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On Discrete Functions and Repetitive Arrangements with Algorithms to Construct All Discrete Functions and a Practical Problem

Nicolae Popoviciu

Department of Informatics, Faculty of Informatics, Hyperion University, Bucharest, Romania

Email address:
popoviciunicolae15@yahoo.ro

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Abstract: The main idea of this work is based on the question: how can we control the electric circuits between a number of electric bulbs and a number of electric sources. This generates the correspondences between two discrete sets. The correspondence is based on the notion of discrete function and repetitive arrangements. The normal construction and notions of the work are introduced gradually and are detailed at every stage. Our constant endevour has been to ensure that every sentence in the work has a logical position. Here appears many questions: how to construct all discrete fuctions, which is the total number of these funtions, which is the relation between the number of bulbs and the number of sources, can we constract and control only a partial number of electric circuits (by direct access method) etc. The work answers all these questions by specialised algorithms: the construction algorithm and the decomposition algorithm. The algorithms use the rule from left to right to construct all possille discrete functions and, hence, all electric circuits. The decomposition algorithm supplies an access direct method. So we can control any part of the whole set of circuits. A lot of notions and specific notations are used to develop and illustrate the work. For combinations we have to show the constructon elements. A lot of examples explain this important notion. The work contains a lot of numerical examples and applications. The last section of the work deals with the bijective (and invertible) functions. Specialized notions and notations are used. Numerical examples and geometric designs illustrate the theory.

Keywords: Arrangements, Repetitive Arrangements, Discrete Functions, Decomposition Algorithm, Rule Left Right

1. Introduction

How to control k electric light bulbs $\{a_1, a_2, \cdots, a_k\}$ alimented (supplied) by n $\{b_1, b_2, \cdots, b_n\}$ sources of electricity. To control it means to know the form of electric circuit, namely the correspondence between these two sets.

In order to solve the above problem we use the discrtete functions [1, 3, 8] and we use some specific notations and remarks.

We denote $f: A \to B$, A -domain, B -codomain; with the discrete sets

$$A = \{a_1, a_2, \cdots, a_k\},\ B = \{b_1, b_2, \cdots, b_n\}$$

for any natural non-null numbers k and n; $card\ A = |A| = k$, $card\ B = |B| = n$, $a_i \neq a_j$ for any i and j. The set B is a multiple set [10].

The set of all functions $f: A \to B$ has the cardinal

$$card\,\{f\} = card\ B^{card\ A} = n^k \tag{1}$$

In applications we use $A = \{1, 2, \cdots, i, \cdots, k\}$, $B = \{1, 2, \cdots, j, \cdots, n\}$.

There are three cases: $k < n$, $k = n$, $k > n$ [10, 13].

The injective functions f could be obtained for $k \leq n$.

The surjective functions f could be obtained for $k \geq n$.

The bijective functions f could be obtained for $k = n$.

Our aim is to construct all n^k functios $f: A \to B$. Then we will analise the bijectice functions [5].

Remark 1. Permutations and arrangements are ordinate sets, while combinations are subsets of a set.

We present a short comparison between the usual arrangements of n objects taken k at a time and the repetitive arrangements (arrangements with repetition) of n objects taken k at a time [6, 7].

Usual arrangements. Example 1. $B = \{1,2,3\}$; $n = 3$, $k = 2$; $n \ge k$ (always).

Permutations $n = 3$; 123; 132; 213; 231; 312; 321; $n! = 3! = 6$.

Arrangements of $n = 3$ taken $k = 2$; 12; 21; 13; 31; 23; 32; $A_n^k = A_3^2 = 6$.

Combinations $n = 3$ taken $k = 2$; 12; 13; 23; $C_n^k = C_3^2 = 3$.

Repetitive arrangements [2, 11, 12]. Example 2. $B = \{1,2,3\}$; $n = 3$, $k = 2$ (any value n, k)

11; 12; 13; 21 22; 23;; 31; 32; 33; $n^k = 3^2 = 9$. Denote $n^k = N$.

Example 3. $B = \{1,2\}$; $n = 2$, $k = 3$; $n \le k$.

111; 112; 121; 211; 122; 221; 212; 222; $n^k = 2^3 = 8$.

Another notation for the total number of all repetitive arrangements is $\overline{A}_n^k = n^k = a_n^k = N$.

2. Problem Formulation

Related with this work we have two aims.

