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Euler-Maruyama Scheme for SDEs with Dini Continuous Coefficients

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Abstract: In the study of Euler-Maruyama scheme for Stochastic Differential Equations, researchers focus on the convergence rate under different conditions, using analytical methods and Stochastic Partial Differential Equation. One of them is to study the Lipschitz continuous, mainly from drift coefficient and diffusion coefficient. The other is the study of non-Lipschitz continuous, since most of the real life is not Lipschitz continuous. Therefore, most researchers are looking at non-Lipschitz continuous. In my study, without loss of generality, we are also a continuous study of non-Lipschitz and a faster convergence rate. In this paper, we show the convergence rate of Euler-Maruyama scheme for non-degenerate SDEs where the drift term b and the diffusion term σ are the uniformly bounded, b and σ satisfy correlated conditions of Dini-continuous, by the aid of the regularity of the solution to the associated Kolmogorov equation of SPDE and common methods in stochastic analysis, including Itô's formula, Jensen's inequality, Hölder inequality BDG's inequality, Gronwall's inequality. We obtain the same conclusions by weakening the conditions of previous research using the properties of Dini continuous and Taylor expansion. At the same time, we also reached the same conclusion under local boundedness and local Dini-continuous. Moreover, my research results have laid the groundwork for the follow-up research.

Keywords: Non-Degenerate, Stochastic Differential Equation, Euler-Maruyama Scheme, Dini Continuous, Kolmogorov Equation

1. Introduction

Let fix $T > 0$. Considering the following stochastic differential equations in $\mathbb{R}^{\mathrm{d}}$:

$$X_t = \mathrm{X}_0 + \int_0^t b(\mathrm{s}, \mathrm{X}_s)\mathrm{ds} + \int_0^t \sigma(\mathrm{s}, \mathrm{X}_s)\mathrm{dW}_s,\ \mathrm{X}_0 \in \mathbb{R}^{\mathrm{d}} \quad (1)$$

where b: $[0, T] \times \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$ and σ: $[0, T] \times \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}} \otimes \mathbb{R}^{\mathrm{d}}$ are two Borel measurable functions, $\{W_t, t \in [0, T]\}$ is an d-dimensional standard Brown motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathrm{P}; (\mathcal{F}_{\mathrm{t}})_{\mathrm{t}\geq 0})$, and the initial value X_0 is $\mathcal{F}_0$-measurable $\mathbb{R}^{\mathrm{d}}$-valued random variable.

The Euler-Maruyama scheme of (1) is

$$Y_t = \mathrm{Y}_0 + \int_0^t b(\eta_{\delta(s)}, \mathrm{Y}_{\eta\delta(s)})\mathrm{ds} + \int_0^t \sigma(\eta_{\delta(s)}, \mathrm{Y}_{\eta\delta(s)})\mathrm{dW}_s$$

$$X_0 = \mathrm{Y}_0 \quad (2)$$

where $\delta := \frac{T}{N} \in (0,1), \eta_\delta(s) := \left[\frac{s}{\delta}\right]\delta, s \in \left[\left[\frac{s}{\delta}\right]\delta, \left(\left[\frac{s}{\delta}\right] + 1\right)\delta\right),$ for the sufficiently large integer $N \in \mathbb{N}$. And the discrete scheme of (2) is analytically tractable on computer application of engineering, physical, finance, biology, etc.

If the coefficient of SDEs is Lipschitz continuous, there are many previous research results. If σ is an identity matrix and the coefficient b is Lipschitz continuous in space and $\frac{1}{2}$-Hölder continuous in time then for any $p > 0$, there exists $C_p > 0$ such that the Euler-Maruyama scheme is the strong rate of $\frac{1}{2}$(see for example [5]). Yan [14] proved the rate of convergence in L_1-norm sense for a range of SDEs, where the drift coefficient is Lipschitz and the diffusion coefficient is Hölder continuous, by means of the Meyer Tanaka formula. Gyöngy and Rásonyi [4] extended Yan [13] to the convergence rate in L_p-norm, by the Yamada-Watanabe approximation. Our finding partly improves upon results in [7, 9, 10], as well as the as well as the well-known ones in [2, 3]. Based on previous research [6, 7], Leobacher et al. proved the Euler-Maruyama approximation converges at the rate of order 1/4−ε in the L_2-norm for small ε > 0 in [8], when the drift coefficient is piecewise Lipschitz continuous and bounded and the diffusion coefficient is Lipschitz continuous, bounded and non-paraller.

