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Homogenization for a Parabolic Partial Differential Equation with Two-scaled Advection

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Abstract: Advection-diffusion equations are frequently encountered in many fields and have become a very active research area. However, in most cases, these equations concern different time and space scales, which make it impossible to derive explicit solutions to these equations. To overcome this difficulty, the theory of homogenization aims to approximate the original differential equation with rapidly oscillating coefficients by an effective homogenized equation with constant or slowly varying coefficients. The homogenized equation is often quite suitable for theoretical analysis or numerical methods. This paper investigates the homogenization principle of an advection-diffusion partial differential equation. The novelty of the parabolic partial differential equation we consider is that the advection term in the equation is two-scaled, which is rarely considered by others for the homogenization of advection-diffusion equation. Under certain proper assumptions on the coefficient functions of the original advection-diffusion partial differential equation, which ensure the variable elimination, we derive the homogenized equation, which is also an advection-diffusion equation, by the technique of multi-scale expansion. It is shown that the coefficient functions of the original two-scaled equation have different influence on the coefficient functions of the homogenized equation.

Keywords: Homogenization, Partial Differential Equation, Multi-scale Expansion

1. Introduction

The homogenization theory investigates the effective dynamical property of composite materials in engineering mechanics. The huge number of heterogeneity of the composite materials makes it impossible to give a rigorous theoretical analysis of the problem. Moreover, direct numerical simulation is extremely difficult in consideration of the tremendous load of computation. Substantial phenomenon in physics and engineering concerning different time and space scales, such as composite materials, flow in porous media, atmospheric turbulence, should be treated within this framework.

From the viewpoint of mathematics, the purpose of theory of homogenization is to replace the original differential equation with rapidly oscillating coefficients by an effective differential equation with constant or slowly varying coefficients which is called homogenized equation. The advantage of the homogenized equation lies in the fact that, even when the explicit form of the solution to the homogenized equation is hard to obtain, it is often quite suitable for theoretical analysis or numerical methods. Therefore a crucial part of the homogenization theory is to prove the convergence of the solution of the original equation converges to the solution of the homogenized equation as the scale ratio tends zero [25].

Many researchers discussed periodic homogenization for gradient flows [14, 23, 25, 27]. Homogenization for incompressible, periodic flows was considered in the theory of turbulent diffusion [11, 12, 13, 18, 18, 19]. In the research of atmospheric transport phenomena, McLaughlin and Forest studied homogenization for compressible flows [20]. Researchers interested in the applications to materials sciences and elasticity could refer to monographs [1, 6, 10, 22]. For detailed introduction on homogenization, see standard books [6, 9, 15, 26] and the references therein. Beside the important work we mentioned above, advection-diffusion problems are frequently encountered in many fields and has become a very active research area.

However, as far as we know, rare result has been obtained for advection-diffusion equation with two-scaled advection. To fill the gap in this field of research, we consider the following parabolic equation with two-scaled advection term:

$$\begin{cases} \dfrac{\partial u^{\varepsilon}}{\partial t} = (b_1(x) + \dfrac{1}{\varepsilon} b_2(\dfrac{x}{\varepsilon})) \cdot \nabla u^{\varepsilon} + \triangle u^{\varepsilon}, for\ (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, & (1) \\ u^{\varepsilon}(x,0) = g(x), for\ (x,t) \in \mathbb{R}^d \times \{0\} & (2) \end{cases}$$

where $\varepsilon > 0$ is a small parameter.

Many researchers made significant contributions in the process of developing the method of multiple scales. See [2, 3, 4, 5, 7, 16, 17] and the references therein. The method of multiple scales can also be used to study the problem of homogenization for parabolic PDEs with time dependent coefficients which are periodic in both x and t [8, 14, 21, 24, 27].

The paper is organized as follows. Section 2 contains some notations, hypothesis and useful results we need in later sections. We present the main result of this in Section 3. Section 4 is devoted to proving our main result.

2. Preliminaries

For the convenience of later expression, we introduce some notations here. Denote the d-dimensional unit torus by $\mathbb{T}^d$, which is found by identifying the opposite faces of a unit cube in $\mathbb{R}^d$. For two matrices $A = (a_{ij}), B = (b_{ij})$, denote the *inner product* by $A : B = tr(A^T B) = \sum_{i,j} a_{ij} b_{ij}$. Note that $S : T = S^T : T = \frac{1}{2}(S + S^T) : T$, if matrix T is symmetric. For vectors $a, b, c \in \mathbb{R}^d$, define the *outer product*, which is a matrix, of a and b by $(a \otimes b)c = (b \cdot c)a$.

