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Continuous-Time Difference Equations and Distributed Chaos Modelling

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Abstract: The article aims to attract the attention of researchers, experts and those interested in nonlinear dynamics and chaos theory to the not well known field of continuous-time difference equations, in the hopes of opening new doors into the study of chaotic system. Deterministic chaos and related notions are used in an increasing number of scientific works. There are a lot of problems associated with the mathematical aspects of the fine structure of chaos. Just as discrete-time difference equations have proven to be excellent models of temporal (discrete) chaos, so continuous-time difference equations provide new elegant mechanisms for onset and inside reconstructions of spatio-temporal (distributed) chaos. Distributed chaos is usually described by boundary value problems for partial differential equations. A number of these boundary value problems can be reduced to continuous-time difference equations, which enable one to build new chaos scenarios arising from the properties of the equations. Whereas the emergence of deterministic chaos is usually attributed to the complex structure of attractors, these new scenarios are based on a highly complex structure of spatially extended “points” of the attractor. Examples of reducible boundary value problems are set forth in the article, but the main focus is on a very elementary overview of the principal features of solutions of the simplest nonlinear continuous-time difference equations: loss of continuity, asymptotic periodicity, gradient catastrophe, fractal geometry, space-filling property, going beyond the horizon of predictability, self-stochasticity (deterministic solutions are asymptotically described by random processes), formation of hierarchical structures (down to arbitrarily small scales). Here we have a wonderful example of how very complex phenomena can be described with very simple equations. The use of continuous-time difference equations in the study of reducible and close-to-reducible boundary value problems migh help to advance in understanding possible mathematical mechanisms for distributed chaos.

Keywords: Continuous-Time Difference Equation, Boundary Value Problem, Deterministic Chaos

1. Introduction

When using the term “difference equation”, it is usually thought of as a discrete-time equation. But this article is about continuous-time difference equations. In many cases, such equations appear as models for systems where the future is determined not only by their current state but by a part of their history. Over the past decades, the asymptotic dynamics of continuous-time difference equations was being the focus of Sharkovsky’s scientific team (Institute of Mathematics, National Academy of Sciences of Ukraine). Although the main ideas of their research were published about 30 years ago [1-3], certain fundamental steps were made relatively recently [4-6], in particular it was clearly shown that such equations are very suitable for numerical and substantive analytical discussion of chaotic dynamics. Now it can already be stated that the primary theory of continuous-time difference equations is completed [7]. This theory is paradigmatic for the modeling of complexity and chaos; in this line, the theory has given rise to the concept of ideal turbulence — a whole new kind of distributed chaos [8-10].

The motivation for this article came from the fact that the above-mentioned research on continuous-time difference equations and recent results on their application in nonlinear dynamics are not very known among scientists active in fields where chaotic phenomena are of importance (the more so since two of the above books are for now avairlable only in Russian). This article is not an overview of all continuous-time difference equations studies, but rather a call (along with its rationale) for academic and applied researchers to pay special close attention to these equations in the context of modeling and simulation of distributed chaos. When writing, little use was made of mathematical formalisms; instead the emphasis was placed on an informal qualitative aspects.

Research over the resent decades have discovered that complexity is inherent to low-dimensional dynamical systems. Now it is well known that the trajectories of the one-dimensional dynamical system $w \to f(w)$, $w \in \mathbb{R}$, and hence the solutions of the discrete-time difference equation $w(n+1) = f(w(n))$, $n \in \mathbb{Z}^+$, can behave very irregular producing deterministic chaos. For example, it may occur that their large-time behavior is practically indistinguishable from that of random variables. However, in this case, we can only speak about the temporal chaos.

And what can we expect from "almost the same" continuous-time difference equation

$$w(t+1) = f(w(t)),\ t \in \mathbb{R}^+, \tag{1}$$

does it have some radically new properties? Every solution of this equation is determined by its values on the interval $0 \le t < 1$ (and not only its value at $t = 0$, as in the case of discrete time). Hence, the equation generates the infinite-dimensional dynamical system $\varphi \mapsto f(\varphi)$ on the space of functions φ: $[0,1) \to \mathbb{R}$. And now that the points of the phase space are functions, we can and must think about the spatio-temporal behavior of trajectories, i.e., about the evolution of functions φ with time (when they are "moving along their own trajectories"). In this case, all complexities in the behavior of trajectories of the one-dimensional dynamical system $w \mapsto f(w)$, are transformed into a very intricate evolution of the functions φ. Indeed, the trajectory of the "point" $\varphi(t)$ consists of the functions $\varphi(t)$, $f(\varphi(t))$, $f^2(\varphi(t))$, $f^3(\varphi(t))$, … Consequently, for any non-constant function $\varphi(t)$ the dynamics of its trajectory can be regarded as the dynamics of a continuum of uncoupled identical oscillators, namely: at each point $t_* \in [0,1)$, there is suspended a "pendulum" that oscillates with the law.

$$w_n \mapsto w_{n+1} = f(w_n), \text{ where } w_0 = \varphi(t_*). \tag{2}$$

The oscillations of every individual "pendulum" do not depend on the "pendulums" at other points. Therefore, with increasing time, pendulum states that were close to each other at the initial moment can by turns be far apart and get closer. Systematic "divergences" and "convergences" of pendulum states erase the initial information and lead to an increase in uncertainty in the collective dynamics of the "pendulums", which results in the highly irregular character of the functions $f^n(\varphi(t))$ when n is large enough. The solutions $w(t)$ can be written as follows.

$$w(t) = f^n(\varphi(t-n)) \text{ for } t \in [n, n+1),\ n = 0,1,..., \tag{3}$$

$$w(t) = \varphi(t) \text{ for } t \in [0,1), \tag{4}$$

hence their large-time behavior also becomes highly irregular as $t \to \infty$. Typical solutions of (1) are depicted in Figure 1 (such is the case for solutions of any smoothness class).

Both computer visualization and theoretical studies have shown that even the simplest continuous-time difference equation (1) readily leads to chaotic dynamics, its solutions are very well suited to a mathematical simulation of nonlinear phenomena such as large-to-small cascades of structures, intermixing, formation of fractals, intermittency.[1]

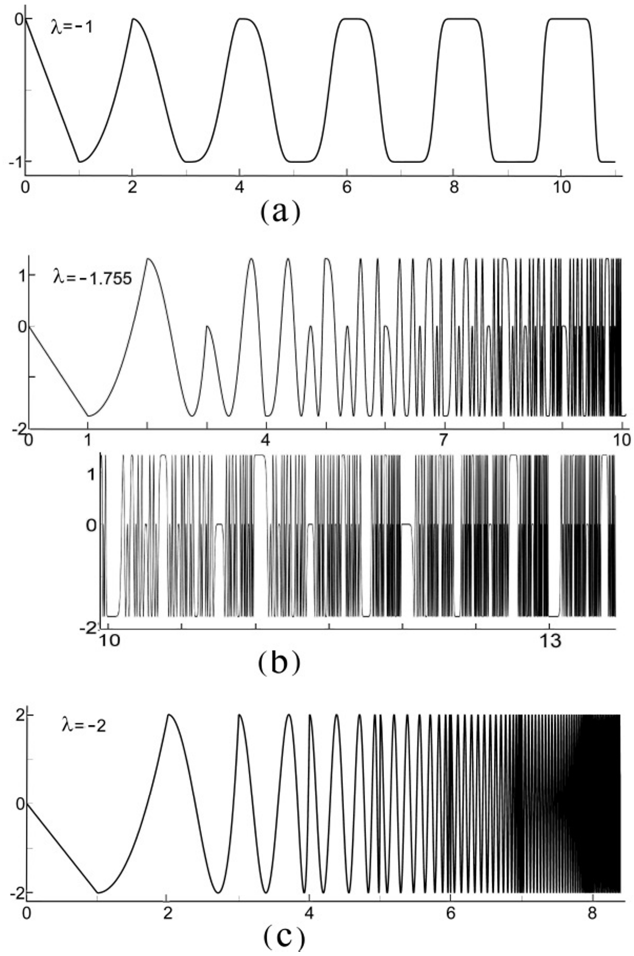

Figure 1. *Typical solutions of the equation $x(t+1) = (x(t))^2 + \lambda$ for $\lambda = -1$, $\lambda = -1.755$ and $\lambda = -2$.*