However, the coefficients b and σ are not Lipschitz continuous in practical applications. Zhang [15] has proved Euler-Maruyama approximation for SDEs to converge uniformly to the solution in L_p-space with respect to the time and starting points under non-Lipschitz coefficients. If the drift coefficient is the Dini-Hölder continuous, Gyöngy and Rásonyi [4] implied the order of strong rate of convergence for one dimensional SDEs. Hoang-Long and Taguchi [11] studied the strong rates of the Euler–Maruyama approximation for one-dimensional stochastic differential equations whose drift coefficient may be neither continuous nor one-sided Lipschitz and whose diffusion coefficient is Hölder continuous. And the case of d-dimension is introduced in [10], using a Yamada-Watanabe approximation technique. In [14], the diffusion coefficient σ is an identity matrix, the drift coefficient b is bounded β-Hölder continuous with $\beta \in (0, 1)$ in space and η-Hölder continuous in time $\eta \in [1/2, 1]$, then for any $p > 1$, the strong rate of can be obtained. The strong rate of convergence of the Euler–Maruyama approximation for stochastic differential equations was obtained by Hoang-Long and Taguchi [12] with possibly discontinuous drift and Hölder continuous diffusion coefficient. Bao, Huang and Yuan [1] discussed the strong convergence rate of Euler-Maruyama for non-degenerate SDEs with rough coefficients, where the drift term is Dini-continuous and unbounded, by the regularity of non-degenerate Kolmogrov equation.

In this paper, we first study the convergence rate of the Euler-Maruyama scheme of (2), where the drift term b and the diffusion term σ are the uniformly bounded, b and σ satisfy correlated conditions of Dini-continuous (see Assumption 2.1), which is weaken the the conditions, simply the proof and obtains the same results in [1]. In addition, we also prove the convergence rate for the non-degenerate SDEs with unbounded coefficients, which method is mainly based on the regularity of the solution to Kolmogorov equation associated to the SDEs (1).

This paper is structured as follows. In the next section, we introduce some notations and the main results. All proofs are deferred to Section 3.

2. Main Results

2.1. Notations

In this section, we recall the foundational definition and notations involved in the paper.

Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ algebra on $\mathbb{R}^d$. Set $\nabla := D = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)^*$, $D^2 = \left(\frac{\partial^2}{\partial x_i \partial y_j}\right)_{1\le i,j\le d}$ and $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i}$, where $*$ is the transpose of a vector or matrix. Take $\|\cdot\|$ and $\|\cdot\|_{HS}$ stand for the usual operator norm and the Hilbert -Schmidt norm, respectively.

Meanwhile, we introduce some space of function:

$\|f\|_{T,\infty} = \sup_{t\in[0,T],x\in\mathbb{R}^d} \|f(t, x)\|$, where an operator-valued map f is on $[0, T]\times\mathbb{R}^d$.

$\mathbb{M}^d_{non}$ denotes the collection of all non-singular d × d-matrices.

$C_b^\beta(\mathbb{R}^d,\mathbb{R}^k)$, $\beta \in (0, 1)$ denotes the set of all function from $\mathbb{R}^d$ to $\mathbb{R}^k$ which are bounded and β-Hölder continuous functions. Hence if $f \in C_b^\beta(\mathbb{R}^d,\mathbb{R}^k)$, then

$$\sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

For a < b, we write $C_b^\beta([a, b])$ for $C([a, b]; C_b^\beta(\mathbb{R}^d,\mathbb{R}^d))$ and define the norm $\|\cdot\|_{C_b^\beta([a,b])}$ on $C_b^\beta([a, b])$ by $\|f\|_{C_b^\beta([a,b])} := \sup_{x,y\in\mathbb{R}^d, t\in[a,b]} |f(t,x)| + \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta}$.

Throughout the paper, we denote the constant as C, the shorthand notation $a \lesssim b$ stands for $a \le Cb$. And C represents a positive constant although its value may change from one appearance to the next.

2.2. Main Results

In this paper, we study the convergence rate of Euler -Maruyama scheme, under the following non-Lipschitz condition. In this section, we state the related assumptions and main theorems of this paper.

Let $\mathcal{D}_0$ be the family of Dini function, i. e.,

$$\mathcal{D}_0 := \{\varphi | \varphi: \mathbb{R}_+ \to \mathbb{R}_+ \text{ is increasing and } \int_0^1 \frac{\varphi(s)}{s} ds < \infty\}.$$

A function f: $\mathbb{R}^d \to \mathbb{R}^d$ is called Dini-continuity if there exists $\varphi \in \mathcal{D}_0$ such that for any $x, y \in \mathbb{R}^d$,

$$|f(x) - f(y)| \le \varphi(|x-y|).$$

It is well known that every Dini-continuous function is continuous and every Lipschitz continuous function is Dini-continuous. Moreover, if f is Hölder continuous, then f is Dini-continuous, but not vice versa. And set $\mathcal{D} := \{\varphi \in \mathcal{D}_0 | \varphi^2 \text{is concave}\}$, for instance, a function f is Hölder-Dini continuous of order $\alpha \in (0, 1)$.

The non-Lipschitz assumptions is following:

Assumption 2.1.

(a) For every $t \in [0, T]$ *and* $x \in \mathbb{R}^d$, $\sigma(t, x) \in \mathbb{M}^n_{non}$, *and*

$$\|b\|_{T,\infty} + \|\sigma\|_{T,\infty} < +\infty,$$

where $\|\sigma\|_{T,\infty} := \sup_{0\le t\le T} \|\sigma(t,x)\|_{HS}$.

