_1 \in w_{h_1}} \{(\gamma_1, w_{\gamma_1})\}$ and $w_{h_2} = \cup_{\gamma_2 \in w_{h_2}} \{(\gamma_2, w_{\gamma_2})\}$ some operations as follows:

$$w_{h^c} = \cup_{\gamma \in w_h} \{(1 - \gamma, w_\gamma)\}; \quad (2)$$

$$w_{h_1} \cup w_{h_2} = \cup_{\gamma_1 \in w_{h_1}, \gamma_2 \in w_{h_2}} \{(\max\{\gamma_1, \gamma_2\}, w_{\gamma_1} \cdot w_{\gamma_2})\}; \quad (3)$$

$$w_{h1} \cap w_{h2} = \bigcup_{\gamma_1 \in w_{h1}, \gamma_2 \in w_{h2}} \{(\min\{\gamma_1, \gamma_2\}, w_{\gamma_1} \cdot w_{\gamma_2})\}. \quad (4)$$

$$w_{h^\lambda} = \bigcup_{\gamma \in w_h} \{(\gamma^\lambda, w_\gamma)\}; \quad (5)$$

$$\lambda w_h = \bigcup_{\gamma_1 \in w_{h1}, \gamma_2 \in w_{h2}} \{(1 - (1 - \gamma)^\lambda, w_\gamma)\}; \quad (6)$$

$$w_{h1} \oplus w_{h2} = \bigcup_{\gamma_1 \in w_{h1}, \gamma_2 \in w_{h2}} \{(\gamma_1 + \gamma_2 - \gamma_1 \gamma_2, w_{\gamma_1} \cdot w_{\gamma_2})\}; \quad (7)$$

$$w_{h1} \oplus w_{h2} = \bigcup_{\gamma_1 \in w_{h1}, \gamma_2 \in w_{h2}} \{(\gamma_1 \gamma_2, w_{\gamma_1} \cdot w_{\gamma_2})\}. \quad (8)$$

By taking the above mathematical operations into consideration, one can easily find that Zhang and Wu [1] were careless about their definition of operations because such definitions inherit some fundamental disadvantages: Disadvantage 1. Zhang and Wu's union and intersection operations given by (3) and (4) are not idempotent, that is, for any Z-WHFE $w_h = \bigcup_{\gamma \in w_h} \{(\gamma, w_\gamma)\} = \{(\gamma_1, w_{\gamma_1}), \dots, (\gamma_L, w_{\gamma_L})\}$

$$w_h \cup w_h = \bigcup_{1 \leq j \leq l} \{(\gamma_j, f_j(w_{\gamma_1}, \dots, w_{\gamma_L}))\} \neq \bigcup_{1 \leq j \leq l} \{(\gamma_j, w_{\gamma_j})\} = w_h; \quad (9)$$

$$w_h \cap w_h = \bigcup_{1 \leq j \leq l} \{(\gamma_j, g_j(w_{\gamma_1}, \dots, w_{\gamma_L}))\} \neq \bigcup_{1 \leq j \leq l} \{(\gamma_j, w_{\gamma_j})\} = w_h; \quad (10)$$

where f_j and g_j are real functions of $w_{\gamma_1}, \dots, w_{\gamma_L}$ such that $f_j(w_{\gamma_1}, \dots, w_{\gamma_L}) \neq w_{\gamma_j}$ and $g_j(w_{\gamma_1}, \dots, w_{\gamma_L}) \neq w_{\gamma_j}$ for $1 \leq j \leq l$.

To more explanation, assume that a company wants to classify some different cars. It asks 10 experts to provide their evaluation information of a car with respect to the safety criterion. 6 experts express their evaluation information by the value "70 percent", and others by the value "80 percent". Keeping in mind that these 10 experts cannot persuade each other to change their opinions. In such a situation, their evaluation information can be described by a Z-WHFE as $w_h = \{\langle 0.7, 0.6 \rangle, \langle 0.8, 0.4 \rangle\}$. If we apply Zhang and Wu's union and intersection definitions given by (3) and (4) to w_h , it results in

$$w_h \cup w_h = \{\langle 0.7, 0.84 \rangle, \langle 0.8, 0.16 \rangle\}.$$

From $w_h \cup w_h$, one finds that near 4 experts are confident with "70 percent" about the safety of a car, and near 6 experts are confident with "80 percent". But, as observed from definition of WHFE $w_h = \{\langle 0.7, 0.6 \rangle, \langle 0.8, 0.4 \rangle\}$, the number of experts who are confident with "70 percent" and "80 percent" are respectively 6 and 10. Such a comparison of confidence level can be made for $w_h \cup w_h$ where near 8 experts are confident with "70 percent" about the safety of a car, and near 2 experts are confident with "80 percent", meanwhile, these numbers of experts have been already mentioned as 6 and 10 in the WHFE w_h .

Disadvantage 2. By applying Zhang and Wu's addition and multiplication definitions given by (7) and (8) to any Z-WHFE w_h does not give a reasonable result, that is,

$$w_h \cup w_h = \{\langle 0.7, 0.36 \rangle, \langle 0.8, 0.64 \rangle\};$$

$$w_h \oplus w_h = \bigcup_{1 \leq j \leq l} \{(2\gamma_j, \gamma_j^2, f_j(w_{\gamma_1}, \dots, w_{\gamma_L}))\} \neq \bigcup_{1 \leq j \leq l} \{(2\gamma_j, \gamma_j^2, w_{\gamma_j})\} = 2w_h$$

$$w_h \oplus w_h = \bigcup_{1 \leq j \leq l} \{(\gamma_j^2, g_j(w_{\gamma_1}, \dots, w_{\gamma_L}))\} \neq \bigcup_{1 \leq j \leq l} \{(\gamma_j^2, w_{\gamma_j})\} = 2w_{h^2},$$

Where f_j and g_j are real functions of $w_{\gamma_1}, \dots, w_{\gamma_L}$ such that $f_j(w_{\gamma_1}, \dots, w_{\gamma_L}) \neq w_{\gamma_j}$ and $g_j(w_{\gamma_1}, \dots, w_{\gamma_L}) \neq w_{\gamma_j}$ for $1 \leq j \leq l$.