This fact assumes a particular importance considering that many of boundary value problems for partial differential equations are reduced to continuous-time difference or similar equations. For example, the boundary value problem.

$$u_\tau - u_x = 0,\ x \in [0,1],\ \tau \in \mathbb{R}^+, \tag{5}$$

$$u|_{x=1} = f(u)|_{x=0} \tag{6}$$

reduces exactly to the difference equation (1). Indeed, the general solution of the partial differential equation has the form $u(x,\tau) = w(\tau + x)$ with w being an arbitrary function; the

1 The entire cascade process of structures emergence can be described using just one difference equation (1), whereas the use of systems of ordinary differential equations requires an increase in their dimension to take into account structures of each subsequent scale and one has to consider ultra-high dimensional systems ("curse of dimensionality"). The explanation is that the state space of continuous-time difference equations is infinite-dimensional.

substitution of this formula into the boundary condition gives the difference equation $w(t+1) = f(w(t))$, $t \in \mathbb{R}^+$. Therefore, an individual solution $u(x,\tau)$ can be written as

$$u(x,\tau) = w(\tau + x) \tag{7}$$

with w being such that $w(x) = u(x,0)$, $x \in [0,1)$.

What does the relationship (7) mean for the boundary value problem? Figure 2 illustrates how the chaotic solutions of the difference equation turn into the chaotic solutions of the boundary value problem.

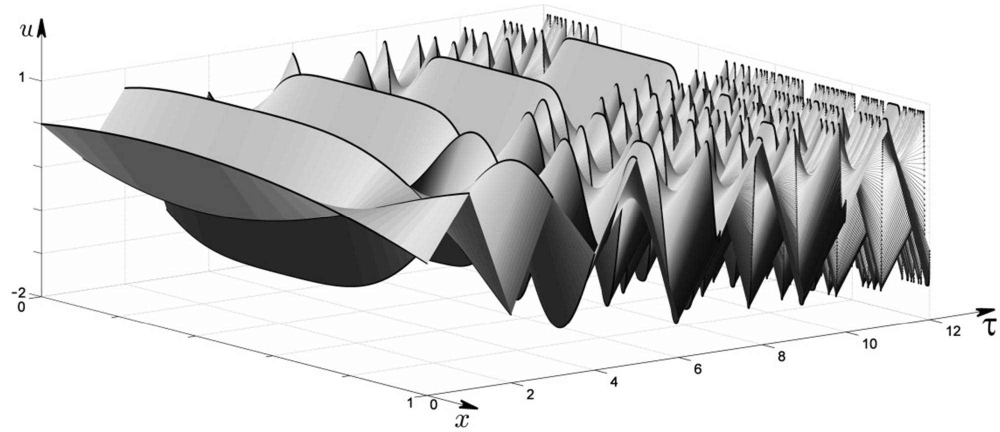

Figure 2. *Typical solution of the boundary value problem* $u_\tau - u_x = 0$, $u(x,1) = (u(x,0))^2 - 1.755$.

The reduction of boundary value problems to continuous-time difference equations made possible new spatio-temporal chaos scenarios (different from scenarios based on the strange attractors concept and from scenarios for discrete ensembles of coupled maps) covering the issues:

a) What causes cascades of structures that come on at ever-decreasing scales and what are the principles of the hierarchy of structures that emerge?
b) Why a system "moves" to the chaotic mixing?
c) How the self-stochastization (or self-randomization) of a deterministic system can happen with time?

Here we have an excellent example of how very complex phenomena can be described with the help of very simple models. In particular, not only does the reduction to (1) provide scenarios for the onset of spatio-temporal chaos, that are roughly comprehensible even to senior schoolchildren, but it also describes the self-stochasticity phenomenon in favor of the hypothesis (growing ever more convincing) that chaos is not completely disordered and obeys certain regularities.

2. Boundary Value Problems Reducible to Continuous-Time Difference Equations

Problems that can be reduced to continuous-time difference equations occupy a special place among boundary value problems for partial differential equations. Examples of this sort problems, which are called reducible, have been known long ago [11–14], but at that time the reduction method did not attract the attention of specialists because the resulting equations seemed no less difficult than the original boundary value problems. A real chance of progress on this issue appeared relatively recently thanks to the development of the qualitative theory of nonlinear continuous-time difference equations [1, 7].

The mathematical concept of chaos in reducible boundary value problems was developed as a result of investigating a number of nonlinear boundary value problems, which are very simple and highly idealized models describing systems without internal resistance but nevertheless exhibiting the essential part of dynamics in more complicated and general models in electrodynamics, radiophysics and other disciplines related to the study of electromagnetic and acoustic vibrations. These are mainly the linear hyperbolic equations.

$$\frac{\partial u_k}{\partial \tau} + \sum_{j=1}^{n} a_{kj} \frac{\partial u_k}{\partial x_j} = 0,\ k = 0, 1, \ldots, m, \tag{8}$$

where $(x_1, \ldots, x_n) \in G \subset \mathbb{R}^n$, $\tau \in \mathbb{R}^{+,}$ $a_{kj} \in \mathbb{R}$, with the nonlinear boundary condition.

$$H(u_1, \ldots, u_m) = 0 \text{ for } (x_1, \ldots, x_n) \in \partial G, \tag{9}$$

and also the wave equation and others that somehow lead to equations of the form (8). The linearity of (8) makes it possible to write out the general formula for solutions. The substitution this formula into the boundary condition gives a nonlinear continuous-time difference equation or differential-difference equation.

The transition to a continuous-time difference equation in many cases enables not only to find analytical formulas for solutions of reducible boundary value problems but also to "convert" them into rigorous qualitative-analytical results. In addition, formulas for solutions are usually well adapted to computer calculations and make it easy to visualize the dynamics of reducible problems. For example, the solutions of problems reducible to (1) are determined via the iterations of the map f. In particular, when solving the problem (5),(6) with the initial condition $u(x,0) = \varphi(x)$, we arrive at the representation.

$$u(x,\tau) = f^n\big(\varphi(\tau + x - n)\big),\ \tau + x \in [n, n + 1),\ n = 0,1,\ldots, \quad (10)$$

where $f^0(z) = z$, $f^n(z) = f\big(f^{n-1}(z)\big)$, which follows from (7) and shows that computer calculations are quite simple to perform. Along with the problem (5),(6), there are still many examples of reducible problems. Such examples are discussed in many works ([15-17] and references therein), and just a few of them will be shared here.