(b) For any $t \in [0, T]$, $\beta \in (0, 1)$ *and* $x, y \in \mathbb{R}^d$, *there exists* $\varphi \in \mathcal{D}$ *such that (regularity of b and σ w. r. t. spatial variables)*

$$|b(t, x) - b(t, y)| \le |x-y|^\beta \varphi(|x-y|),$$
$$\|\sigma(t,x) - \sigma(t,y)\|_{HS} \le \varphi(|x-y|).$$

(c) For any $s, t \in [0, T]$ *and* $x \in \mathbb{R}^d$, *there exists* $\varphi \in \mathcal{D}$ *such that (regularity of b and σ w. r. t. time variables)*

$$|b(s, x) - b(t, x)| + \|\sigma(t,x) - \sigma(t,y)\|_{HS} \le \varphi(|s-t|).$$

The following main results are stated including the convergence rate of SDEs.

Theorem 2.1. *Suppose that Assumption 2.1 holds. For* $p \ge 1$ *and* $\beta \in (0, 1)$, *there exists the constant* $C > 1$ *depending* $T, p, d, \|\sigma\|_{T,\infty}, M, \beta$, *then*

$$\mathbb{E}\left(\sup_{0\le t\le T}|X_t-Y_t|^p\right)\le C\delta^{p/2}.$$

Remark 2.1. *In* [1], *the diffusion term is the rough coefficient which refers to second order continuous differentiable. However, we check the results with the regularity of the solution to Kolmogorov equation associated to the SDE* (1), *by the properties of Dini continuous and Taylor expansion instead of second order continuous differentiable, which is a simple and clever way to simplify the proof and weaken the conditions in* [1].

Remark 2.2. *In the article* [1], *the convergence rate is verified from the perspective of norm, while we not only get similar results but also do better conclusions using the properties of dimension. At the same time, we can also do the degenerate result with Hamiltonian system. Because the method is similar, we will not elaborate here.*

It seems to be a little bit stringent that the coefficients are uniformly bounded, and the drift b is global Dini-continuous, in Theorem 2.1. Therefore, the above conditions can be weakened by the means of uniform boundedness instead of local boundedness and global Dini-continous instead of local Dini-continuous, respectively.

Theorem 2.2. *Assume that for any s, t* $\in$ [0, *T*], β $\in$(0, 1) *and for every* $x\in\mathbb{R}^{\mathrm{d}}$ *and* $\sigma(t, x)\in\mathbb{M}^{\mathrm{d}}_{\mathrm{non}}$, *there exists the constant* C_T, *b and σ are Borel measurable functions such*

$$|b(t, x)| + ||\sigma(t,x)||_{\mathrm{HS}}\le C_T\ (1 + x),\ x\in\mathbb{R}^{\mathrm{d}},$$

And if b and σ satisfy

$$|b(t, x) - b(t, y)|\le |x-y|^{\beta}\varphi_k(|x-y|),\ |x|\vee|y|\le k,$$

$$||\sigma(t,x)-\sigma(t,y)||_{\mathrm{HS}}\le\varphi_k(|x-y|),\ |x|\vee|y|\le k,$$

$$|b(s, x) - b(t, x)| + ||\sigma(s,x)-\sigma(t,x)||_{\mathrm{HS}}\le\varphi_k(|s-t|),$$

$$|x|\le k,$$

where $\varphi_k\in\mathcal{D}$. Then for all $p\ge 1$ and $E|X_0|^p<\infty$, it holds that

$$\lim_{\delta\to 0}\mathbb{E}\left(\sup_{0\le t\le T}|X_t-Y_t|^p\right)=0.$$

Remark 2.3. *We verify this conclusion by a method similar to Theorem 1.2 in* [1]. *In* [1], *the diffusion term is the uniformly bounded and second order continuous differentiable. However, in this paper, σ is the uniformly bounded, which is weaken the conditions of Theorem 1.2 in and obtains the same conclusions.*

3. Proofs of Main Results

We also need the following lemma for the proof.

Lemma 3.1. *Let the coefficients b, σ is the uniformly bounded. For* $p\ge 1$ *and* $t\in$[0, *T*], *there exists a positive constant* $C>0$ *depending on T, M, p, d, it holds that*

$$\mathbb{E}\left[\varphi\left(Y_t-Y_{\eta_\delta(t)}\right)\right]\le C\delta^{p/2}.$$

Proof. Owing to $\varphi\in\mathcal{D}$, based on Taylor expansion and properties of Dini function, we have $\varphi(0)=0$, $\varphi'>0$ and $\varphi''<0$, so that

$$\varphi(t)=\varphi(0)+\varphi'(0)t+\frac{\varphi''(\theta t)}{2!}t^2\le\varphi'(0)t:=Mt,$$

$$\theta\in(0,t),t\in\mathbb{R}_+$$

Thus, for any $t\in\mathbb{R}$,

$$\varphi(|t|)\le M|t|.$$

For $p\ge 1$, noticing that

$$\varphi\left(|Y_t-Y_{\eta_\delta(t)}|\right)^p\le M^p|Y_t-Y_{\eta_\delta(t)}|^p.$$