1) Aim 1. We have to construct the set of all discrete functions $f: A \to B$,

$f: \{a_1, a_2, \cdots, a_k\} \to \{b_1, b_2, \cdots, b_n\}$ for any non-zero natural numbers n, k .

The total number of these functions is denoted $\overline{A}_n^k = n^k = N$.

There are several methods to constract all discrete functions f .

We propose a method based on direction left to right in the set B , and elaborate the algorithm left – right (algorithm 1). It is a sequential method.

2) Aim 2. We make a decomposition of $\overline{A}_n^k = n^k$ based on the decomposition algorithm (algorithm 2). It is a direct acces method, i.e. we can construct any subset of the hole set with $\overline{A}_n^k = n^k$ functions.

Examples. Some particular cases and the total number of functions.

Nr	k	n	n^k	$f: A \to B$	
1	1	2	nu	$1 \to 1, 2$	$N = 2$.
2	2	1	1	$1, 2 \to 1$	$N = 1$.
3	2	2	4	$1, 2 \to 1, 2$	$N = 4$.
4	3	2	8	$1, 2, 3 \to 1, 2$	$N = 8$.
5	4	2	16	$1, 2, 3, 4 \to 1, 2$	$N = 16$.
6	5	2	32	$1, 2, 3, 4, 5 \to 1, 2$	$N = 32$.
7	2	3	9	$1, 2 \to 1, 2, 3$	$N = 9$.
8	3	3	27	$1, 2, 3 \to 1, 2, 3$	$N = 27$.
9	4	3	81	$1, 2, 3, 4 \to 1, 2, 3$	$N = 81$.
10	5	3	243	$1, 2, 3, 4, 5 \to 1, 2, 3$	$N = 243$.
11	2	4	16	$1, 2 \to 1, 2, 3, 4$	$N = 16$.
12	3	4	64	$1, 2, 3 \to 1, 2, 3, 4$	$N = 48$.
13	4	4	256	$1, 2, 3, 4 \to 1, 2, 3, 4$	$N = 256$.
14	5	4	1024	$1, 2, 3, 4, 5 \to 1, 2, 3, 4$ etc.	$N = 1024$.

Remark 2. Tthe total number of functions increases very quicly with *k* and *n*.

3. The Algorithm Left - Right to Construct All Discrete Functions

The functions are $f: A \to B$, $A = \{a_1, a_2, \cdots, a_k\}$, $B = \{b_1, b_2, \cdots, b_n\}$, where $k \ge 1$ and $n \ge 1$ are given natural numbers.

The total numner of functions f is the total number of repetitive arrangements.

The rule from left to right [7] means: if the fix value is j or b_j , then, for combinations one uses only the values S ={ b_{j+1} , b_{j+2} , $\cdots, b_n$ } from the right part of b_j .

Remark 3. The algorithm is based on combibations and the moving rule down-up.

Example.

1) 1, 2, 3 $\to$ 1, 2, 3; C_3^1 =3; $f(1) = 2 \quad 1 \quad 3;$
$f(2) = 1 \quad 3 \quad 2;$
$f(3) = 3 \quad 2 \quad 1.$

2) 1, 2, 3 $\to$ a, b, c, d; C_3^1 =3; $f(1) = \mathrm{b} \quad \mathrm{b} \quad \mathrm{d};$
$f(2) = \mathrm{b} \quad \mathrm{d} \quad \mathrm{b}$
$f(3) = \mathrm{d} \quad \mathrm{b} \quad \mathrm{b}.$

3.1. Exemples of All Discrete Functions for Small Numbers k and n

Application 3.1. We use the algorithm 1. It is a sequential method. (Version 1).

1) $f:\ 1, 2 \rightarrow 1, 2,\ N = 2^2 = 4$

Nr	1	2 3	4
$f(1) =$	1	1 2	2
$f(2) =$	1	2 1	2

2) $f:\ 1, 2, 3 \rightarrow 1, 2,\ N = 2^3 = 8$

Nr	1	2 3 4	5 6 7	8
$f(1) =$	1	1 1 2	1 2 2	2
$f(2) =$	1	1 2 1	2 2 1	2
$f(3) =$	1	2 1 1	2 1 2	2

On the line Nr we count the current number of function f or its address.

3) $f:\ 1, 2, 3, 4 \rightarrow 1, 2,\ N = 2^4 = 16$ functions (small number).