Consider the equation (5) combined with another boundary condition containing derivatives, say

$$u_\tau|_{x=1} = h(u)\, u_\tau|_{x=0.} \quad (11)$$

This problem is reducesed to a family of difference equations. Indeed, the integration of (11) brings the boundary condition to the form.

$$w|_{x=1} = f(w)|_{x=0} + \lambda, \quad (12)$$

where f is a primitive of h and λ is an arbitrary constant. Substituting the general solution $u(x,\tau) = w(\tau + x)$ with w being an arbitrary C^0-function into (12), we obtain for w the one-parameter family of difference equations.

$$w(t + 1) = f(w(t)) + \lambda,\ t \in \mathbb{R}^+. \quad (13)$$

Since the solutions of the original boundary value problem are continuous, every individual solution $u(x,\tau)$ can be represented in the form.

$$u(x,\tau) = w(\tau + x), \quad (14)$$

where $w(t)$ is the solution to the only one of the difference equations (13). Indeed, let $u(x,0) = \varphi(x)$, then $w(t)$ is found from the equation.

$$w(t + 1) = f\big(w(t)\big) + \varphi(1) - f\big(\varphi(0)\big),\ t \in \mathbb{R}^+, \quad (15)$$

with the initial condition $w(t)|_{t=[0,1)} = \varphi(t)$.

And some words about the wave equation. Even the one-dimensional wave equation $u_{\tau\tau} - c^2 u_{xx} = 0$ in combination with various nonlinear boundary conditions provides a wide range of reducible boundary value problems. One of the simplest is the problem.

$$u_{\tau\tau} - u_{xx} = 0,\ x \in [0,1],\ \tau \in \mathbb{R}^+, \quad (16)$$

$$u|_{x=0} = 0,\ u_x|_{x=1} = f(u_\tau)|_{x=1}. \quad (17)$$

The substitution of the general solution formula.

$$u(x,\tau) = w(\tau + x) + \hat{w}(\tau - x), \quad (18)$$

where w, $\hat{w}$ are arbitrary C^2-functions, in (17) yields.

$$\hat{w}(t) = -w(t),$$

$$w'(t + 1) - \hat{w}'(t - 1) = f\big(w'(t + 1) + \hat{w}'(t - 1)\big),\ t \in {}^{+}.$$

It follows that the function $v(t)$ given by

$$v(t) = -\hat{w}'(t - 1) \text{ for } 0 \le t < 1,$$

$$v(t) = w'(t - 1) \text{ for } t \ge 1,$$

must satisfy the difference equation.

$$v(t + 2) + v(t) = f\big(v(t + 2) - v(t)\big), \quad (19)$$

where the dependence of $v(t + 2)$ on $v(t)$ is defined implicitly and is for the most part multivalued. Thus, every individual solution $u(x,\tau)$ can be represented in the form.

$$u(x,\tau) = \int_{\tau-x+t}^{\tau+x+t} v(t)\,dt, \quad (20)$$

where $v(t)$ is the solution (if exists) of (19) with the initial condition.

$$v(t)\,|_{[0,1)} = \tfrac{1}{2}\big(-\varphi'(1 - t) + \psi(1 - t)\big),$$

$$v(t)\,|_{[1,2)} = \tfrac{1}{2}\big(\varphi'(t - 1) + \psi(t - 1)\big),$$

$$\varphi(x) = u(x,0),\ \psi(x) = u_\tau(x,0).$$

If the boundary conditions (17) are replaced as follows:

$$u_x|_{x=0} = a\, u_\tau|_{x=0},\ u_x|_{x=1} = f(u)|_{x=0},\ a > 1, \quad (21)$$

then the problem reduces not to one but to several difference equations. Combining (18) and (17) after certain manipulations gives.

$$w(t) = \tfrac{1}{2}(1 + a)\, z(t + 1) - A,\ \hat{w}(t) = \tfrac{1}{2}(1 - a)\, z(t + 1) + A,$$

where A is an arbitrary constant, and $z(t)$ is a solution of the second-order difference equation.

$$(a + 1)\, z(t + 2) = 2 f\big(z(t + 1)\big) + (a - 1)\, z(t).$$

Setting

$$y(\tau) = z(t + 1)/b.\ h(y) = 2 f(by)/(a - 1),\ b = (a - 1)/(a + 1),$$

we obtain the system of first-order difference equations.

$$y(t + 1) = h\big(y(t)\big) + z(t), \quad (22)$$

$$z(t + 1) = by(t).$$

To (22) there corresponds the two-dimensional map.

$$y \mapsto h(y) + z,$$

$$z \mapsto by,$$

which is the Henon map for $h(y) = 1 - My^2$, and the Lozi map for $h(y) = 1 - M|y|$.

The above reduction method has a clear physical meaning. Hyperbolic equations are marked by the existence of real and distinct characteristics — curves, along which the partial differential equation becomes an ordinary differential equation. From a physical point of view, the characteristics are space-time curves along which signals propagate in the medium described by the boundary value problem. If the medium is linear (as in the case of problems like (8),(9)), then the signal $u(x,\tau)$ moves along the characteristics without changing its waveform. This is exactly what the formal procedure of “substitution into boundary conditions” is. In particular, it is possible that the signal, after reflection and distortion at the boundary, returns to its original position (again, without changing its new waveform when moving along the characteristics). Then, to find the signal value at a point x_0, we compose the difference equation $u(x_0, \tau + \theta) = f\big(u(x_0, \tau)\big)$, $\tau \in \mathbb{R}^+$, where f is the function describing the effect of the boundary on the signal waveform, and θ is the signal return time. For example, such is observed in the problem (5),(6), where the straight lines $\tau + x =$ const are characteristics, and the signal return time equals 1.

Thus, for a successful study of reducible and related boundary value problems, a good understanding of the specifics of continuous-time difference equations is necessary.

3. A Little About Continuous-Time Difference Equations

Now that we are aware of the potential benefits of using continuous-time difference equations when modeling complex dynamics, we give a very brief and simplified description of the principal features of the equation.

$$w(t + 1) = f(w(t)),\ t \in \mathbb{R}^+, \tag{23}$$

where f is a nonlinear continuous map of a closed interval I into itself (precise formulations of all notions, arguments, and conclusions are represented in the completed form in the author's book [7]).

Solutions of (23) are generated by functions φ: $[0,1) \to I$. Every initial condition $w(t) = \varphi(t)$, $t \in [0,1)$ defines a unique solution, which we denote by $w_\varphi(t)$. This solution takes values in the interval I and, as mentioned above, can be written in the form.

$$w_\varphi(t) = f^n\big(\varphi(t - \mathrm{n})\big),\ t \in [n, n + 1),\ n = 1, 2,...$$

Continuous-time difference equation by themselves do not impose any regularity conditions on their solutions. We are primarily be interested in continuous and smooth solutions. As it follows from the above formula for $w_\varphi(t)$, continuous solutions are generated by just those φ that satisfy the *consistency conditions*:

$$\varphi \in C([0, 1), I),\ \varphi(1{-}0) = f\big(\varphi(0)\big).$$

If f is C^k-smooth, $k \geq 1$, then a good many solutions are also smooth. More precisely, for any integer $1 \leq i \leq k$, the solution $w_\varphi(t)$ is C^i-smooth if and only if.

$$\varphi \in C^i([0, 1), I),\ d^s\varphi(1{-}0)/dt^s = d^s f\big(\varphi(0)\big)/dt^s,\ s = 0, 1,..., i.$$

From now on, the solutions are understood as continuous (including smooth) solutions of (23) and, in order to avoid degenerate situations, the initial functions are assumed to take values in the interior of I.