Using Assumption 2.1 *(a)-(b)*, we deduce that

$$|Y_t-Y_{\eta_\delta(t)}|^p\preccurlyeq$$

$$\left|\int_{\eta_\delta(t)}^t b(\eta_\delta(s),Y_{\eta_\delta(s)})\mathrm{d}s\right|^p+\left|\int_{\eta_\delta(t)}^t\sigma(\eta_\delta(s),Y_{\eta_\delta(s)})\mathrm{dW}_s\right|^p$$

$$\preccurlyeq\delta^p+|W_t-W_{\eta_\delta(t)}|^p.$$

Hence there exists a positive constant $C=C(T,M,p,d)$ such that, for $p\ge 1$,

$$\mathbb{E}\left[\varphi\left(|Y_t-Y_{\eta_\delta(t)}|\right)^p\right]\le M\mathbb{E}\left[|Y_t-Y_{\eta_\delta(t)}|^p\right]\le C\delta^{p/2}.$$

The following lemma is taken from Theorem 2.8 in [3], which provides the regularity of solution to Kolmogorov equation associated to the SDEs (1).

Lemma 3.2. *Let* $T>0$, *for any* $\varepsilon\in(0,1)$, *there exists* $m\in\mathrm{N}$ *such that* $0=t_0<t_1<\cdots<t_m=T$, *for any* $\phi\in C([t_{j-1},\ t_j];\ C_b^{\beta}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$, $\beta\in(0,1)$, *j* = 1, · · · *m*, *there is least one solution u to Backward Kolmogorov equation*

$$\frac{\partial u}{\partial t}+\nabla u\cdot b+\frac{1}{2}\Delta u\cdot\sigma^2=-\phi,\ \text{on}\ \left[t_{j-1},t_j\right]\times\mathbb{R}^{\mathrm{d}},\ u(t_j,x)=0$$

of class

$$u\in C([t_{j-1},\ t_j];\ C_b^{2,\beta'}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})\cap C^1([t_{j-1},\ t_j];\ C_b^{\beta'}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}}).$$

For some constant K depending on j and for all $\beta'\in(0,\beta)$, *we have*

$$||D^2u||_{C_b^{\beta'}([t_{j-1},t_j])}\le K||\phi||_{C_b^{\beta}([t_{j-1},t_j])}$$

and for some constant C_0, *it holds that*

$$||\nabla u||_{C_b^{\beta}([t_{j-1},t_j])} \leq C_0(t_j - t_{j-1})^{1/2} \, || \phi||_{C_b^{\beta}([t_{j-1},t_j])}$$

At same time, we can obtain

$$C_0(t_j - t_{j-1})^{1/2} \, || \phi||_{C_b^{\beta}([0,T])} \leq \varepsilon.$$

Now we can give

Proof of Theorem 2.1. Let $T > 0$, for any $\varepsilon \in (0,1)$, there is $m \in \mathrm{N}$, such that $0 = T_0 < T_1 < \cdots < T_m = T$. For $i =$ and, $j = 1,2,\cdots,m$ using Lemma 3.2, we can get

$$\frac{\partial u}{\partial t} + \nabla u \cdot b + \frac{1}{2}\Delta u \cdot \sigma^2 = -b, \text{ on } [T_{j-1}, T_j] \times \mathbb{R}^{\mathrm{d}}, u(T_j, x) = 0 \tag{3}$$

and u satisfies

$$||\nabla u||_{C_b^{\beta}([T_{j-1},T_j])} \leq C_0(T_j - T_{j-1})^{1/2} \, || b||_{C_b^{\beta}([T_{j-1},T_j])} \leq \varepsilon. \tag{4}$$

For $t \in [T_{j-1}, T_j]$, by Itô's formula and (3), we have

$$u(t, X_t) = u(T_{j-1}, X_{j-1}) + \int_{T_{j-1}}^{t} \frac{\partial u}{\partial t}(s, X_s)\,\mathrm{d}s + \int_{T_{j-1}}^{t} \nabla u(s, X_s)\mathrm{d}X_s + \frac{1}{2}\int_{T_{j-1}}^{t} \Delta u(s, X_s)\mathrm{d}\langle X_s, X_s\rangle$$
$$= u(T_{j-1}, X_{j-1}) - \int_{T_{j-1}}^{t} b(s, X_s)\mathrm{d}s + \int_{T_{j-1}}^{t} \langle \nabla u(s, X_s), \sigma(s, X_s)\rangle\,\mathrm{d}W_s.$$