Nr	1	2 3 4 5	6 7 8 9 0 1	2 3 4 5	16
$f(1) =$	1	1 1 1 2	1 1 2 2 1 2	1 2 2 2	2
$f(2) =$	1	1 1 2 1	1 2 2 1 2 1	2 2 2 1	2
$f(3) =$	1	1 2 1 1	2 2 1 1 1 2	2 2 1 2	2
$f(4) =$	1	2 1 1 1	2 1 1 2 2 1	2 1 2 2	2

4) $f:\ 1, 2 \rightarrow 1, 2, 3, 4,\ N = 4^2 = 16$ functions (small number).

Nr	1	23	45	67	8	90	12	3	45	16
$f(1) =$	1	12	13	14	2	23	24	3	34	4
$f(2) =$	1	21	31	41	2	32	42	3	43	4

5) $f:\ 1, 2, 3, 4, 5 \rightarrow 1, 2,\ N = 2^5 = 32$ functions (smal number).

Nr	1	2 3 4 5 6	7 8 9 0 1 2 3 4 5 16	17 8 9 0 1 2 3 4 5 26
$f(1) =$	1	1 1 1 1 2	1 1 1 2 2 1 1 2 1 2	1 1 2 2 2 1 2 1 2 2
$f(2) =$	1	1 1 1 2 1	1 1 2 2 1 1 2 1 2 1	1 2 2 2 1 2 1 2 2 1
$f(3) =$	1	1 1 2 1 1	1 2 2 1 1 2 1 2 1 1	2 2 2 1 1 1 2 2 1 2
$f(4) =$	1	1 2 1 1 1	2 2 1 1 1 1 2 1 1 2	2 2 1 1 2 2 2 1 2 1
$f(5) =$	1	2 1 1 1 1	2 1 1 1 2 2 1 1 2 1	2 1 1 2 2 2 1 2 1 2

Nr	27 8 9 0 1	32
$f(1) =$	1 2 2 2 2	2
$f(2) =$	2 2 2 2 1	2
$f(3) =$	2 2 2 1 2	2
$f(4) =$	2 2 1 2 2	2
$f(5) =$	2 1 2 2 2	2

6) $f:\ 1, 2, 3 \rightarrow 1, 2, 3,\ N = 3^3 = 27$ functions (small number).

Nr	1	2 3 4	5 6 7	8 9 0	1 2 3	4 5 6	7 8 9	0	1 2 3	4 5 6	7=27
$f(1) =$	1	1 1 2	1 1 3	1 2 2	1 2 3	1 3 2	1 3 3	2	2 2 3	2 3 3	3
$f(2) =$	1	1 2 1	1 3 1	2 2 1	2 3 1	3 2 1	3 3 1	2	2 3 2	3 3 2	3
$f(3) =$	1	2 1 1	3 1 1	2 1 2	3 1 2	2 1 3	3 1 3	2	3 2 2	3 2 3	3

Remark 4. For $k = 3$ and $n = 3$ a short summary of the basic ideea of computations has the following form

$f(1) = 1$	: 1	: 1	: 2	: 2	: 2	: 3
$f(2) = 1$	: 1	: x	: 2	: 2	: x	: 3
$f(3) = 1$	: x	: y	: 2	: x	: y	: 3

$C_3^3(1;\theta)$; $C_3^2(1;x)$; $C_3^1(1;x,y)$; $C_3^3(2;\theta)$; $C_3^2(2;x)$; $C_3^1(2;x,y)$; $C_3^3(3;\theta)$

The symbol θ is empty set and $x, y, \cdots \in S$.

3.2. Exemples of All Discrete Functions for Great Numbers k and n

We use a modified sequential method. (Version 2) [4, 5].

Use successively the rules a), b), c), d), e).

Use the subset $S \subset B$ with all elements from the right part of j.

Denote by L the completion length, $0 \le L \le k-1$.

Construct all repetitive arrangements of S having the length L, $L = 0, 1, 2 \cdots$ etc.

Compute the total number of arrangements $(n-j)^L$. Denote $(n-j)^L = R$.

Increase the last address with the value $R\,x\,C_k^{k-L}$

Construct all bijective functions corresponding to the number $(n-j)^L$.

Application 3.2.1. $f:\ 1, 2, 3 \rightarrow 1, 2, 3, 4, 5,\ k = 3; n = 5$; $N = 5^3 = 125$ functions (great number).