Define the operator

$$\mathcal{L}_0 = b_2(y) \cdot \nabla_y + \Delta_y \tag{1}$$

on $[0,1]^d$ and its L^2-adjoint $\mathcal{L}_0^*$, both with periodic boundary conditions.

We impose the following assumptions:

A1 (Periodicity Condition): The coefficient functions $b_1(x), b_2(x)$ are smooth and periodic in all directions with period 1.

A2 (Centering Condition):

$$\int_{\mathcal{Y}} b_2(y)\rho(y)dy = 0. \tag{2}$$

Note that the operator $\mathcal{L}_0$ should be viewed as a differential operator in y, with x be a parameter. For variable elimination in the later derivation, we impose the natural *ergodicity assumption* that

$$\begin{aligned} \mathcal{L}_0 I(y) &= 0, & (3) \\ \mathcal{L}_0^* \rho(y) &= 0. & (4) \end{aligned}$$

Here $I(y)$ stands for constants in y and $\rho(y)$ is the density function of an ergodic measure $\mu(dy) = \rho(y)dy$. By the assumption (A1), this ergodicity assumption is validated by the following result (Theorem 6.16 in [25]):

Lemma 2.1. Equip $\mathcal{L}_0, \mathcal{L}_0^*$ on $\mathbb{T}^d$ with periodic boundary conditions. Then

1. $\mathcal{N}(\mathcal{L}_0) = span\{I\}$;
2. $\mathcal{N}(\mathcal{L}_0^*) = span\{\rho\}, inf_{z \in \mathbb{T}^d} \rho(z) > 0$.

3. Main Result

Note that the vector field $b_2(y)$ satisfies the centering condition (A2). Define the solution function $\chi(y)$ of the *cell problem* , which will be crucial to the later derivation, as follows:

$$-\mathcal{L}_0 \chi(y) = b_2(y), \int_{\mathbb{T}^d} \chi(y)\rho(y)dy = 0, \chi(y) \text{ is 1-periodic}. \tag{5}$$

Define

$$b = b_1(x) + \int_{\mathbb{T}^d} (b_1(x) \otimes \nabla_y \chi)\rho(y)dy \tag{6}$$

Define the *effective diffusivity* as

$$\mathcal{K} = I + \int_{\mathbb{T}^d} (b_2(y) \otimes \chi + 2\nabla_y \chi^T)\rho(y)dy \tag{7}$$

where I denotes the identity matrix.

Now we state our main result of this paper, the derivation of which is in the next section.

Theorem 3.1. Assume that the centering condition(A2) holds. For $0 < \varepsilon \ll 1$ and times t up to $\mathcal{O}(1)$, the solution u^{ε} of (1) is approximated by the solution of the homogenized equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= b\cdot\nabla_x u+\mathcal{K}:\nabla_x\nabla_x u, for(x,t)\in\mathbb{R}^d\times\mathbb{R}^+, &(8)\\
u &= g, for(x,t)\in\mathbb{R}^d\times\{0\}. &(9)
\end{aligned}
$$

Remark 3.1. Note that the homogenized equation is also an advection-diffusion equation. And the definitions of $b, \mathcal{K}$ indicate that the coefficient functions of the original two-scaled equation have different influence on the coefficient functions of the homogenized equation.

4. Derivation

The aim of this section is to derive the equation satisfied by the limit of $u^\varepsilon(x,t)$, the solution of (1), as $\varepsilon\to 0$, by the method of multiple scales. Let $\phi=\phi(x,\frac{x}{\varepsilon})$ be a scalar-valued function and introduce the auxiliary variable $y=\frac{x}{\varepsilon}$. By the chain rule, we obtain

$$\nabla\phi=\nabla_x\phi+\frac{1}{\varepsilon}\nabla_y\phi, \tag{10}$$

and

$$\Delta\phi=\Delta_x\phi+\frac{2}{\varepsilon}\nabla_x\cdot\nabla_y\phi+\frac{1}{\varepsilon^2}\Delta_y\phi. \tag{11}$$

Thus the operator of the right-hand side of Equation (1) becomes

$$\mathcal{L}=\frac{1}{\varepsilon^2}\mathcal{L}_0 v+\frac{1}{\varepsilon}\mathcal{L}_1 v+\mathcal{L}_2, \tag{12}$$

where

$$
\begin{aligned}
\mathcal{L}_0 &= b_2(y)\cdot\nabla_y+\Delta_y, &(13)\\
\mathcal{L}_1 &= b_2(y)\cdot\nabla_x+b_1(x)\cdot\nabla_y+2\nabla_x\cdot\nabla_y, &(14)\\
\mathcal{L}_2 &= b_1(x)\cdot\nabla_x+\Delta_x. &(15)
\end{aligned}
$$