Once again consider the Z-WHFE $w_h = \{\langle 0.7, 0.6 \rangle, \langle 0.8, 0.4 \rangle\}$. Then,

$$w_h \oplus w_h = \{\langle 0.91, 0.36 \rangle, \langle 0.94, 0.48 \rangle, \langle 0.96, 0.16 \rangle\} \neq 2w_h = \{\langle 0.91, 0.6 \rangle, \langle 0.96, 0.4 \rangle\};$$

$$w_h \oplus w_h = \{\langle 0.49, 0.36 \rangle, \langle 0.56, 0.48 \rangle, \langle 0.64, 0.16 \rangle\} \neq w_{h^2} = \{\langle 0.49, 0.6 \rangle, \langle 0.64, 0.4 \rangle\}.$$

Here, in order to avoid the disadvantages arising from Zhang and Wu's definition of WHFS and mathematical operations on WHFSs, we redefine a weighted hesitant fuzzy set as follows.

Definition 2.5 Let X be the universe of discourse. A weighted hesitant fuzzy set (WHFS) on X is defined as

$$w_A = \{\langle x, w_{hA}(x) \rangle : x \in X\} = \left\{ \langle x, \bigcup_{1 \leq j \leq L_x} \{h^{\sigma(j)}(x), w_A^{\sigma(j)}(x)\} \rangle : x \in X \right\}, \quad (11)$$

Where $w_{h_A}(x)$, referred to as the weighted hesitant fuzzy element (WHFE), is a set of some different values in $[0, 1]$, denoting all possible membership degrees of the element $x \in X$ to the set w_A , $w_A^{\sigma(j)}(x) \in [0, 1]$ is the weight of $h_A^{\sigma(j)}(x)$ such that $\sum_{1 \leq j \leq L_x} w_A^{\sigma(j)}(x) = 1$ for any $x \in X$.

It is interesting to note that if we take $w_A^{\sigma(1)}(x) = \dots = w_A^{\sigma(L_x)}(x) = \frac{1}{L_x}$ for any $x \in X$, then the WHFS w_A is reduced to a typical HFS.

Hereafter, for the convenience of representation, we denote the WHFE $w_{h_A}(x)$ by $w_{h_A} = \bigcup_{1 \leq j \leq L_x} \{h_A^{\sigma(j)}, w_A^{\sigma(j)}\}$.

Assumption 2.6 Notice that the number of values in different WHFEs may be different. Suppose that $l(w_{h_1}(x))$ stands for the number of values in $w_{h_1}(x)$. Hereafter, the following assumptions are made: (A1) All the first component of elements in each $w_{h_1}(x)$ are arranged in

increasing order, and then $h_1^{\sigma(j)}(x)$ is referred to as the j th largest value in $w_{h_1}(x)$. (A2) If, for some $x \in X$, $l(w_{h_1}(x)) \neq l(w_{h_2}(x))$, then $L_x = \max \{l(w_{h_1}(x)), l(w_{h_2}(x))\}$.

To have a correct comparison, the two WHFEs $w_{h_1}(x)$ and $w_{h_2}(x)$ should have the same length L_x . If there are fewer elements in $w_{h_1}(x)$ than in $w_{h_2}(x)$, an extension of $w_{h_1}(x)$ should be considered optimistically by repeating the maximum first component of elements associated with zero weight until it has the same length with $w_{h_2}(x)$. This kind of extension is quite reasonable since the added element with zero weight is meant to be an element that does not really exist.

Throughout this paper, we assume that all WHFEs have the same length N , and let $w_h = \bigcup_{1 \leq j \leq N} \{h^{\sigma(j)}, w^{\sigma(j)}\}$.

Definition 2.7 Let

$$w_h = \bigcup_{1 \leq j \leq N} \{h^{\sigma(j)}, w^{\sigma(j)}\}, w_{h_1} = \bigcup_{1 \leq j \leq N} \{h_1^{\sigma(j)}, w_1^{\sigma(j)}\} \text{ and } w_{h_2} = \bigcup_{1 \leq j \leq N} \{h_2^{\sigma(j)}, w_2^{\sigma(j)}\}$$

be three WHFEs. Then, some operations on the WHFEs w_h, w_{h_1} and w_{h_2} are defined as the following:

$$w_{h^c} = \bigcup_{1 \leq j \leq N} \{(1 - h^{\sigma(j)}), w^{\sigma(j)}\}; \quad (12)$$

$$w_{h_1} \cup w_{h_2} = \bigcup_{1 \leq j \leq N} \{(\max\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}, \overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})})\}; \quad (13)$$

$$w_{h_1} \cap w_{h_2} = \bigcup_{1 \leq j \leq N} \{(\min\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}, \overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})})\}; \quad (14)$$

$$w_{h^\lambda} = \bigcup_{1 \leq j \leq N} \{(h^{\sigma(j)})^\lambda, w^{\sigma(j)}\}; \quad (15)$$

$$\lambda w_h = \bigcup_{1 \leq j \leq N} \{(1 - (1 - h^{\sigma(j)})^\lambda), w^{\sigma(j)}\}; \quad (16)$$

$$w_{h_1} \oplus w_{h_2} = \bigcup_{1 \leq j \leq N} \{(h_1^{\sigma(j)} + h_2^{\sigma(j)} - h_1^{\sigma(j)} h_2^{\sigma(j)}, \overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})})\}; \quad (17)$$

$$w_{h_1} \oplus w_{h_2} = \bigcup_{1 \leq j \leq N} \{(h_1^{\sigma(j)} h_2^{\sigma(j)}, \overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})})\},$$