Let us first turn to the representative example shown in Figure 1, which we will constantly refer to and which is related to the quadratic equation.

$$w(t + 1) = \big(w(t)\big)^2 + \lambda,\ t \in \mathbb{R}^+. \tag{24}$$

In all three cases in Figure 1, there exist points $t \in [0,1)$ such that the solution derivative at $t = t + n$ increases ad infinitum as $n \to \infty$, and hence the solution is asymptotically discontinuous. In the case (a), there is just one such point, but in the cases (b) and (c), there are infinitely many such points; moreover, these are all points from $[0,1)$ in the case (c). On closer inspection of the drawing, the solutions in the cases (a), (b) and (c) look asymptotically periodic with period 2, 3 and 1, respectively. This example illustrates the general research finding that solutions (even C^∞-smooth) tend to periodic discontinuous functions (interval-valued at discontinuity points). These limit functions often have an infinite number of discontinuities on any unit-length interval. If so, the solutions behave chaotically over time but with the peculiarity of nearly perfect periodicity: the pattern of irregular behavior persists on larger scales and becomes more complicated on smaller scales (just as in Figures 1(b), (c)). Therefore, it is meaningful to speak of the dual tendency for solutions to exhibit both deterministic periodicity and unpredictability.

Without going into much mathematics, we can explain why this is so in terms of the above interpretation of the solution $w_\varphi(t)$ as the continuums of oscillators (2), which do not interact, differ from each other only in the initial state and follow the same law $w_n \mapsto w_{n+1} = f(w_n)$, $w_0 = \varphi(t\ )$. Namely:

a) The discontinuity of the limit functions arises from the imbalance of some or all of the individual oscillators, which is usually caused by the lack of coupling of the oscillators.
b) The periodicity of the limit functions is the result of the increasing similarity (up to nearly perfect periodicity) in the collective behavior of all the oscillators considered as a whole.

Whereas the former statement is intuitive, the latter requires some kind of comment on how the individual oscillators oscillate independently and yet "work" together to produce a periodic pattern. So, taken together, the oscillators (2) constitute a holistic system represented by the continual family of trajectories $w_n\langle t\rangle = f^n\big(\varphi(t)\big)$, $t \in [0,1)$. In order to investigate the system dynamics (especially where the divergence of trajectories occurs), it is necessary to analyze not every individual trajectory $w_n\langle t_*\rangle = f^n\big(\varphi(t_*)\big)$, $t_* \in [0, 1)$, but its associated beam of nearby trajectories $w_n\langle t\rangle = f^n\big(\varphi(t)\big)$, $t \in (t_* - \varepsilon, t_* + \varepsilon)$; and what matters is that each such a beam must be analyzed first when $n \to \infty$, and then when $\varepsilon \to 0$ (and not vice versa).[2] Thus, the factor responsible for the asymptotic behavior of the uncoupled-oscillators system (2) is precisely the dynamics of neighborhoods of points, and not the dynamics of individual points (whereas the latter is true

2 This approach is in tune with the concept of holism, which studies an object as a whole (focusing on the relationships between the object parts) and stands in contrast to the reductionism, which believes that a whole is simply the sum of its parts (see, e.g., Ostreng, W. Reductionism versus Holism – Contrasting Approaches. In. Consilience. Interdisciplinary Communications 2005/2006, Centre for Advanced Study, Oslo, 2007, p. 11–14).

for the discrete-time equation $w(n+1) = f\big(w(n)\big)$, $n \in \mathbb{Z}^+$). The trajectories of neighborhoods are typically asymptotically periodic [18], and therefore the holistic system (2) also behaves asymptotically periodically.

While the theory of linear continuous-time difference equations appeared in the early 1900s (mostly due to the direct analogy with linear differential equations), things are different with the nonlinear continuous-time difference equations. Until recently there was not even a systematic theory of the simplest nonlinear equation (24). This is because the crucial differences between continuous-time difference equations and ordinary differential equations are fully manifested in the case of these being nonlinear. Visual evidence is provided in Figure 1; naturally, these kinds of solutions are fundamentally infeasible for ordinary differential equations.

Below we list and briefly discuss the main specific properties of (23), which, we emphasize once again, are mathematically formalized and fully justified.

3.1. Loss of Continuity: Asymptotically Discontinuous Solutions

In all the cases of Figure 1, there are regions where the solution changes very fast; its rate of change tends to infinity and the transition interval tends to zero as $t \to \infty$. This suggest an idea of the asymptotic loss of continuity. Under the assumptions made, all solutions of (23) are continuous (with some being smooth) and bounded, but would they be uniformly continuous? The answer depends heavily on the set

$$D(f) = \big\{\, z \in I\text{: the trajectory } f^n(z),\ n = 0, 1, \ldots, \text{ is Liapunov unstable} \,\big\},$$

which is referred to as *separator*.[3] In the vicinity of $D(f)$, there is the divergence of initially nearby trajectories, i.e., for every point $z \in D(f)$ there exists $d(z) > 0$ such that whatever $\varepsilon > 0$, one can find $m > 0$ and $\hat{z} \in (z-\varepsilon, z+\varepsilon)$ with the property: $|f^m(z) - f^m(\hat{z})| > d(z)$.

If $D(f)$ contains a positive Lebesgue-measure subset $D_0(f)$ such that $\inf d(z) > 0$ for $z \in D_0(f)$, then f is said to be sensitively dependent on initial data (the butterfly effect).

The set $D(f)$ has the following implication for the solution $w_\varphi(t)$ of (23). Let $z_0 \in D(f)$ and let for simplicity z_0 be a fixed (unstable) point. If there exists $t_0 \in [0,1)$ such that $\varphi(t_0) = z_0$, $\varphi'(t_0) \neq 0$, then whatever be a neighborhood V_n of the point $t_n = t_0 + n$, the inequality $\operatorname{diam} w_\varphi(V_n) > d(z_0)$ holds true from a certain n onwards. This implies that near the points $t = t_n$ the graph of $w_\varphi(t)$ will look more and more like a vertical segment of length $> d(z_0)$ as n increases. Hence, the solution $w_\varphi(t)$ is not uniformly continuous on the whole semiaxis $\mathbb{R}^+$ and is asymptotically discontinuous. All the solutions shown in Figure 1 have this property. But if $\varphi(t)$ takes no values from $D(f)$, then the solution $w_\varphi(t)$ is uniformly continuous. Thus, all the solutions are uniformly continuous on $\mathbb{R}^+$ only if $D(f) = \varnothing$. Of course, (23) can have uniformly continuous solutions when $D(f) \neq \varnothing$. The simplest example is afforded by considering the constant solutions, which have the form $w(t) \equiv c$ with c being a fixed point of f. Every constant solution is uniformly continuous on $\mathbb{R}^+$ regardless of whether or not its associated fixed point is stable. For example, (24) has exactly two constant solutions $w(t) \equiv \alpha$ and $w(t) \equiv \alpha_0$, where $\alpha = ½\,(1 - \sqrt{b})$ and $\alpha_0 = ½\,(1 + \sqrt{b})$, $b = 1 - 4\lambda$. If $1/4 < \lambda < 3/4$, then the fixed point α is attracting $\big($hence $\alpha \in I \setminus D(f)\big)$ and the fixed point α_0 is repelling $\big($hence $\alpha_0 \in D(f)\big)$. In general, typical uniformly continuous solutions are those tending to constant ones.