Step $j = 1$, $L = 0$; $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 1 (first address)

$f(1) = 1$

$f(2) = 1$

$f(3) = 1$

$j = 1$, $L = 1$, $S = \{2, 3, 4, 5\}$, $(n-j)^L = (5-1)^1 = 4 = R$; use 2 3 4 5

and the arrangements 2, 3, 4, 5; $C_k^{k-L} = C_3^{3-L} = 3$.

Increase the last address 1 with the value R x3=12; 1+12=13; true.

Nr	2 3 4	5 6 7	8 9 10	11 12 13 (addresses)
$f(1) =$	1 1 2	1	1	1 1 5
$f(2) =$	1 2 1	1	1	1 5 1
$f(3) =$	2 1 2	3	4	5 1 1

$j = 1$, $L = 2$, $S = \{2, 3, 4, 5\}$, $(n-j)^L = (5-1)^2 = 16 = R$; use 2 3 4 5 and the arrangements

2 2 2 2 3 3 3 3 4 4 4 4 5 5 5 5 ; $C_k^{k-L} = C_3^{3-L} = 3$.

2 3 4 5 2 3 4 5 2 3 4 5 2 3 4 5 ;

Increase the last address 13 with the value R x3=48; 13+48=61; true.

Nr	14 15 16	17 18 19	20 21 22	23 24 25	……	56 57 58	59 60 61 (addresses)
$f(1) =$	1 2 2	1	1	1 3 2		1 5 4	1 5 5
$f(2) =$	2 2 1	2	2	3 2 1		5 4 1	5 5 1
$f(3) =$	2 1 2	3	4	2 1 3		4 1 5	5 1 5

Step j=2, $L=0$. $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 62 (new address)

$f(1)=2$

$f(2)=2$

$f(3)=2$

j=2, $L=1$, S={3, 4, 5}, $(n-j)^L=(5-2)^1=3=R$; use 3, 4, 5

and the arrangements 3, 4, 5; $C_k^{k-L} = C_3^{3-L} = 3$.

Increase the last address 62 with the value R x3=9; 62+9=71; true.

Nr	63 64 65	66 67 68	69 70 71	(addresses)
$f(1)$ =	2 2 3	2	2 2 5	
$f(2)$ =	2 3 2	2	2 5 2	
$f(3)$ =	**3** 2 2	**4**	**5** 2 2	

j=2, $L=2$, S={ 3,4,5 }, $(n-j)^L=(5-2)^2=9=R$, use 3, 4, 5 and the arrangements

3 3 3 4 4 4 5 5 5 ; $C_k^{k-L} = C_3^{3-L} = 3$.

3 4 5 3 4 5 3 4 5 ;

Increase the last address 71 with the value R x3=9; 71+27=98; true.

Nr	72 73 74	75 76 77	78 79 80 ······ .93 94 95	96 97 98	(addresses)
$f(1)$ =	2 3 3	2	2 3 5 ······ 2	2 5 5	
$f(2)$ =	**3** 3 2	**3**	**3** 5 2 ······ **5**	**5** 5 2	
$f(3)$ =	**3** 2 3	**4**	**5** 2 3 ······ **4**	**5** 2 5	

Step j=3, $L=0$; $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 99 (new address)

$f(1)=3$

$f(2)=3$

$f(3)=3$

j=3, $L=1$, S={ 4,5 }, $(n-j)^L=(5-3)^1=2=R$; use 4,5

and the arrangements 4, 5; $C_k^{k-L} = C_3^{3-L} = 3$.

Increase the last address 99 with the value R x3=6; 99+6=105; true.

Nr	100 101 102	103 104 105	(addresses)
$f(1)$ =	3 3 4	3 3 5	
$f(2)$ =	**3** 4 3	**3** 5 3	
$f(3)$ =	**4** 3 3	**5** **3** 3	

j=3, $L=2$, S={ 4,5 }, $(n-j)^L=(5-3)^2=4=R$; use 4,5

and the arrangements 4 4 5 5 ; $C_k^{k-L} = C_3^{3-L} = 3$.

4 5 4 5 ;

Increase the last address 105 with the value R x3=12; 105+12=117; true.

Nr	106 107 108	109 110 111	112 113 114	115 116 117	(addresses)
$f(1)$ =	3 4 4	3 4 5	3	3 5 5	
$f(2)$ =	**4** 4 3	**4** 5 3	**5**	**5** 5 3	
$f(3)$ =	**4** 3 4	**5** **3** 4	**4**	**5** 3 5	

Step j=4, $L=0$; $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 118 (new address)

$f(1)=4$

$f(2)=4$

$f(3)=4$

j=4, $L=1$, S={ 5 }, $(n-j)^L=(5-4)^1=1=R$; use 5

and the arrangements 5; $C_k^{k-L} = C_3^{3-L} = 3$.