In the above formulas, $\overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})}$ for $1 \leq j \leq N$, referred to as the normalized weights, are determined in two steps: (i) We first calculate the weight of j th component of the binary operation $w_{h_1} \odot w_{h_2}$ by simply adding the weights $w_1^{\sigma(j)}$ and $w_2^{\sigma(j)}$ for $1 \leq j \leq N$; (ii) After the whole components of $w_{h_1} \odot w_{h_2}$ are to be obtained, their weights are considered again and then normalized. In this regard, the normalized weights of the above binary operations are defined as follows:

$$\overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})} = \frac{(w_1^{\sigma(j)} + w_2^{\sigma(j)})}{\sum_{K=1}^N (w_1^{\sigma(K)} + w_2^{\sigma(K)})}, 1 \leq j \leq N \quad (19)$$

In the case that the associative binary operation \odot is iterated on the finite set of WHFEs $w_{h_1}, w_{h_2}, \dots, w_{h_m}$, that is, we are interested to obtain $w_{h_1} \odot w_{h_2} \odot \dots \odot w_{h_m} = (\dots ((w_{h_1} \odot w_{h_2}) \odot w_{h_3}) \dots \odot w_{h_m})$, the normalized weights are constructed as

$$\overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)} + \dots + w_m^{\sigma(j)})} = \frac{(\dots ((w_1^{\sigma(j)} + w_2^{\sigma(j)}) + w_3^{\sigma(j)}) + \dots + w_m^{\sigma(j)})}{\sum_{K=1}^N (w_1^{\sigma(K)} + w_2^{\sigma(K)} + \dots + w_m^{\sigma(K)})}, 1 \leq j \leq N$$

Example 2.8 Suppose that

$w_{h_1} = \{\langle 0.2, 0.1 \rangle, \langle 0.4, 0.3 \rangle, \langle 0.5, 0.6 \rangle\}$ and $w_{h_2} = \{\langle 0.3, 0.5 \rangle, \langle 0.7, 0.5 \rangle\}$ are two given WHFEs. Bearing Assumption 2.6 in mind, w_{h_1} should be first extended as $w_{h_2} = \{\langle 0.3, 0.5 \rangle, \langle 0.7, 0.5 \rangle, \langle 0.7, 0.0 \rangle\}$. Then, one gets

$$\begin{aligned}
w_{h_1^c} &= \{\langle 0.5, 0.6 \rangle, \langle 0.6, 0.3 \rangle, \langle 0.8, 0.1 \rangle\}; \\
w_{h_1} \cup w_{h_2} &= \left\{ \langle \max\{0.2, 0.3\}, \frac{(0.1 + 0.5)}{2} \rangle, \langle 0.7, 0.4 \rangle, \langle 0.7, 0.3 \rangle \right\}; \\
w_{h_1} \cap w_{h_2} &= \left\{ \langle \min\{0.2, 0.3\}, \frac{(0.1 + 0.5)}{2} \rangle, \langle 0.4, 0.4 \rangle, \langle 0.5, 0.3 \rangle \right\}; \\
(w_{h_1} \cup w_{h_2}) \cup w_{h_1} &= \left\{ \langle \max\{\max\{0.2, 0.3\}, 0.2\}, \frac{((0.1 + 0.5) + 0.1)}{3} \rangle, \langle 0.7, \frac{1.1}{3} \rangle, \langle 0.7, \frac{1.2}{3} \rangle \right\}; \\
(w_{h_1} \cap w_{h_2}) \cap w_{h_1} &= \left\{ \langle \min\{\min\{0.2, 0.3\}, 0.2\}, \frac{((0.1 + 0.5) + 0.1)}{3} \rangle, \langle 0.4, \frac{1.1}{3} \rangle, \langle 0.5, \frac{1.2}{3} \rangle \right\}; \\
w_{h_1^\lambda} &= \{\langle 0.2^\lambda, 0.1 \rangle, \langle 0.4^\lambda, 0.3 \rangle, \langle 0.5^\lambda, 0.6 \rangle\}; \\
\lambda w_{h_1} &= \{\langle 1 - 0.8^\lambda, 0.1 \rangle, \langle 1 - 0.6^\lambda, 0.3 \rangle, \langle 1 - 0.5^\lambda, 0.6 \rangle\}; \\
w_{h_1} \oplus w_{h_2} &= \{\langle 0.44, 0.3 \rangle, \langle 0.82, 0.4 \rangle, \langle 0.85, 0.3 \rangle\}; \\
w_{h_1} \otimes w_{h_2} &= \{\langle 0.06, 0.3 \rangle, \langle 0.28, 0.4 \rangle, \langle 0.35, 0.3 \rangle\}.
\end{aligned}$$

Theorem 2.9 Let

$w_h = \bigcup_{1 \leq j \leq N} \{ \langle h^{\sigma(j)}, w^{\sigma(j)} \rangle \}$, $w_{h_1} = \bigcup_{1 \leq j \leq N} \{ \langle h_1^{\sigma(j)}, w_1^{\sigma(j)} \rangle \}$ and $w_{h_2} = \bigcup_{1 \leq j \leq N} \{ \langle h_2^{\sigma(j)}, w_2^{\sigma(j)} \rangle \}$ be three WHFEs. Then, all operations $w_{h_1^c}$, $w_{h_1} \cup w_{h_2}$, $w_{h_1} \cap w_{h_2}$, $w_{h_1^\lambda}$, λw_{h_1} , $w_{h_1} \oplus w_{h_2}$, $w_{h_1} \otimes w_{h_2}$ given in Definition 2.7 are also WHFEs.