In terms of x and y, Equation (1) becomes

$$\frac{\partial u^\varepsilon}{\partial t}=(\frac{1}{\varepsilon^2}\mathcal{L}_0 v+\frac{1}{\varepsilon}\mathcal{L}_1 v+\mathcal{L}_2)u^\varepsilon. \tag{16}$$

By multiple-scales expansion of the solution, we have

$$u^\varepsilon(x,t)=u_0^\varepsilon(x,y,t)+\varepsilon u_1^\varepsilon(x,y,t)+\varepsilon^2u_2^\varepsilon(x,y,t)+\dots \tag{17}$$

where $u_j(x,y,t), j=1,2,\dots$ are all periodic functions with period 1 in y. Substituting (17) into Equation (1) and equating terms of equal powers in ε leads to the following sequence of equations:

$$
\begin{aligned}
\mathcal{O}(\frac{1}{\varepsilon^2}):\ -\mathcal{L}_0u_0 &= 0, &(18)\\
\mathcal{O}(\frac{1}{\varepsilon}):\ -\mathcal{L}_0u_1 &= \mathcal{L}_1u_0, &(19)\\
\mathcal{O}(1):\ -\mathcal{L}_0u_2 &= -\frac{\partial u_0}{\partial t}+\mathcal{L}_1u_1+\mathcal{L}_2u_0. &(20)
\end{aligned}
$$

Note that the differential operator $\mathcal{L}_0$ acts in y only.

By the fact that the null space of $\mathcal{L}_0$ is one-dimensional, Equation (18) indicates that the first term $u_0^\varepsilon(x,y,t)$ in the expansion is actually independent of y, so that $u_0=u(x,t)$ only. We proceed now with Equation (19).

Notice that

$$\mathcal{L}_1u_0=b_2(y)\cdot\nabla_x u(x,t). \tag{21}$$

The centering condition assumption (A2) ensures that Equation (19) has a solution, by the Fredholm alternative. By separation of variables, it is natural to write the solution of Equation (19) as follows:

$$u_1(x,y,t)=\chi(y)\cdot\nabla_x u(x,t). \tag{22}$$

with $\chi(y)$ solving the cell problem (5). The assumptions (A2) ensures that there is a solution to the cell problem and the uniqueness is guaranteed by the normalization condition, i.e., the second equation in (5).

Turning now to Equation (20). The solvability condition reads

$$\int_{\mathbb{T}^d}(\frac{\partial u_0}{\partial t}-\mathcal{L}_2u_0-\mathcal{L}_1u_1)\rho(y)dy=0. \tag{23}$$

Again, due to that $u_0=u(x,t)$ is independent of y, we can transform the preceding equation into

$$\frac{\partial u}{\partial t}=\mathcal{L}_2u+\int_{\mathbb{T}^d}(\mathcal{L}_1u_1)\rho(y)dy \tag{24}$$

with

$$\mathcal{L}_2u=b_1(x)\cdot\nabla_xu+\Delta_xu, \tag{25}$$

$$
\begin{aligned}
\mathcal{L}_1u_1 &= b_2(y)\cdot\nabla_x(\chi\cdot\nabla_xu)+b_1(x)\cdot\nabla_y(\chi\cdot\nabla_xu)+2\nabla_x\cdot\nabla_y(\chi\cdot\nabla_xu)\\
&= (b_1(x)\otimes\nabla_y\chi)\cdot\nabla_xu+(b_2(y)\otimes\chi+2\nabla_y\chi^T):\nabla_x\nabla_xu.
\end{aligned} \tag{26}
$$

In view of the preceding calculation, Equation (24) becomes

$$\frac{\partial u}{\partial t}=b\cdot\nabla_xu+\mathcal{K}:\nabla_x\nabla_xu \tag{27}$$

with

$$b = b_1(x) + \int_{\mathbb{T}^d} (b_1(x) \otimes \nabla_y \chi)\rho(y)dy \tag{28}$$

$$\mathcal{K} = I + \int_{\mathbb{T}^d} (b_2(y) \otimes \chi + 2\nabla_y \chi^T)\rho(y)dy \tag{29}$$

where I denotes the identity matrix.

This completes the derivation.

5. Conclusion

This paper investigates the homogenization principle of an advection-diffusion equation. The novelty of the parabolic partial differential equation is the two-scaled advection term. By the technique of multi-scale expansion, the homogenized equation under certain proper conditions is derived, which is also an advection-diffusion equation. The homogenized equation is suitable for theoretical analysis or numerical methods, compared with the original equation. And the coefficient functions of the original two-scaled equation have different influence on the coefficient functions of the homogenized equation.

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Conflict of Interest

The author states no conflict of interest.

Data Availability Statement

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