Increase the last address 118 with the value R x3=3; 118+3=121; true.

Nr	119 120 121	(addresses)
$f(1)$ =	4 4 5	
$f(2)$ =	4 5 4	
$f(3)$ =	**5** 4 4	

j=4, $L=2$, S={ 5 }, $(n-j)^L=(5-4)^2=1=R$; use 5

and the arrangements 5 ; $C_k^{k-L} = C_3^{3-L} = 3$.

5 ;

Increase the last address 121 with the value R x3=3; 121+3=124; true.

Nr	122 123 124	(addresses)
$f(1)$ =	4 5 5	
$f(2)$ =	**5** 5 4	
$f(3)$ =	**5** 4 5	

Step j=, $L=0$; $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 125 (last address). End of computing. Stop. N=125.

$f(1)=5$

$f(2)=5$

$f(3)=5$

Application 3.2.2. f: 1, 2, 3, 4, 5 $\to$ 1, 2, N=32

Nr	1	2 3 4 5 6	7 8 9 0 1 2 3 4 5 16	17 8 9 0 1 2 3 4 5 6=26
$f(1)$ =	1	1 1 1 1 2	1 1 1 2 2 1 1 2 1 2	1 1 2 2 2 1 2 1 2 2
$f(2)$ =	1	1 1 1 2 1	1 1 2 2 1 1 2 1 2 1	1 2 2 2 1 2 1 2 2 1
$f(3)$ =	1	1 1 2 1 1	1 2 2 1 1 2 1 2 1 1	2 2 2 1 1 1 2 2 1 2
$f(4)$ =	1	1 2 1 1 1	2 2 1 1 1 1 2 1 1 2	2 2 1 1 2 2 2 1 2 1
$f(5)$ =	1	2 1 1 1 1	2 1 1 1 2 2 1 1 2 1	2 1 1 2 2 2 1 2 1 2

Nr	27 8 9 0 1	2=32
$f(1)$ =	1 2 2 2 2	2
$f(2)$ =	2 2 2 2 1	2
$f(3)$ =	2 2 2 1 2	2
$f(4)$ =	2 2 1 2 2	2
$f(5)$ =	2 1 2 2 2	2

$$N = C_5^5(1;\theta) + C_5^4(1;2) + C_5^3(1;2,2) + C_5^2(1;2,2,2) + C_5^1(1;2,2,2,2) + C_5^5(2;\theta)$$

N=1+5+10+10+5+1=32 positions=32 functions.

Application 3.2.3. $f:\ 1, 2, 3 \rightarrow 1, 2, 3,\ N=27$

Nr	1 234 567 890 123 456 789 0 123 456 7=27
$f(1)=$	1 112 113 122 123 132 133 2 223 233 3
$f(2)=$	1 121 131 221 231 321 331 2 232 332 3
$f(3)=$	1 211 311 212 312 213 313 2 322 323 3

$$N = C_3^3(1;\theta) +(C_3^2(1;2) + C_3^2(1;3))+(C_3^1(1;2,2) + C_3^1(1;2,3) + C_3^1(1;3,2) + C_3^1(1;3,3))+$$

$$+ C_3^3(2;\theta) + C_3^2(2;3) + C_3^1(2;3,3) + C_3^3(3;\theta)$$

N =1+(3+3)+(3+3+3+3)+1+3+3+1=27 positions=27 functions.

Application 3.2.4. $f:\ 1, 2, 3 \rightarrow 1, 2, 3, 4,\ N=64$

Nr	1 234 567 890 123 456 789 012 345 678 901 234 567
$f(1)=$	1 112 113 114 122 123 124 132 133 134 142 143 144
$f(2)=$	1 121 131 141 221 231 241 321 331 341 421 431 441
$f(3)=$	1 211 311 411 212 312 412 213 313 413 214 314 414