Proof. We only prove that $w_{h_1} \cup w_{h_2}$ is also WHFE. Known by the definition of $w_{h_1} \cup w_{h_2}$ from (13), i.e., $w_{h_1} \cup w_{h_2} = \bigcup_{1 \leq j \leq N} \{ \langle \max\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}, (w_1^{\sigma(j)}, w_2^{\sigma(j)}) \rangle \}$, we need to show that

$$\sum_{j=1}^N \overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})} = 1.$$

By definition of the normalized weight $\overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})}$, one can get

$$\sum_{j=1}^N \overline{(w_1^{\sigma(j)} + w_2^{\sigma(j)})} = \sum_{j=1}^N \frac{(w_1^{\sigma(j)}, w_2^{\sigma(j)})}{\sum_{k=1}^N (w_1^{\sigma(k)}, w_2^{\sigma(k)})} = \frac{\sum_{j=1}^N (w_1^{\sigma(j)}, w_2^{\sigma(j)})}{\sum_{k=1}^N (w_1^{\sigma(k)}, w_2^{\sigma(k)})} = 1.$$

This completes the proof.

Theorem 2.10 Let

$w_h = \bigcup_{1 \leq j \leq N} \{ \langle h^{\sigma(j)}, w^{\sigma(j)} \rangle \}$, $w_{h_1} = \bigcup_{1 \leq j \leq N} \{ \langle h_1^{\sigma(j)}, w_1^{\sigma(j)} \rangle \}$ and $w_{h_2} = \bigcup_{1 \leq j \leq N} \{ \langle h_2^{\sigma(j)}, w_2^{\sigma(j)} \rangle \}$ be three WHFEs. Then,

$$(w_{h^c})^\lambda = (\lambda w_h)^c; (w_{h^\lambda})^c = \lambda(w_{h^c}); \quad (20)$$

$$(w_{h_1} \cup w_{h_2})^c = w_{h_1^c} \cap w_{h_2^c}; (w_{h_1} \cap w_{h_2})^c = w_{h_1^c} \cup w_{h_2^c}; \quad (21)$$

$$(w_{h_1} \otimes w_{h_2})^c = w_{h_1^\lambda} \otimes w_{h_2^\lambda}; \lambda(w_{h_1} \oplus w_{h_2}) = \lambda w_{h_1} \oplus \lambda w_{h_2}; \quad (22)$$

$$(w_{h_1} \oplus w_{h_2})^c = w_{h_1^c} \otimes w_{h_2^c}; (w_{h_1} \otimes w_{h_2})^c = w_{h_1^c} \oplus w_{h_2^c}; \quad (23)$$

$$w_{h_1} \oplus w_{h_2} = w_{h_2} \oplus w_{h_1}; w_{h_1} \otimes w_{h_2} = w_{h_2} \otimes w_{h_1}; \quad (24)$$

$$w_{h_1} \cup w_{h_2} = w_{h_2} \cup w_{h_1}; w_{h_1} \cap w_{h_2} = w_{h_2} \cap w_{h_1}; \quad (25)$$

$$w_h \cup (w_{h_1} \cup w_{h_2}) = (w_h \cup w_{h_1}) \cup w_{h_2}; w_h \cap (w_{h_1} \cap w_{h_2}) = (w_h \cap w_{h_1}) \cap w_{h_2} \quad (26)$$

$$w_h \cup w_h = w_h; w_h \cap w_h = w_h; \quad (27) \quad w_h \oplus w_h = 2w_h; w_h \otimes w_h = w_{h^2} \quad (28)$$

It is noteworthy to say that properties given by (27)-(28) show the superiority of WHFS definition proposed here over that of Z-WHFS suggested by Zhang and Wu [1].

2.1. Correlation Measures for WHFSs

By correlation analysis in both classical set theory and fuzzy set theory, we can examine the joint relationship of two sets with the aid of a measure of interdependency of the two sets. In the following, the axiomatic definitions of correlation measure for WHFSs are described as:

Definition 2.11 A real-valued function ρ is called a correlation measure for WHFSs, if for WHFSs w_A, w_B on X , ρ satisfies the following properties:

$$(\rho 1) \quad 0 \leq \rho(w_A, w_B) \leq 1;$$

$$(\rho 2) \quad \rho(w_A, w_B) = \rho(w_B, w_A);$$

$$(\rho 3) \quad \rho(w_A, w_B) = 1 \text{ if } w_A = w_B.$$

By assuming that

$$w_A = \{ \langle x, w_{h_A}(x) \rangle : x \in X \} = \left\{ \langle x, 1 \leq j \leq N \{ \langle h_A^{\sigma(j)}(x), w_A^{\sigma(j)}(x) \rangle \} \rangle : x \in X \right\}, \quad (29)$$

$$w_B = \{ \langle x, w_{h_B}(x) \rangle : x \in X \} = \left\{ \langle x, 1 \leq j \leq N \{ \langle h_B^{\sigma(j)}(x), w_B^{\sigma(j)}(x) \rangle \} \rangle : x \in X \right\}, \quad (30)$$

we define the following correlation measure formulas for any two WHFSs w_A and w_B as

$$\rho_{WHFS1}(w_A, w_B) = \frac{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right)}{\sqrt{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i))^2 \right)} \sqrt{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i))^2 \right)}}, \quad (31)$$

$$\rho_{WHFS2}(w_A, w_B) = \frac{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right)}{\max \left\{ \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i))^2 \right), \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i))^2 \right) \right\}}, \quad (32)$$