Similarly, we have

$$u(t, Y_t) = u(T_{j-1}, Y_{j-1}) + \int_{T_{j-1}}^{t} \frac{\partial u}{\partial t}(s, Y_s)\,\mathrm{d}s + \int_{T_{j-1}}^{t} \nabla u(s, Y_s)\mathrm{d}Y_s + \frac{1}{2}\int_{T_{j-1}}^{t} \Delta u(s, Y_s)\mathrm{d}\langle Y_s, Y_s\rangle$$
$$= u(T_{j-1}, Y_{j-1}) - \int_{T_{j-1}}^{t} b(s, Y_s)\mathrm{d}s + \int_{T_{j-1}}^{t} \langle \nabla u(s, Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle \mathrm{d}W_s + \int_{T_{j-1}}^{t} \langle \nabla u(s, Y_s), b(\eta_\delta(s), Y_{\eta_\delta(s)}) - b(s, Y_s)\rangle\,\mathrm{d}s.$$

Hence, we can get

$$\int_{T_{j-1}}^{t} b(s, X_s)\mathrm{d}s = u(T_{j-1}, X_{j-1}) - u(t, X_t) + \int_{T_{j-1}}^{t} \langle \nabla u(s, X_s), \sigma(s, X_s)\rangle\,\mathrm{d}W_s \tag{5}$$

and

$$\int_{T_{j-1}}^{t} b(s, Y_s)\mathrm{d}s = u(T_{j-1}, Y_{j-1}) - u(t, Y_t) + \int_{T_{j-1}}^{t} \langle \nabla u(s, Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle \mathrm{d}W_s + \int_{T_{j-1}}^{t} \langle \nabla u(s, Y_s), b(\eta_\delta(s), Y_{\eta_\delta(s)}) - b(s, Y_s)\rangle\,\mathrm{d}s. \tag{6}$$

Combining with (5) and (6), we have

$$X_t - Y_t = X_{T_{j-1}} - Y_{T_{j-1}} + \int_{T_{j-1}}^{t} (b(s, X_s) - b(\eta_\delta(s), Y_{\eta_\delta(s)})\,\mathrm{d}s + \int_{T_{j-1}}^{t} (\sigma(s, X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}))\mathrm{d}W_s$$
$$= X_{T_{j-1}} - Y_{T_{j-1}} + \left(u\left(T_{j-1}, X_{T_{j-1}}\right) - u\left(T_{j-1}, Y_{T_{j-1}}\right)\right) - \left(u(t, X_t) - u(t, Y_t)\right)$$
$$+ \int_{T_{j-1}}^{t} \left[\langle \nabla u(s, X_s), \sigma(s, X_s)\rangle - \langle \nabla u(s, Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle\right]\mathrm{d}W_s + \int_{T_{j-1}}^{t} \left[\langle \nabla u(s, Y_s), b(s, Y_s) - b(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle\right]\mathrm{d}s$$
$$+ \int_{T_{j-1}}^{t} (b(s, Y_s) - b(\eta_\delta(s), Y_{\eta_\delta(s)})\,\mathrm{d}s + \int_{T_{j-1}}^{t} (\sigma(s, X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}))\mathrm{d}W_s.$$

By (4) and the mean-value theorem, we have:

$$|X_t - Y_t| \leq |X_{T_{j-1}} - Y_{T_{j-1}}| + |u(T_{j-1}, X_{T_{j-1}}) - u(T_{j-1}, Y_{T_{j-1}})| + |u(t, X_t) - u(t, Y_t)|$$
$$+ \left|\int_{T_{j-1}}^{t} \left[\langle \nabla u(s, X_s), \sigma(s, X_s)\rangle - \langle \nabla u(s, Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle\right]\mathrm{d}W_s\right| + ||\nabla u||_{C_b^{\beta}[T_{j-1},T_j]} \int_{T_{j-1}}^{t} \left|b(s, Y_s) - b(\eta_\delta(s), Y_{\eta_\delta(s)})\right|\mathrm{d}s$$
$$+ \int_{T_{j-1}}^{t} \left|b(s, Y_s) - b(\eta_\delta(s), Y_{\eta_\delta(s)}\right|\,\mathrm{d}s + \left|\int_{T_{j-1}}^{t} (\sigma(s, X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}))\mathrm{d}W_s\right|$$

$$\leq (1+\varepsilon)|X_{T_{j-1}} - Y_{T_{j-1}}| + \varepsilon|X_t - Y_t| + \left|\int_{T_{j-1}}^{t}\left[\langle \nabla u(s,X_s), \sigma(s,X_s)\rangle - \langle \nabla u(s,Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle\right]\mathrm{d}W_s\right|$$

$$+(1+\varepsilon)\left[\int_{T_{j-1}}^{t}\left|Y_s - Y_{\eta_\delta(s)}\right|^{\beta}\varphi\left(\left|Y_s - Y_{\eta_\delta(s)}\right|\right)\mathrm{d}s + \int_{T_{j-1}}^{t}\varphi(|s-\eta_\delta(s)|)\mathrm{d}s\right] + \left|\int_{T_{j-1}}^{t}(\sigma(s,X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}))\mathrm{d}W_s\right|$$