Nr	38 901 234 567 890 123 456 7 890 123 4=64
$f(1)=$	2 223 224 233 234 243 244 3 334 344 4
$f(2)=$	2 232 242 332 342 432 442 3 343 443 4
$f(3)=$	2 322 422 323 423 324 424 3 433 434 4

$$N = C_3^3(1;\theta) +[C_3^2(1;2) + C_3^2(1;3) + C_3^2(1;4)]+[C_3^1(1;2,2) + C_3^1(1;2,3) + C_3^1(1;2,4) +$$

$$+ C_3^1(1;3,2) + C_3^1(1;3,3) + C_3^1(1;3,4) + C_3^1(1;4,2) + C_3^1(1;4,3) + C_3^1(1;4,4)]+$$

$$+ C_3^3(2;\theta) +[C_3^2(2;3) + C_3^2(2;4)]+[C_3^1(2;3,3) + C_3^1(2;3,4) + C_3^1(2;4,3) +$$

$$+ C_3^1(2;4,4)]+\ C_3^3(3;\theta) + C_3^2(3;4) + C_3^1(3;4,4) + C_3^3(4;\theta)$$

There are a number of 10 groups of combinations.
N =1+[3+3+3]+[3+3+3+3+3+3+3+3+3]+1+[3+3]+[3+3+3+3]+1+3+3+1==64 positions=64 functions. True.

Application 3.2.5. $f:\ 1, 2, 3, 4 \rightarrow 1, 2, 3,\ N=81$

Nr	1 2345 6789 012345 678901 234567 890123 4567=37
$f(1)=$	1 1112 1113 112212 112312 113213 113313 1222
$f(2)=$	1 1121 1131 122121 123121 132131 133131 2221
$f(3)=$	1 1211 1311 221112 231113 321112 331113 2212
$f(4)=$	1 2111 3111 211221 311231 211321 311331 2121 etc .

Now, for this case, we describe the decomposition of repetitive arrangements.

$$N = C_4^4(1;\theta) +[C_4^3(1;2) + C_4^3(1;3)]+[C_4^2(1;2,2) + C_4^2(1;2,3) + C_4^2(1;3,2) + C_4^2(1;3,3)]+$$

$$+[C_4^1(1;2,2,2) + C_4^1(1;2,2,3) + C_4^1(1;2,3,2) + C_4^1(1;2,3,3) + C_4^1(1;3,2,2) +$$

$$+ C_4^1(1;3,2,3) + C_4^1(1;3,3,2) + C_4^1(1;3,3,3)]+ C_4^4(2;\theta) + C_4^3(2;3) + C_4^2(1;3,3) +$$

$$+ C_4^1(1;3,3,3) + C_4^4(3;\theta)\ .$$

There are a number of 9 groups of combinations.
N =1+[4+4]+[6+6+6+6+6]+[4+4+4+4+4+4+4+4]+1+4+6+4+1
N =1+[8]+[24]+[32]+1+4+6+4+1 = 81 positions=81 functions. True

4. General Case for Left – Right Algorithm and Decomposition Algorithm

The bijective functions are $f: A \to B$, $A = \{a_1, a_2, \cdots, a_k\}$, $B = \{b_1, b_2, \cdots, b_n\}$.

All combinations have the form C_k^i, $i = 1, k$; C_k^k, C_k^{k-1}, $C_k^{k-2}, \cdots, C_k^1$=k; C_k^1=0.

and the function f is defined for $f(a_1)$, $f(a_2)$, …, $f(a_k)$.

Remark 5. [7, 9]

$C_k^k(b_1;\theta)$, $k + card\{\theta\} = k$;

$C_k^{k-1}(b_1;x)$, for x: b_2, b_3, …, b_n; $k-1+card\{x\} = k$;

$C_k^{k-2}(b_1;x,y)$; for x,y: $b_2,b_2;b_2,b_3;etc.$; $k-2+card\{x,y\} = k$;

$C_k^{k-3}(b_1;x,y,z)$, $k-3+card\{x,y,z\} = k$ etc.

We intruduce some adequate notations and we analyze the algorithm for $k=4$ and $n=4$, $N=256$. (Version 3).