$$\rho_{WHFS3}(w_A, w_B) = \frac{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N |w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - \overline{h_A}(x_i)| |w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) - \overline{h_B}(x_i)| \right)}{\sqrt{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - \overline{h_A}(x_i))^2 \right)} \sqrt{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) - \overline{h_B}(x_i))^2 \right)}}, \quad (33)$$

$$\rho_{WHFS4}(w_A, w_B) = \frac{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N |w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - \overline{h_A}(x_i)| |w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) - \overline{h_B}(x_i)| \right)}{\max \left\{ \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - \overline{h_A}(x_i))^2 \right), \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) - \overline{h_B}(x_i))^2 \right) \right\}}, \quad (34)$$

$$\rho_{WHFS5}(w_A, w_B) = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N \frac{\Delta_{\min}(x_i) + \Delta_{\max}(x_i)}{\Delta_j(x_i) + \Delta_{\max}(x_i)} \right),$$

$$\overline{h_A}(x_i) = \frac{1}{N} \sum_{j=1}^N w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i),$$

$$\overline{h_B}(x_i) = \frac{1}{N} \sum_{j=1}^N w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i),$$

$$\Delta_j(x_i) = |w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i)|,$$

$$\Delta_{\min}(x_i) = \min_j \left\{ |w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i)| \right\},$$

$$\Delta_{\max}(x_i) = \max_j \left\{ |w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i)| \right\},$$

It is noted that if WHFSs w_A and w_B are reduced to HFSs A and B , then the above correlation measures are reduced to Xu and Xia's measures [19] and Chen et al.'s measures [18].

In order to equip the WHFS theory with further correlation measures, we presented two other correlation measures for WHFSs by extending Jaccard's [21] and Dice's [22] correlation measures defined on the vector space as follows:

$$\rho_{WHFS5}(w_A, w_B) = \left\{ \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right) \right\} /$$

$$\left\{ \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N \left(w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) \right)^2 + \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N \left(w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right)^2 - \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right) \right\}, \quad (36)$$

$$\rho_{WHFS5}(w_A, w_B) = \frac{2 \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right)}{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i))^2 \right) + \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i))^2 \right)}. \quad (37)$$

Theorem 2.12 The measure functions ρ_{WHFSi} (w_A, w_B) ($i = 1, \dots, 7$) given respectively by (31)-(37) are correlation measures for WHFSs w_A and w_B .

Proof. It is necessary to show that each measure function satisfies the requirements (ρ_1)-(ρ_3) listed in Definition 2.11. The proof of (ρ_1) and (ρ_3) for ρ_{WHFS1} given by (31) is straightforward and we prove only (ρ_1). The left-hand side inequality is obvious where $0 \leq \rho_{WHFS1}(w_A, w_B)$. To show the assertion of the right-hand side inequality, recall the Cauchy-Schwarz inequality that

$$\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2},$$

where $a_i, b_i \in \mathbb{R}$ for $i = 1, \dots, n$.

In view of the above inequality relation and letting

$$a_i := \sum_{j=1}^n \frac{w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i)}{\sqrt{N}}, \quad (i = 1, \dots, n), \quad (38)$$

$$b_i := \sum_{j=1}^n \frac{w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i)}{\sqrt{N}}, \quad (i = 1, \dots, n), \quad (39)$$

one can easily verify that

$$\begin{aligned} & \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right) \\ & \leq \sqrt{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i))^2 \right)} \sqrt{\sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N (w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i))^2 \right)}, \end{aligned}$$

which implies that $\rho_{WHFS1}(w_A, w_B) \leq 1$. Thus, $\rho_{WHFS1}(w_A, w_B)$ is a correlation measure for WHFSs w_A and w_B .

The correlation proof of $\rho_{WHFS2}(w_A, w_B)$ is much like that of $\rho_{WHFS1}(w_A, w_B)$. The only difference is that instead of the Cauchy-Schwarz inequality one should consider the following inequality

$$\sum_{i=1}^n a_i b_i \leq \max \left\{ \sum_{i=1}^n a_i^2, \sum_{i=1}^n b_i^2 \right\},$$

Where $a_i, b_i \in \mathbb{R}$ for $i = 1, \dots, n$ are those set as (38) and (39).

The correlation proof of $\rho_{WHFS3}(w_A, w_B)$ and $\rho_{WHFS4}(w_A, w_B)$ are respectively much like that of $\rho_{WHFS1}(w_A, w_B)$ and $\rho_{WHFS2}(w_A, w_B)$

The correlation proof of $\rho_{WHFS5}(w_A, w_B)$ is straightforward.

$$\rho_{WHFS1}(w_A, w_B) = \rho_{WHFS3}(w_A, w_B) = 1, \quad (40)$$

$$\rho_{WHFS2}(w_A, w_B) = \rho_{WHFS4}(w_A, w_B) = \begin{cases} k, & \text{if } 0 \leq k \leq 1; \\ \frac{1}{k}, & k \geq 1, \end{cases} \quad (41)$$

The correlation proof of $\rho_{WHFS6}(w_A, w_B)$ and $\rho_{WHFS7}(w_A, w_B)$ comes from the fact that

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{2} \left(\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 \right),$$

Where $a_i, b_i \in \mathbb{R}$ for $i = 1, \dots, n$ are those set as (38) and (39).