The intuition tells us that the condition $D(f) = \varnothing$. holds true in exceptional cases, and the existence of asymptotically discontinuous solutions is therefore distinctive of (23). In the broadest strokes, *for almost all equations of the form (23), all solutions other than asymptotically constant ones are asymptotically discontinuous.*

3.2. Periodicity in the Asymptotic Dynamics of Solutions

The limit behavior of an asymptotically discontinuous solution cannot be described in terms of continuous functions. In order to handle this situation, we need the space of upper semicontinuous functions from $\mathbb{R}^+$ to 2^I, that is endowed with metric Δ: $\mathbb{R}^+ \times I \to \mathbb{R}$ given by the Hausdorff distance between the graphs of functions. A solution tending to a periodic upper semicontinuous function in the metric Δ will be called Δ-asymptotically periodic. Very simplified, *for almost all equations of the form (23), almost all solutions are Δ-asymptotically periodic.* Their limit functions can be described by a single formula using the set

$$Q_f(z) = \cap_{\delta>0} \cap_{j>0} \operatorname{cl}\big(\cup_{i>j} f^i(V_\delta(z))\big),\ V_\delta(z) = (z-\delta, z+\delta),$$

which is referred to as *domain of influence of a point* under the map f and shows how far from the trajectory of the point z the trajectories of its nearby points go. *There exists an integer $p = p(f) > 0$ such that almost every solution $w_\varphi(t)$ tends to the upper semicontinuous p-periodic function.*

$$\mathcal{H}_\varphi(t) = f^n\Big(Q_{f^p}\big(\varphi(t-n)\big)\Big) \text{ for } t \in [n, n+1), \tag{25}$$

$$n = 0, 1, \ldots, p-1.$$

Let << >> be for the integer part of a number. The value of $\mathcal{H}_\varphi(t)$ is a closed interval if $\varphi(\langle\langle t\rangle\rangle) \in D(f)$ and a one-point set (singleton) otherwise.[4]

As an illustration we take the solutions of (24), which are plotted in Figure 1. Denote these by $w_{(a)}$, $w_{(b)}$, $w_{(c)}$ and their initial functions by $\varphi_{(a)}$, $\varphi_{(b)}$, $\varphi_{(c)}$ for the cases (a), (b) and (c), respectively. The solutions behavior is determined by the dynamics of the map g: $z \to z^2 + \lambda$. For the above values of λ, the map g has the fixed points $z = \alpha$ and $z = \alpha_0$, each repelling,

3 The notion of separator is directly related to the Julia set $J(f)$. Namely, $\operatorname{cl}(D(f)) = J(f)$ and $D(f)$ is typically closed (here cl() is for the closure of a set).

4 Formula (25) strongly indicates that the asymptotic behavior of $w_\varphi(t)$ is determined by the collective dynamics of the neighborhoods of points $z \in \varphi([0,1])$ under the map f.

and the invariant interval $I = [-\alpha_0, \alpha_0]$. The initial functions $\varphi_{(a)}$, $\varphi_{(b)}$, and $\varphi_{(c)}$ take values from I (as seen from the drawing), hence so do the solutions $w_{(a)}$, $w_{(b)}$ and $w_{(c)}$. We will therefor consider the map g only on I (see Figure 3).

As unexpected as it may be, the simplest to describe is the most chaotic solution $w_{(c)}(t)$, that occurs at $\lambda = -2$. In this case, $I = [-2, 2]$, all the trajectories of the map g are unstable and any neighborhood of every point covers I in a finite number of iterations. Hence, $D(g) = [-2, 2]$ and $Q_g(z) = [-2, 2]$, $z \in I$. To us this means that the limit function of $w_{(c)}(t)$ has the form.

$$\mathcal{H}\varphi_{(c)}(t) = [-2, 2],\ t \in [0,1],$$

it is discontinuous at every point and is periodic with period 1.

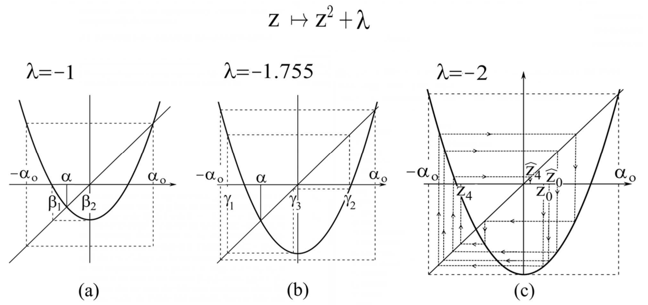

Figure 3. *Three types of dynamical behavior of quadratic maps.*

The most "well behaving" solution $w_{(a)}(t)$ occurs at $\lambda = -1$. In this case, the map g has an attracting period-2 cycle, namely, the cycle $\{\beta_1, \beta_2\}$ with $\beta_1 = -1$, $\beta_2 = 0$. This cycle attracts all the points of I except for the set $D(g)$ consisting of the (repelling) fixed points α, α_0, and their preimages (condensed towards the ends of I). Therefore, $Q_g(z) = \{\beta_1\} \cup \{\beta_2\}$ for $z \in I \setminus D(g)$. As for the rest of the points, whatever be the neighborhood of a point $z \in D(g)$, $z \neq -\alpha_0, \alpha_0$, it expands in the limit to the interval $[\beta_1, \beta_2]$; hence, $Q_g(z) = [\beta_1, \beta_2]$. From this we draw the following conclusions about the solution $w_{(a)}(t)$. As Figure 1(a) suggests, there is only one point $t_* \in [0,1)$ such that $\varphi_{(a)}(t_*) \in D(g)$, namely, $\varphi_{(a)}(t_*) = \alpha$. If $t \in [0, t_*)$, then $g^{2k}(\varphi_{(a)}(t)) \to \beta_2$ as $k \to \infty$, and if $t \in (t_*,1)$, then $g^{2k}(\varphi_{(a)}(t)) \to \beta_1$ as $k \to \infty$. Consequently, $w_{(a)}(t)$ tends to the upper semicontinuous function.

$$\mathcal{H}\varphi_{(a)}(t) = \begin{cases} \beta_2, & \ll t \gg \in (0, t_*), \ll t \gg \in (t_* + 1, 2), \\ [\beta_1, \beta_2], & \ll t \gg = t_*, \ll t \gg = t_* + 1, \\ \beta_1, & \ll t \gg \in (t_*, t_* + 1), \end{cases}$$

which is periodic with period 2 and has one discontinuity in each unit-length interval (in general, the initial function of every non-constant solution of (24) with $\lambda = -1$ has exactly finitely many discontinuities in an unit-length interval).