For all $p \geq 1$, utilizing Jensen's inequality, Hölder inequality and Lemma 3.1, we can obtain

$$|X_t - Y_t|^p \leq 6^{p-1}(1+\varepsilon)^p|X_{T_{j-1}} - Y_{T_{j-1}}|^p + 6^{p-1}\varepsilon^p|X_t - Y_t|^p$$

$$+6^{p-1}\left|\int_{T_{j-1}}^{t}\left[\langle \nabla u(s,X_s), \sigma(s,X_s)\rangle - \langle \nabla u(s,Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle\right]\mathrm{d}W_s\right|^p$$

$$+6^{p-1}(1+\varepsilon)^p(t-T_{j-1})^{p-1}M^p\int_{T_{j-1}}^{t}\left|Y_s - Y_{\eta_\delta(s)}\right|^{p(\beta+1)}\mathrm{d}s$$

$$+6^{p-1}(1+\varepsilon)^p(t-T_{j-1})^{p-1}M^p\int_{T_{j-1}}^{t}|s-\eta_\delta(s)|^p\mathrm{d}s$$

$$+6^{p-1}\left|\int_{T_{j-1}}^{t}(\sigma(s,X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}))\mathrm{d}W_s\right|^p.$$

Because ε is arbitrary, there exists $c(p,\varepsilon) := 6^{p-1}\varepsilon^p < 1$. Then we know

$$|X_t - Y_t|^p \leq \frac{6^{p-1}(1+\varepsilon)^p}{1-c(p,\varepsilon)}|X_{T_{j-1}} - Y_{T_{j-1}}|^p + \frac{6^{p-1}(1+\varepsilon)^p}{1-c(p,\varepsilon)}\left|\int_{T_{j-1}}^{t}\left[\langle \nabla u(s,X_s), \sigma(s,X_s)\rangle - \langle \nabla u(s,Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})\rangle\right]\mathrm{d}W_s\right|^p$$

$$+\frac{6^{p-1}(1+\varepsilon)^p(t-T_{j-1})^{p-1}M^p}{1-c(p,\varepsilon)}\int_{T_{j-1}}^{t}\left[\left|Y_s - Y_{\eta_\delta(s)}\right|^{p(\beta+1)} + |s-\eta_\delta(s)|^p\right]\mathrm{d}s + \frac{6^{p-1}}{1-c(p,\varepsilon)}\left|\int_{T_{j-1}}^{t}(\sigma(s,X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}))\mathrm{d}W_s\right|^p. \quad (7)$$

Taking the supremum, expectation on both sides of the above inequality, and using BDG's inequality, for $t \in (T_{j-1}, T_j]$, we have

$$\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq t}|X_u - Y_u|^p\right] \leq \frac{6^{p-1}(1+\varepsilon)^p}{1-c(p,\varepsilon)}\mathbb{E}\left[|X_{T_{j-1}} - Y_{T_{j-1}}|^p\right]$$

$$+\frac{6^{p-1}C(p,d)T^{p/2-1}}{1-c(p,\varepsilon)}\times\int_{T_{j-1}}^{t}\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\left\|\nabla u(u,X_u)\sigma(u,X_u) - \nabla u(u,Y_u)\sigma(\eta_\delta(u), Y_{\eta_\delta(u)})\right\|_{\mathrm{HS}}^p\right]\mathrm{d}s + \frac{6^{p-1}T^{p-1}M^p(1+\varepsilon)^p}{1-c(p,\varepsilon)}$$

$$\times\left[\int_{T_{j-1}}^{t}\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\left|Y_s - Y_{\eta_\delta(s)}\right|^{p(\beta+1)}\right]\mathrm{d}s + T\delta^p\right] + \frac{6^{p-1}T^{p/2-1}C(p,d)}{1-c(p,\varepsilon)}\times\int_{T_{j-1}}^{t}\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\left\|\sigma(u,X_u) - \sigma(\eta_\delta(u), Y_{\eta_\delta(u)})\right\|_{HS}^p\right]\mathrm{d}s$$

$$= \sum_{i=1}^{4}\mathbb{I}_i,$$

where $C(p,d)$ is the constant in BDG's inequality. With the help of lemma 3.1 and the Assumption 2.1 (*a*), in $\mathbb{I}_2$, we have

$$\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\left\|\nabla u(u,X_u)\sigma(u,X_u) - \nabla u(u,Y_u)\sigma(\eta_\delta(u), Y_{\eta_\delta(u)})\right\|_{\mathrm{HS}}^p\right]$$

$$= \mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\|\nabla u(u,X_u)\sigma(u,X_u) - \nabla u(u,Y_u)\sigma(u,X_u) + \nabla u(u,Y_u)\sigma(u,X_u) - \nabla u(u,Y_u)\sigma(u,Y_u)\right.$$

$$\left.+\nabla u(u,Y_u)\sigma(u,Y_u) - \nabla u(u,Y_u)\sigma(\eta_\delta(u),Y_u) + \nabla u(u,Y_u)\sigma(\eta_\delta(u),Y_u) - \nabla u(u,Y_u)\sigma(\eta_\delta(u), Y_{\eta_\delta(u)})\|_{HS}^p\right]$$