Proposition 2.1 Let $\rho_{WHFSi}(w_A, w_B)$ ($i = 1, \dots, 7$) be the correlation measures for WHFSs w_A and w_B , given respectively by (31)-(37). If $w_A^{\sigma(j)}(x_i) = w_B^{\sigma(j)}(x_i) = \frac{1}{N}$ and the values of $h_B^{\sigma(j)}(x_i)$ in B are k times the values of $h_A^{\sigma(j)}(x_i)$ in A such that $0 \leq h_B^{\sigma(j)}(x_i) = k h_A^{\sigma(j)}(x_i) \leq 1$ for $1 \leq j \leq N$ and $1 \leq i \leq n$, then

$$\rho\text{WHFS5}(w_A, w_B) = \begin{cases} \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N \frac{\min_j \{h_A^{\sigma(j)}(x_i)\} + \max_j \{h_A^{\sigma(j)}(x_i)\}}{h_A^{\sigma(j)}(x_i) + \max_j \{h_A^{\sigma(j)}(x_i)\}} \right), & \text{if } k \neq 1; \\ \frac{0}{0}, & \text{if } k = 1. \end{cases} \quad (42)$$

$$\rho\text{WHFS6}(w_A, w_B) = \frac{k}{1+k^2-k}, \quad (43)$$

$$\rho\text{WHFS7}(w_A, w_B) = \frac{2k}{1+k^2}, \quad (44)$$

Proof. We only prove (42), and the others can be easily verified.

Suppose that $w_A^{\sigma(j)}(x_i) = w_B^{\sigma(j)}(x_i) = \frac{1}{n}$ and the values of $h_B^{\sigma(j)}(x_i)$ in B are k times the values of $h_A^{\sigma(j)}(x_i)$ in A such that $0 \leq h_B^{\sigma(j)}(x_i) = k h_A^{\sigma(j)}(x_i) \leq 1$ for $1 \leq j \leq N$ and $1 \leq i \leq n$. By taking the definition of correlation measures $\rho\text{WHFS5}(w_A, w_B)$ into account, we find for $k \neq 1$ that

$$\begin{aligned} \Delta_j(x_i) &= \left| w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right| = \frac{1}{N} |1 - k| h_A^{\sigma(j)}(x_i), \\ \Delta_{\min(x_i)} &= \min_j \left\{ \left| w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right| \right\} = \frac{1}{N} |1 - k| \min_j \{ h_A^{\sigma(j)}(x_i) \}, \\ \Delta_{\max(x_i)} &= \max_j \left\{ \left| w_A^{\sigma(j)}(x_i) h_A^{\sigma(j)}(x_i) - w_B^{\sigma(j)}(x_i) h_B^{\sigma(j)}(x_i) \right| \right\} = \frac{1}{N} |1 - k| \max_j \{ h_A^{\sigma(j)}(x_i) \}. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho\text{WHFS5}(w_A, w_B) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N \frac{\Delta_{\min(x_i)} + \Delta_{\max(x_i)}}{\Delta_j(x_i) + \Delta_{\max(x_i)}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N \frac{\frac{1}{N} |1 - k| \min_j \{ h_A^{\sigma(j)}(x_i) \} + \frac{1}{N} |1 - k| \max_j \{ h_A^{\sigma(j)}(x_i) \}}{\frac{1}{N} |1 - k| h_A^{\sigma(j)}(x_i) + \frac{1}{N} |1 - k| \max_j \{ h_A^{\sigma(j)}(x_i) \}} \right) \end{aligned}$$

Trivially, for $k = 1$ we find that $\rho\text{WHFS5}(w_A, w_B) = \frac{0}{0}$. On the other hand, for $k \neq 1$

$$\rho\text{WHFS5}(w_A, w_B) = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{N} \sum_{j=1}^N \frac{\min_j \{ h_A^{\sigma(j)}(x_i) \} + \max_j \{ h_A^{\sigma(j)}(x_i) \}}{h_A^{\sigma(j)}(x_i) + \max_j \{ h_A^{\sigma(j)}(x_i) \}} \right)$$

3. Medical Diagnosis with Weighted Hesitant Fuzzy Information

In this portion, we implement the following medical diagnosis problem to illustrate the efficiency of the correlation measures for WHFSs. Although, the problem of medical diagnosis can be similarly re-modeled with WIVHFS data, we do not consider here such a problem because of having the same solution procedure.

Example 3.1 Consider the set of diagnoses $D = \{\text{Viral fever, Malaria, Typhoid, Stomach problem, Chest problem}\}$. The aim here is to assign a patient with the given values of the symptoms, $S = \{\text{Temperature, Headache, Cough, Stomach pain, Chest pain}\}$ to one of the aforementioned diagnoses. Three medical experts E_l ($l = 1, 2, 3$) are invited to provide their possible assessment of diagnoses with respect to symptoms. For each diagnosis with respect to each symptom, all of the medical experts provide anonymously their evaluated values. As an example, for the diagnosis "Viral fever" with respect to the symptom "Temperature", the

evaluation value provided by medical experts E_1 and E_3 is 0.5; and E_2 's evaluation value is 0.7. In this regard, and noting that the weights of three medical experts are unknown, the evaluation of "Viral fever" with respect to "Temperature" can be represented by a WHFE as

$$w_h(\text{Viral fever, Temperature}) = w_{h11} = \left\{ \langle 0.5, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle \right\}.$$

Note that the characteristics of the diagnosis "Viral fever" with respect to the symptoms "Headache", "Cough", "Stomach pain", "Chest pain", denoted respectively by WHFEs w_{h1j} ($j = 2, 3, 4, 5$), form the WHFS w_{h1} which is indicated in the first row of Table 1. The results evaluated for other diagnoses with respect to symptoms are contained in a weighted hesitant fuzzy decision matrix, shown in Table 1.

Furthermore, suppose that the set of patients is $P = \{\text{Al, Bob, Joe, Ted}\}$, and the symptoms characteristic for the considered patients are evaluated and given by the three medical experts in the form of a weighted hesitant fuzzy matrix demonstrated in Table 2. Here, the main task is to seek a diagnosis for each patient.