Now, let us look at the case of $\lambda = -1.755$, which is represented by the solution $w_{(b)}(t)$. With this value of λ, the map g is very nearly the same as in the case of $\lambda = -1$, but with the following differences. The map g again has an attracting cycle, but of period 3; let it be denoted by $\{\gamma_1, \gamma_2, \gamma_3\}$.[5] The basin of this cycle is formally written in the same form as previously: $I \setminus D(g)$, but now the separator has a much more complicated structure: $D(g)$ is a Cantor-like set. For every point $z \in D(g)$, $z \neq -\alpha_0, \alpha_0$, its neighborhood expands in the limit to the interval $[\lambda, \lambda+\lambda^2]$. Therefore, $Q_g(z) = \{\gamma_1\} \cup \{\gamma_2\} \cup \{\gamma_3\}$ for $z \in I \setminus D(g)$ and $Q_g(z) = [\lambda, \lambda+\lambda^2]$ for $z \in D(g)$, $z \neq -\alpha_0, \alpha_0$. The limit function of $w_{(b)}(t)$ can be written out in the same way as it does in the case of $\lambda = -1$, but now in general form. Indeed, let $B_k(g)$ be for the basin of the point $z = \gamma_k$, $k = 1, 2, 3$, under the map g^3. The wanted function is given by

$$\mathcal{H}\varphi_{(b)}(t) = \begin{cases} \gamma_k, & \varphi_{(b)}(\ll t \gg) \in B_k(g), \\ [\lambda, \lambda + \lambda^2], & \varphi_{(b)}(\ll t \gg) \in D(g). \end{cases}$$

Hence, $\mathcal{H}\varphi_{(b)}(t)$ is periodic with period 3 and has an uncountable number of discontinuities in each unit-length interval (such are the limit functions corresponding to all but the constant solutions of (24) with $\lambda = -1.755$). Figure 4 shows how complicated the geometry of $\mathcal{H}\varphi_{(b)}(t)$ is.

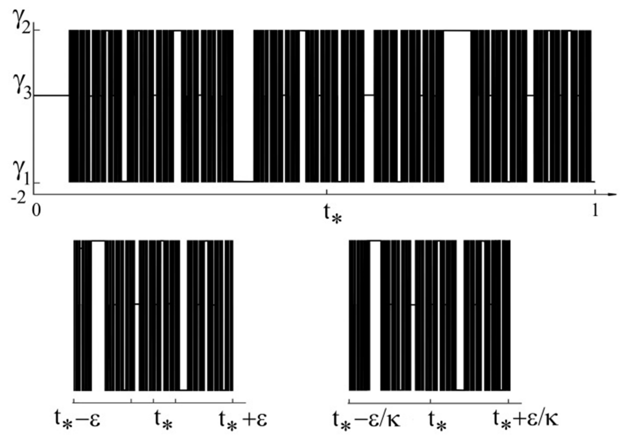

Figure 4. *Typical limit function for $\lambda = -1.755$.*

The graph is locally self-similar at discontinuity points, its fractal dimension is greater than 1 and less than 2 ($\varphi_{(b)}(t) = \lambda t$, $t_* = 0.52192605461574$, $\varepsilon = 0.00000003$, $\kappa = 1.83$)

The graph of $\mathcal{H}\varphi_{(a)}(t)$ consists of horizontal and vertical segments; horizontal segments "multiply" in the vicinity of each vertical segment (and thus, between any two horizontal segments, there is always a third one), moreover, the graph is locally self-similar at each discontinuity point (with its own coefficient of similarity).

3.3. Non-standard Properties of Solutions

As was noted above, solutions tending to discontinuous periodic functions are typical of (23). On the one hand, this fact points to the simple dynamics of typical solutions (they are asymptotically periodic), but on the other hand, it shows that typical solutions behave in complex and intricate ways on large time scales (because of the very complicated geometrical structure of their limit functions). Equation (24) has demonstrated that a step-by-step increasing in complexity of the solution $w_\varphi(t)$ depends crucially on the topological

5 The numerical values of β_1, β_2 and γ_1, γ_2, γ_3 can be found as being the zeros of the polynomial $(g^2(x) - x)/(g(x) - x)$ and $(g^3(x) - x)/(g(x) - x)$, respectively.

structure of the discontinuity set of its limit function $\mathcal{H}\varphi(t)$. It is the set

$$D(f, \varphi) = \{t = t_* + n: t_* \in \varphi^{-1}(D(f)), n = 0, 1, \ldots, p\}$$

with p being the period of $\mathcal{H}\varphi(t)$. Thus, the structure of $D(f, \varphi)$ is completely dominated by the set $\varphi^{-1}(D(f))$, that "takes by inheritance" the structure of the separator $D(f)$ and is the root cause of unusual-looking solutions. The main properties of these solutions are listed below; all but the first property occur only when the map f has a cycle of period different from 2^k, $k = 0, 1, 2, \ldots$

3.3.1. Gradient Catastrophe

The asymptotic discontinuity of solutions causes the gradient catastrophe: the derivative of a bounded solution increases indefinitely.

Think, for example, of the solutions in Figure 1. As for the solution $w_{(a)}(t)$, no matter how large $L > 0$ and small $\delta > 0$, one will find a number $T_* > 0$ such that if $T > T_*$ there exists a point $t_* \in [T, T+1]$ for which $|w'_{(a)}(t_*)| > \mathrm{L}$ and the oscillation ω_* of $w_{(a)}(t)$ in the vicinity of $t = t_*$ is δ-nearly equal to 1, i.e., $1 - \delta < \omega \ \leq 1$. Besides, for the solutions $w_{(b)}(t)$ and $w_{(c)}(t)$ (with the sets $D(f, \varphi_{(b)})$ and $D(f, \varphi_{(c)})$ containing infinitely many poins from any unit-length interval), the number of "gradient catastrophe" points and hence the number of undamped oscillations on $[T, T+1]$ increases ad infinitum as $T \to \infty$. This property is inherent in every non-asymptotically-, constant solution of (23). The oscillation build-up adds considerable complexity and leads to chaotization of the solutions behaviors.

3.3.2. Fractal Geometry of Solutions

In the above line of thought, it is natural to evaluate the degree of chaos in the solution $w_\varphi(t)$ based on its limit function $\mathcal{H}\varphi(t)$ and the topological structure of the set $\varphi^{-1}(D(f))$

If $\varphi^{-1}(D(f))$ is of positive fractal dimension, we should expect very high chaos. Indeed, as a set on the plane, the graph of $\mathcal{H}\varphi(t)$ is locally self-similar and, furthermore, fractal, i.e., its fractal dimension is greater than unity (see Figure 4 for illustration). This fact can be briefly described as follows. Let $\dim_{\text{box}}$ denote the box-counting dimension (a version of fractal dimension) and gr denote the graph of a function. If look at the limit functions $\mathcal{H}\varphi_{(a)}$, $\mathcal{H}\varphi_{(b)}$, $\mathcal{H}\varphi_{(c)}$ constructed above, it becomes seemingly obvious that

$$\dim_{\text{box}} \operatorname{gr} \mathcal{H}\varphi\big|_{[0,1]} = \dim_{\text{box}} \varphi^{-1}(D(f)) + 1.$$

Hence, if $\dim_{\text{box}} \varphi^{-1}(D(f)) > 0$, then the fractal dimension of gr $\mathcal{H}\varphi\big|_{[0,1]}$ — and hence of gr $\mathcal{H}\varphi\big|_{[T,T+1]}$ with any $T > 0$ — is greater than 1 and can even be equal to 2. In particular,

$$1 < \dim_{\text{box}} \operatorname{gr} \mathcal{H}\varphi_{(b)}\big|_{[T,T+1]} < 2, \ \dim_{\text{box}} \operatorname{gr} \mathcal{H}\varphi_{(c)}\big|_{[T,T+1]} = 2,$$

whereas $\dim_{\text{box}} \operatorname{gr} \mathcal{H}\varphi_{(a)}\big|_{[T,T+1]} = 1$.

These facts speak of the asymptotically fractal geometry of solutions. Figure 5 shows just how it runs. The solution $w_{(b)}(t)$ is again used as an example. The graph fragment (a) "under a magnifying glass" — at horizontal magnification — prove to be almost a copy of the graph on the interval [6, 7]. The fragment (b) "under a magnifying glass", in turn, looks like almost a copy of the graph on the interval [9, 10]. This same situation continues to repeat with step 3 along the t-axis (the step size is equal to the period of the limit function of $w_{(b)}(t)$, and the number of oscillations grows exponentially[6]).