$$\leq 4^{p-1}2^p\varepsilon^p\|\sigma\|_{T,\infty}^p + 4^{p-1}\varepsilon^p\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\varphi(|X_u - Y_u|)^p\right] + 4^{p-1}\varepsilon^p\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\varphi(|u-\eta_\delta(u)|)^p\right]$$

$$+4^{p-1}\varepsilon^p\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}\varphi(|Y_u - Y_{\eta_\delta(u)}|)^p\right]$$

$$\leq 4^{p-1}2^p\varepsilon^p\|\sigma\|_{T,\infty}^p + 4^{p-1}\varepsilon^pM^p\mathbb{E}\left[\sup_{T_{j-1}\leq u\leq s}|X_u - Y_u|^p\right] + 4^{p-1}\varepsilon^pM^p\delta^p + 4^{p-1}\varepsilon^pC^p\delta^{p/2}.$$

In $\mathbb{I}_4$, from the properties of Dini-function, it may be chosen the constant C_0,C_1, such that

$$\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\left\|\sigma(u,X_u)-\sigma(\eta_\delta(u),Y_{\eta_\delta(u)})\right\|_{HS}^p\right]$$

$$= \mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\|\sigma(u,X_u)-\sigma(\eta_\delta(u),X_u)+\sigma(\eta_\delta(u),X_u)-\sigma(\eta_\delta(u),Y_u)\ +\sigma(\eta_\delta(u),Y_u)-\sigma(\eta_\delta(u),Y_{\eta_\delta(u)})\ \|_{HS}^p\right]$$

$$\le 3^{p-1}\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\varphi(|u-\eta_\delta(u)|)^p\right]+3^{p-1}\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\varphi(|X_u-Y_u|)^p\right]+3^{p-1}\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\varphi(|Y_u-Y_{\eta_\delta(u)}|)^p\right].$$

$$\le 3^{p-1}M^p\delta^p+3^{p-1}M^p\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}|X_u-Y_u|^p\right]+3^{p-1}M^pC^p\delta^{p/2}.$$

Thus, for $\delta\in(0,1)$ and $p\ge 1$, there exists the constant $C_2,\ C_3,\ C_4$, we know

$$\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}|X_u-Y_u|^p\right]\le\frac{6^{p-1}(1+\varepsilon)^p}{1-c(p,\varepsilon)}\mathbb{E}\left[|X_{T_{j-1}}-Y_{T_{j-1}}|^p\right]+\frac{6^{p-1}c(p,d)T^{p/2-1}(\varepsilon^pM^p4^{p-1}+3^{p-1}M^p)}{1-c(p,\varepsilon)}$$

$$\times\int_{T_{j-1}}^t\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}|X_u-Y_u|^p\right]\mathrm{d}s$$

$$+\frac{6^{p-1}c(p,d)T^{p/2-1}}{1-c(p,\varepsilon)}\times\left[4^{p-1}\varepsilon^pC^p\delta^{p/2}+4^{p-1}\varepsilon^pM^p\delta^p+\ 4^{p-1}2^p\varepsilon^pK^p+3^{p-1}C^p\delta^{p/2}+3^{p-1}M^p\delta^p\right]$$

$$+\frac{6^{p-1}T^p(1+\varepsilon)^pM^p}{1-c(p,d,\varepsilon)}(C\delta^{p(\beta+1)/2}+T\delta^p)$$

$$\le C_2\ \mathbb{E}\left[|X_{T_{j-1}}-Y_{T_{j-1}}|^p\right]+C_3\int_{T_{j-1}}^t\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}|X_u-Y_u|^p\right]\mathrm{d}s+C_4\delta^{p/2}.$$

Next, we prove by the Lemma 3.1 that for each $j=1,2\cdots,m$,

$$\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}|X_u-Y_u|^p\right]\le A_j\delta^{p/2},\qquad t\in[T_{j-1},T_j],\tag{8}$$

where $A_1=C_4e^{C_3T}$ and $A_j=(C_2A_{j-1}+C_4)e^{C_3T}$, for $j=2,\cdots,m$. If $j=1$, since $T_0=0$, $\forall t\in(0,T_1]$, we have

$$\mathbb{E}\left[\sup_{0\le u\le t}|X_u-Y_u|^p\right]\le C_3\int_0^t\mathbb{E}\left[\sup_{0\le u\le s}|X_u-Y_u|^p\right]\mathrm{d}s+C_4\delta^{p/2}.$$

Using Gronwall's inequality, we can get

$$\mathbb{E}\left[\sup_{0\le u\le t}|X_u-Y_u|^p\right]\le C_4e^{C_3T}\delta^{p/2},\qquad t\in(0,T_1].$$

We assume that (8) holds for $j=1,2,\cdots,i-1$ with $2\le i\le m$. Then $\forall t\in(T_{i-1},T_i]$, we realize

$$\mathbb{E}\left[\sup_{T_{i-1}\le u\le t}|X_u-Y_u|^p\right]\le C_2\mathbb{E}\left[|X_{T_{i-1}}-Y_{T_{i-1}}|^p\right]+C_3\int_{T_{i-1}}^t\mathbb{E}\left[\sup_{T_{i-1}\le u\le s}|X_u-Y_u|^p\right]\mathrm{d}s+C_4\delta^{p/2}$$

$$\le C_3\int_{T_{i-1}}^t\mathbb{E}\left[\sup_{T_{i-1}\le u\le s}|X_u-Y_u|^p\right]\mathrm{d}s+(C_2A_{i-1}+C_4)\delta^{p/2}.$$