Table 1. Symptoms characteristic for the considered diagnoses.

	Temperature	Headache	Cough	Stomach pain	Chest pain
Al	$\{\langle 0.4, \frac{3}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$	$\{\langle 0.6, \frac{1}{3} \rangle, \langle 0.7, \frac{2}{3} \rangle\}$	$\{\langle 0.2, \frac{2}{3} \rangle, \langle 0.4, \frac{1}{3} \rangle\}$	$\{\langle 0.1, \frac{1}{3} \rangle, \langle 0.2, \frac{2}{3} \rangle\}$
Bob	$\{\langle 0.6, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$	$\{\langle 0.5, \frac{1}{3} \rangle, \langle 0.8, \frac{2}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.6, \frac{1}{3} \rangle\}$	$\{\langle 0.3, \frac{3}{3} \rangle\}$	$\{\langle 0.4, \frac{2}{3} \rangle, \langle 0.5, \frac{1}{3} \rangle\}$
Joe	$\{\langle 0.2, \frac{2}{3} \rangle, \langle 0.3, \frac{1}{3} \rangle\}$	$\{\langle 0.5, \frac{3}{3} \rangle\}$	$\{\langle 0.2, \frac{2}{3} \rangle, \langle 0.4, \frac{1}{3} \rangle\}$	$\{\langle 0.6, \frac{1}{3} \rangle, \langle 0.7, \frac{2}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$
Ted	$\{\langle 0.4, \frac{3}{3} \rangle\}$	$\{\langle 0.4, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$	$\{\langle 0.3, \frac{1}{3} \rangle, \langle 0.4, \frac{2}{3} \rangle\}$	$\{\langle 0.7, \frac{2}{3} \rangle, \langle 0.8, \frac{1}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.6, \frac{1}{3} \rangle\}$

Table 2. Symptoms characteristic for the considered patients.

	Temperature	Headache	Cough	Stomach pain	Chest pain
Al	$\{\langle 0.4, \frac{3}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$	$\{\langle 0.6, \frac{1}{3} \rangle, \langle 0.7, \frac{2}{3} \rangle\}$	$\{\langle 0.2, \frac{2}{3} \rangle, \langle 0.4, \frac{1}{3} \rangle\}$	$\{\langle 0.1, \frac{1}{3} \rangle, \langle 0.2, \frac{2}{3} \rangle\}$
Bob	$\{\langle 0.6, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$	$\{\langle 0.5, \frac{1}{3} \rangle, \langle 0.8, \frac{2}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.6, \frac{1}{3} \rangle\}$	$\{\langle 0.3, \frac{3}{3} \rangle\}$	$\{\langle 0.4, \frac{2}{3} \rangle, \langle 0.5, \frac{1}{3} \rangle\}$
Joe	$\{\langle 0.2, \frac{2}{3} \rangle, \langle 0.3, \frac{1}{3} \rangle\}$	$\{\langle 0.5, \frac{3}{3} \rangle\}$	$\{\langle 0.2, \frac{2}{3} \rangle, \langle 0.4, \frac{1}{3} \rangle\}$	$\{\langle 0.6, \frac{1}{3} \rangle, \langle 0.7, \frac{2}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$
Ted	$\{\langle 0.4, \frac{3}{3} \rangle\}$	$\{\langle 0.4, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}$	$\{\langle 0.3, \frac{1}{3} \rangle, \langle 0.4, \frac{2}{3} \rangle\}$	$\{\langle 0.7, \frac{2}{3} \rangle, \langle 0.8, \frac{1}{3} \rangle\}$	$\{\langle 0.5, \frac{2}{3} \rangle, \langle 0.6, \frac{1}{3} \rangle\}$

As can be seen from Tables 1 and 2, all WHFEs are not in the same size. To circumvent this issue, we implement Assumption 2.6. In this regard, The WHFEs with fewer elements are extended optimistically by repeating the maximum first component of elements associated with zero weight until it has the same length with others. For

example, the WHFE w_h (Viral fever, Temperature): =

$$\text{Al} = \{\text{Temperature}, \{\langle 0.4, \frac{3}{3} \rangle, \langle 0.4, 0 \rangle\}, \langle \text{Headache}, \{\langle 0.5, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\} \rangle, \langle \text{Cough}, \{\langle 0.6, \frac{1}{3} \rangle, \langle 0.7, \frac{2}{3} \rangle\} \rangle, \\ \langle \text{Stomachpain}, \{\langle 0.2, \frac{2}{3} \rangle, \langle 0.4, \frac{1}{3} \rangle\} \rangle, \langle \text{Chestpain}, \{\langle 0.1, \frac{1}{3} \rangle, \langle 0.2, \frac{2}{3} \rangle\} \rangle\};$$

$$\text{Viral fever} = \{\text{Temperature}, \{\langle 0.5, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\}, \langle \text{Headache}, \{\langle 0.3, \frac{3}{3} \rangle, \langle 0.3, 0 \rangle\} \rangle, \langle \text{Cough}, \{\langle 0.3, \frac{1}{3} \rangle, \langle 0.6, \frac{2}{3} \rangle\} \rangle, \\ \langle \text{Stomachpain}, \{\langle 0.3, \frac{2}{3} \rangle, \langle 0.4, \frac{1}{3} \rangle\} \rangle, \langle \text{Chestpain}, \{\langle 0.6, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle\} \rangle\};$$

are taken into account to determine the correlated degree of Al and viral fever.

In order to proceed, we apply the correlation measures ρ_{WHFS1} , ρ_{WHFS4} , and ρ_{WHFS7} to determine the degree of dependence between diagnoses and patients. The results obtained by the use of these correlation measures are shown in Tables 3-5, respectively.