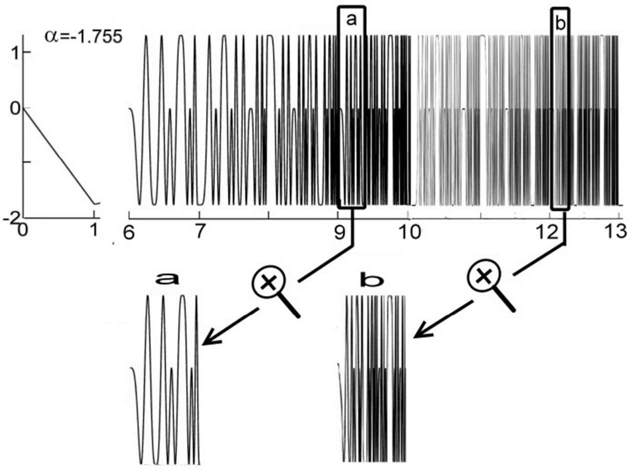

Figure 5. *Asymptotically fractal geometry of solutions for* $\lambda = -1.755$.

3.3.3. Space-Filling Property

If $\varphi^{-1}(D(f))$ is an interval, then the graph of the solution $w_\varphi(t)$ has dimension 1, but the graph of its limit function $\mathcal{H}\varphi(t)$ has dimension 2 and, moreover,

$$\dim_{\text{box}} \operatorname{gr} \mathcal{H}\varphi\big|_{[T,\, T+\delta]} = 2 \text{ for any } T \geq 0, \delta > 0.$$

Therefore, when t is sufficiently large, the curve $s = w_\varphi(t)$ behaves like a planar space-filling curve (i.e., a continuous curve that passes through each point of some square): for any $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that the graph gr w_φ passes at a distance less than $\varepsilon > 0$ from each point of the region $\Pi_{\varphi,T} = \{(t,s) \in \operatorname{gr} \mathcal{H}\varphi\big|_{[T,\infty]}\}$ when $t > T_\varepsilon$. Simply put, the graph of $w_\varphi(t)$ "tries" to completely fill the region $\Pi_{\varphi,T}$ as $T \to \infty$. The perfect example of this is afforded by the solution $w_{(c)}(t)$ of Figure 1(c); the greater T is, the denser gr $w_{(c)}$ fills up the region $\Pi_{\varphi,T} = [\mathrm{T},\infty) \times [-2,2]$. More generally, similar behavior is possessed by every solution $w_\varphi(t)$ for which $\varphi^{-1}(D(f))$ contains at least one interval. We refer to such solutions as *strongly chaotic*.

3.3.4. Going Beyond the Horizon of Predictability

For a strongly chaotic solution, there is no way to reliably calculate its values on large time scales. In this case, *the solution is said to be beyond the horizon of predictability* (where chaos asserts itself). We illustrate this again with the example of the solution $w_{(c)}(t)$ of Figure 1(c). Its limit function has the form $\mathcal{H}\varphi_{(c)} = [-2,2]$. Consequently, whatever $t, t' \in \mathbb{R}^+$, the values $w_{(c)}(t+n)$ and $w_{(c)}(t'+n)$

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6 The number of oscillations grows at a rate proportional to the rate of multiplication of the inverse images of points $z \in D(g)$ and, hence, proportional to $e^{\operatorname{ent} g}$ with $\operatorname{ent} g$ being the topological entropy of g, and $\operatorname{ent} g > 0$ for $\lambda = -1.755$.

"diverge" at a certain $n = n'$ by the amount nearly equal to the diameter of the interval $[-2, 2]$. This reasoning is formalized as follows: given $t \in \mathbb{R}^+$, for arbitrarily small $\varepsilon > 0$ and $\delta > 0$ there exists $K > 0$ such that

$$\sup |w_{(c)}(t + n) - w_{(c)}(t' + n)| \geq 4 - \delta, \text{ for } n > K.\ t': |t - t'| < \varepsilon$$

Thus, the calculation of the numerical values of a strongly chaotic solution becomes meaningless sooner or later and, as a rule, one has to resort to probability and statistics. The property of solutions "to go beyond the predictability horizon" captures the fact that many real-world systems will always behave in unexpected way — no matter how deep a study of them we have made.

3.3.5. Self-stochasticity Property

In its general sense, the self-stochasticity concept means that the asymptotic dynamics of a deterministic chaotic system can be described in probability terms [2, 5]. As regards (23), self-stochasticity consists in the existence of solutions that "go" beyond the predictability horizon but whose behavior is asymptotically accurately described by some random processes. A probabilistic description of unpredictable solutions is possible where the map f possesses a smooth (i.e., absolutely continuous with respect to Lebesgue measure) invariant measure. Self-stochasticity is physically realizable by (23) in the sense that there are natural one-parameter families of chaotic maps having a smooth invariant measure on a positive Lebesgue measure set of parameters.

To be specific, let us take (24) with $\lambda = -2$. Its associated map g: $z \mapsto z^2 - 2$ has the smooth invariant measure $\mu(dz) = -dz/\pi(4 - z^2)^{1/2}$ concentrated on $[-2, 2]$. The statistical behavior of almost all trajectories of g is known to be described by the same random variable with the distribution $F(A) = \mu(A)$, $A \subset I$. This random variable gives an asymptotically accurate estimate for the probability of a trajectory falling into a particular subset of $[-2, 2]$. This temporal stochasticity of the trajectories of g transforms into the spatio-temporal stochasticity of the solutions of (24).

Precisely speaking, every non-constant solution $w_\varphi(t)$ can be thought of as the continual beam of the trajectories $g^n(z)$ emanating from the points $z = \varphi(t)$, $t \in [0, 1]$. If $\varphi(t)$ is non-singular [7], then the ensemble of random variables corresponding to this beam is a random process[8], say $\mathcal{R}\varphi(t)$, which can be expected to describe the statistical behavior of $w_\varphi(t)$. It has been found that $w_\varphi(t)$ tends (in a specially constructed metric) to $\mathcal{R}\varphi(t)$ as $n \to \infty$; in particular, the probability of its value at some distant moment belonging to the interval $[a, b] \subset [-2, 2]$ is nearly equal to $F(t,b) - F(t,a)$, $F(t,s)$ being the one-dimensional distribution of $\mathcal{R}\varphi(t)$. The finite-dimensional distributions of $\mathcal{R}\varphi(t)$ are expressed in terms of the measure μ. The one-dimensional distribution has the form.

$$F(t,s) = -\frac{1}{\pi}\int_{-2}^{s}\frac{dz}{\sqrt{4-z^2}} = -\frac{2}{\pi}\arcsin\frac{1}{2}\sqrt{2-s}, \quad (26)$$

and is hence independent of φ. The higher-order distributions are normally φ-dependent. Thus, different solutions generally (but not always) have different limit random processes. Neglecting of autocorrelations (which are often not needed in applications), the asymptotic dynamics of almost all solutions (including $w_{(c)}(t)$ of Figure 1(c)) is described by one and the same simplified random process — the stationary random process with independent values, that specifies by the one-dimensional distribution (26). Here we observe forgetting initial data in its purest form.