By once more Gronwall's inequality, it holds that

$$\mathbb{E}\left[\sup_{T_{i-1}\le u\le t}|X_u-Y_u|^p\right]\le(C_2A_{i-1}+C_4)e^{C_2T}\delta^{p/2}=A_i\delta^{p/2},\qquad t\in(T_{i-1},T_i].$$

Hence $\forall j=1,\cdots m$, (8) is true. And we draw the conclusion that

$$\mathbb{E}\left[\sup_{0\le s\le T}|X_u-Y_u|^p\right]\le\sum_{j=1}^m\mathbb{E}\left[\sup_{T_{j-1}\le u\le T_j}|X_u-Y_u|^p\right]\le\sum_{j=1}^mA_j\delta^{p/2}:=\beta\,\delta^{p/2}.$$

So the proof is finished.

Proof of Theorem 2.2. Let $\chi \in C_b^{\infty}(\mathbb{R}+)$ is the cut-off function, such that $0\leq \chi \leq 1$, $\chi(r) = 1$ for $r \in(0, 1)$, and $\chi(r) = 0$, for $r\geq 2$. For any $t\in [0, T]$ and $k\geq 1$, let

$$b(t,x) = b(t,x)\chi\left(\frac{|x|}{k}\right), \sigma^{(k)}(t,x) = \sigma(t,\chi\left(\frac{|x|}{k}x\right)), x \in \mathbb{R}^{n}.$$

Fixed $k\geq 1$, we have,

$$X_t^{(k)} = x + \int_0^t b^{(k)}(s, Y_s^{(k)})\mathrm{d}s + \int_0^t \sigma^{(k)}(s, Y_s^{(k)})\mathrm{d}W_s, t \in (0,T].$$

The corresponding continuous-time Euler-Maruyama is

$$Y_t^{(k)} = x + \int_0^t b^{(k)}\left(\eta_\delta(s), Y_{\eta_\delta(s)}^{(k)}\right)\mathrm{d}s + \int_0^t \sigma^{(k)}\left(\eta_\delta(s), Y_{\eta_\delta(s)}^{(k)}\right)\mathrm{d}W_s, \qquad t \in (0,T].$$

Using the BDG, Hölder and Gronwall inequality, for all $p\geq 1$, for some C_T, we have (see the proof Theorem 1.2 in [1])

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|^p\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^p\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|X_t^{(k)}|^p\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^{(k)}|^p\right] \leq C_T(1+\mathbb{E}|X_0|^p) < +\infty. \quad (9)$$

Since

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t - Y_t|^p\right] \leq 3^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t - X_r^{(k)}|^p\right] + 3^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t^{(k)} - Y_t^{(k)}|^p\right] + 3^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^{(k)} - Y_t^{(k)}|^p\right] = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3,$$

applying the Chebyshev inequality and (9), we can deduce

$$\mathbb{I}_1 + \mathbb{I}_3 \preccurlyeq \mathbb{E}\left[\sup_{0\leq t\leq T}|X_t - X_r^{(k)}|^p \mathbb{I}_{\{\sup_{0\leq t\leq T}|X_t|\geq k\}} + \mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t - Y_r^{(k)}|^p \mathbb{I}_{\{\sup_{0\leq t\leq T}|Y_t|\geq k\}}\right]\right.$$

$$\preccurlyeq \left[\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|^p\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|X_t^{(k)}|^p\right]\right]\frac{\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|\right]}{k} + \left[\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^p\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^{(k)}|^p\right]\right]\frac{\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|\right]}{k} \preccurlyeq \frac{1}{k}.$$

For the terms $\mathbb{I}_2$, by the Theorem 2.1, we have

$$\mathbb{I}_2 \preccurlyeq \beta_1\delta^{p/2}.$$

where the constant $\beta_1 > 1$ depending on $T, p, d, \|\sigma\|_{T,\infty}, M, k, \beta$. Consequently, we conclude that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t - Y_t|^p\right] \preccurlyeq \frac{1}{k} + \beta_1\delta^{p/2}.$$

For any $\varepsilon > 0$, taking $k = \frac{1}{\varepsilon}$ and $\delta \to 0$, implies that

$$\lim_{\delta\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t - Y_t|^p\right] = 0.$$

Thus, the proof of Theorem 2.2 can be complete.

4. Conclusion

In this paper, we show the convergence rate of Euler -Maruyama scheme for non-degenerate SDEs with Dini continuous coefficients, by the aid of the regularity of the solution to the associated Kolmogorov equation. We obtain the same conclusions by weakening the conditions in [1] using the properties of Dini continuous and Taylor expansion.

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