Table 3. Values of ρ_{WHFS1} for each patient to the considered set of possible diagnoses.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest Problem
Al	0.7984	0.5376	0.5998	0.6467	0.7906
Bob	0.7261	0.7581	0.8549	0.7881	0.7781
Joe	0.803	0.6568	0.5697	0.6706	0.7356
Ted	0.8513	0.5778	0.8723	0.6798	0.8433

Table 4. Values of ρ_{WHFS4} for each patient to the considered set of possible diagnoses.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest Problem
Al	0.7591	0.4221	0.5378	0.4653	0.7334
Bob	0.6551	0.6939	0.8179	0.6610	0.6192
Joe	0.7858	0.5542	0.5490	0.5185	0.6350
Ted	0.7961	0.5102	0.8650	0.5500	0.6956

Table 5. Values of $_WHFS7$ for each patient to the considered set of possible diagnoses.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest Problem
Al	0.7974	0.5223	0.5963	0.6132	0.7884
Bob	0.7223	0.7551	0.8541	0.7761	0.7582
Joe	0.8028	0.6474	0.5694	0.6490	0.7277
Ted	0.8494	0.5733	0.8722	0.6649	0.8279

By comparing the results listed in Table 3, we observe that AL and Joe suffer from “Viral fever”, Bob and Ted from “Typhoid”.

Tables 3-5 present the same result for all correlation measures.

4. Conclusion

Weighted hesitant fuzzy set (WHFS) is a new extension of hesitant fuzzy set (HFS) where the membership degree of an element to a given set is expressed by several possible values together with their importance weight. In this contribution, we modified and emended a fault of WHFS definition proposed by Zhang and Wu [1] so as the modified definition of WHFS is acceptable in accordance with the well-known axioms for mathematical operations. Because of importance of correlation measure in data analysis, we then developed a series of correlation measures for WHFSs and employed them to solve the weighted hesitant fuzzy multi-attribute group decision making (MAGDM) vproblems. We believe many future works can be developed by the use of the findings of this contribution which support the decision makers in making decisions effectively in WHFS-structured MAGDM problems.

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References

- [1] Zhang Zh., Wu Ch., Weighted hesitant fuzzy sets and their application to multi-criteria decision making, *British Journal of Mathematics and Computer Science* 4(2014) 1091-1123.
- [2] Murthy C.A., Pal S.K., Majumder, D.D., Correlation between two fuzzy membership functions, *Fuzzy Sets and Systems* 17(1985) 23-38.
- [3] Chiang D.A., Lin N.P., Partial correlation of fuzzy sets, *Fuzzy Sets and Systems* 110(2000) 209-215.
- [4] Gerstenkorn T., Manko J., Correlation of intuitionistic fuzzy sets, *Fuzzy Sets and Systems* 44(1991) 39-43.
- [5] Mitchell H.B., A correlation coefficient for intuitionistic fuzzy sets, *International Journal of Intelligent Systems* 19(2004) 483-490.
- [6] Torra V., Hesitant fuzzy sets, *International Journal of Intelligent Systems* 25(2010) 529-539.
- [7] Farhadinia B., Correlation for dual hesitant fuzzy sets and dual interval-valued hesitant fuzzy sets, *International Journal of Intelligent Systems* 29(2014) 184-205.
- [8] Qian G., Wang H., Feng X., Generalized hesitant fuzzy sets and their application in decision support system, *Knowledge Based Systems* 37(2013) 357-365.
- [9] Rodriguez R. M., Martinez L., Herrera F., Hesitant fuzzy linguistic term sets for decision making, *IEEE Transactions on Systems* 20(2012) 109-119.
- [10] Farhadinia B., Distance and similarity measures for higher order hesitant fuzzy sets, *Knowledge-Based Systems* 55(2014) 43-48.
- [11] Farhadinia B., A novel method of ranking hesitant fuzzy values for multiple attribute decision-making problems, *International Journal of Intelligent Systems* 28(2013) 752-767.
- [12] Farhadinia B., Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets, *Information Sciences* 240(2013) 129-144.
- [13] B. Farhadinia, A theoretical development on the entropy of interval-valued fuzzy sets based on the intuitionistic distance and its relationship with similarity measure, *J. Knowledge-Based Systems* 39 (2013) 79-84.
- [14] B. Farhadinia, An efficient similarity measure for intuitionistic fuzzy sets, *J. Soft Computing* 18 (2014) 85-94.
- [15] B. Farhadinia, Fuzzy multicriteria decision-making method based on a family of novel measured functions under vague environment, *J. Intelligent and Fuzzy Systems* 27 (2014) 2797-2808.
- [16] B. Farhadinia, A.I. Ban, Developing new similarity measures of generalized intuitionistic fuzzy numbers and generalized interval-valued fuzzy numbers from similarity measures of generalized fuzzy numbers, *J. Mathematical and Computer Modelling* 57 (2013) 812-825.
- [17] Chen N., Xu Z., Xia M., Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis, *Applied Mathematical Modelling* 37(2013) 2197-2211.
- [18] Xu Z., Xia M., On distance and correlation measures of hesitant fuzzy information, *International Journal of Intelligent Systems* 26(2011) 410-425.
- [19] Xia M., Xu Z., Hesitant fuzzy information aggregation in decision making, *International Journal of Approximate Reasoning* 52(2011) 395-407.
- [20] Jaccard P., Distribution de la flore alpine dans le Bassin des Drouces et dans quelques regions voisines, *Bulletin de la Socit Vaudoise des Sciences Naturelles* 37(1901) 241-272.
- [21] Dice L.R., Measures of the amount of ecologic association between species, *Ecology* 26(1945) 297-302.