3.3.6. Structure Formation

The non-asymptotically constant solutions of (23) have one more (extremely important) property — cascades from large to small structures (down to arbitrarily small scales) in their graphs. This property is clearly demonstrated with the help of (24). Figure 6 reveals how the solution $w_{(b)}(t)$ of Figure 1(b) changes in going from one plot section to another. The drawings show the successive stages of the cascading emergence of coherent (i.e., persisting in their form for relatively long periods) spatial structures. As smaller-scale structures arise, the larger-scale structures continue to exist. This leads to a cascading spatio-temporal hierarchy of coherent structures.

In each hierarchical chain, there are geometric similarities between the structures, which leads to the formation of a complicated fractal-like structure that is locally self-similar at different scale levels (scaling). The major factor responsible for this type of structuring processes is the highly complex topological-dynamical organization of the basin of the attracting cycle $\{\gamma_1, \gamma_2, \gamma_3\}$ of the map g.

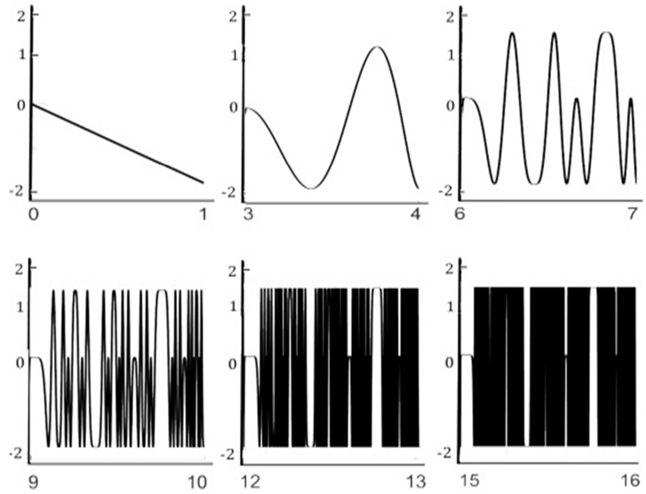

Figure 6. Cascade of structures in solutions for $\lambda = -1.755$.

The solution $w_{(c)}(t)$ of Figure 1(c) exemplifies another type of self-organization, where the emergence of smaller-scale structures is accompanied by the destruction of larger-scale structures (generation of smaller-scale structures from larger-

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7 We call a function $\varphi(t)$ non-singular if mes $\varphi^{-1}(A) = 0$ for mes $A = 0$ (mes is for the Lebesgue measure).

8 The fulfillment of Kolmogorov consistency conditions is ensured by the invariance of the measure μ under the map g and the nonsingularity of the initial functions $\varphi(t)$.

scale ones). As a consequence, from a certain moment the structures become so small that they cease to be coherent and their collective dynamics can be regarded as chaotic mixing. Structuring processes of this type arise because the unstable points of the map g are dense in the invariant interval I.

In case f is more complex than a unimodal map (say, bimodal), (23) usually has solutions with both types of self-structuring and, in addition, there are solutions with self-structuring of the former type on some intervals and of the latter type on others.

4. One Visual Example of Distributed Chaos in Boundary Value Problems

Let us replace (5) with one of its two-dimensional analogue and consider the problem.

$$u_\tau = u_x + u_y, \quad (27)$$

$$v_\tau = -v_x - v_y,\ x \in [0,1],\ y \in \mathbb{R},\ \tau \in \mathbb{R}^+.$$

$$u|_{x=0} = v|_{x=0},\ u|_{x=1} = f(v)|_{x=1}. \quad (28)$$

This problem can be considered as a model of a planar flow of a non-viscous medium in a channel of infinite length with a nonlinear interaction at the boundary. Figure 7, which presents the instantaneous streamlines of this flow, shows how the corresponding vector field $\left(u(x,y,\tau),\ v(x,y,\tau)\right)$ changes with time in the case where $f(z) = z^2 - 2$. Here we see a cascade leading to the emergence of smaller and smaller structures, which is accompanied by the destruction of larger structures. It seems like the flow in any spatial region is absolutely chaotic in the limit (as $\tau \to \infty$) and apparently, chaotic mixing occurs within the flow. All this is well described mathematically by reducing the boundary value problem to a difference equation. The general solution of (27) is given by

$$u(x,y,\tau) = w(x+\tau, y+\tau),\ v(x,y,\tau) = \hat{w}(x-\tau, y-\tau), \quad (29)$$

where w, $\hat{w}$ are arbitrary C^2-functions. The substitution of (29) in the boundary conditions shows that $w(t,\sigma)$ is a solution to the partial difference equation.

$$w(t+2, \sigma+2) = f\left(w(t,\sigma)\right),\ t \in [-1,0) \cup \mathbb{R}^+,\ \sigma \in \mathbb{R}. \quad (30)$$

Hence, every individual solution $u(x,y,\tau)$, $v(x,y,\tau)$ can be written as follows

$$u(x,y,\tau) = w(x+\tau, y+\tau), \quad (31)$$

$$v(x,y,\tau) = w(-x+\tau, y-2x+\tau),$$

where $w(t,\sigma)$ is the solution of (30) with the initial condition.

$$w(t,\sigma)\,|_{[-1,0)} = \tfrac{1}{2}\,\varphi_2(-t, \sigma-2t),\ \ w(t,\sigma)\,|_{[0,1)} = \tfrac{1}{2}\,\varphi_1(t,\sigma),$$

$$\varphi_1(x) = u(x,y,0),\ \varphi_2 = v(x,y,0).$$

Where $f(z) = z^2 - 2$, a typical solutions of (30) is strongly chaotic (it behaves as in Figure 1(c)). Its asymptotic dynamics is described with a random process (self-stochasticity property), and so is the dynamics of components u and v of each typical solution of (27),(28). The mentioned random process is defined by the smooth invariant measure $\mu(dz) = -\,dz/\pi(4 - z^2)^{\frac{1}{2}}$ of the map $z \mapsto z^2 - 2$. If autocorrelations are neglected, then the statistical properties of the vector field $\left(u(x,y,\tau),\ v(x,y,\tau)\right)$ are asymptotically the same as of the random vector field $\left(U(x,y),\ V(x,y)\right)$, both its components having the density $\eta(z) = -\,1/\pi(4 - z^2)^{\frac{1}{2}}$ (independent of x and y). The stream lines of the corresponding random flow satisfy the stochastic differential equations $x' = U(x,y)$, $y' = V(x,y)$.

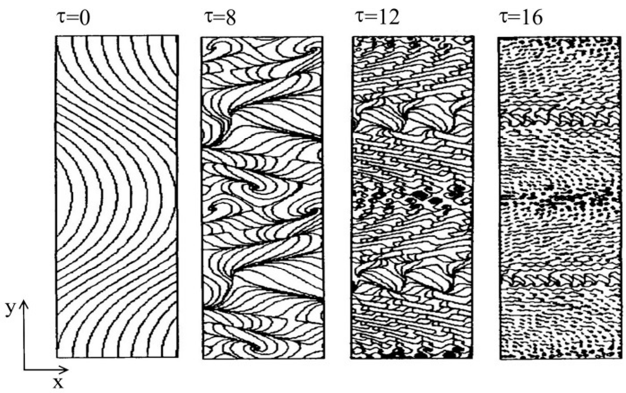

Figure 7. *One of the scenarios for the transition to distributed chaos.*

The application of continuous-time difference equations theory in the study of reducible and close-to-reducible boundary value problems will make it relatively easy to describe very complex space-time dynamics and understand what mathematical mechanisms can give rise to distributed chaos.

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