Number m is used for combinations C_k^m, $m = k, k-1, \cdots, 3, 2, 1$.

j or b_j is the current number from $C_k^m(b_j;\cdots)$, where $m \le k$.

q indicates the number of completion elements taken from the set B in arragements, with $m+q=k$.

p indicates the total elements from set $B = \{b_1, b_2, \cdots, b_j, b_{j+1}, \cdots, b_n\}$ having the position in the right of b_j .; $p = n - j$. We include the parameters p and q in the above notation and obtain

$$C_k^m(b_j;p;q). \tag{2}$$

Verification:

$$p = n - j,\ m + q = k \tag{3}$$

p^q is the partial number of repetitive arrangaments for

$$m, b_j, p, q \tag{4}$$

$T(b_j;p;q)$ (natural number) is the last partial address genarated by combinations

$$C_k^m(b_j;p;q).$$

$$T(b_j;p;q) = C_k^m\ p^q \tag{5}$$

The sum of all $T(b_j;p;q)$ is N. [13, 14]

Application 4.1. Input data: $k=4$ and $n=4$, $N=256$; f: 1, 2, 3, 4 $\to$ 1, 2, 3, 4.

For verification we use the formulas (1), (2), (3), (4).

$$C_4^4(1;\theta) = 1,\ T(1;\theta) = 1.$$

$C_4^3(1;p=3;q=1) = 4$, $p^q = 3$, $T(1;p=3;q=1) = C_4^3\ p^q$ =4 x 3=12 (product).

$p=3, q=1 \Rightarrow$ completion values are 2, 3, 4.

$C_4^2(1;p=3;q=2) = 6$, $p^q = 9$, $T(1;p=3;q=2) = C_4^2\ p^q$ =6 x 9=54.

$p=3, q=2 \Rightarrow$ completion values are 22, 23, 24; 32, 33, 34; 42, 43, 44.

$C_4^1(1;p=3;q=3) = 4$, $p^q = 27$, $T(1;3;q=3) = C_4^1\ p^q$ =4 x 27=108.

$p=3, q=3 \Rightarrow$ completion values are 222, 223, 224; 232, 233, 234; 242, 243, 244 etc.

$$C_4^4(2;\theta) = 1,\ T(2;\theta) = 1.$$

$C_4^3(2;p=2;q=1) = 4$, $p^q = 2$, $T(2;p=2;q=1) = C_4^3\ p^q$ =4 x 2=8.

$p=2, q=1 \Rightarrow$ completion values are 3, 4.

$C_4^2(2;p=2;q=2) = 6$, $p^q = 4$, $T(2;p=2;q=2) = C_4^2\ p^q$ =6 x 4=24.

$p=2, q=2 \Rightarrow$ completion values are, 33, 34; 43, 44.

$C_4^1(2;p=2;q=3) = 4$, $p^q = 8$, $T(2;p=2;q=3) = C_4^1\ p^q$ =4 x 8=32.

$p=2, q=3 \Rightarrow$ completion values are 333, 334; 343, 344; 433, 434; 443, 444.

$$C_4^4(3;\theta) = 1\ ,\ T(3;\theta) = 1\ .$$

$$C_4^3(3; p = 1; q = 1) = 4\ ,\ p^q = 1\ ,\ T(3; p = 1; q = 1) = C_4^3\ p^q = 4 \text{ x } 1 = 4.$$

$p = 1, q = 1 \Rightarrow$ completion values are 4.

$$C_4^2(3; p = 1; q = 2) = 6\ ,\ p^q = 1\ ,\ T(3; p = 1; q = 2) = C_4^2\ p^q = 6 \text{ x } 1 = 6.$$

$$C_4^1(3; p = 1; q = 3) = 4\ ,\ p^q = 1\ ,\ T(3; p = 1; q = 3) = C_4^1\ p^q = 4 \text{ x } 1 = 4.$$

$p = 1, q = 3 \Rightarrow$ completion values are 444.

$$C_4^4(4;\theta) = 1\ ,\ T(4;\theta) = 1\ .$$

Verification.

$$T(1;\theta) + T(1; p = 3; q = 1) + T(1; p = 3; q = 2) + T(1; p = 3; q = 3) ++ T(2;\theta) + T(2; p = 2; q = 1)$$

$$+ T(2; p = 2; q = 2) + T(2; p = 2; q = 3) ++ T(3;\theta) + T(3; p = 1; q = 1) + T(3; p = 1; q = 2) + T(3; p = 1; q = 3) + T(4;\theta) = N\ .$$

$$+ T(3; p = 1; q = 2) + T(3; p = 1; q = 3) + T(4;\theta) = N$$

1+12+54+108+1+8+24+32+1+4+6+4+1=256; N =256. True.

Application 4.2. f : 1, 2, 3, 4 $\rightarrow$ 1, 2, 3, 4, N =256

We describe only the decomposition of repetitive arrangements.

There are a number of 60 groups of combinations.

$$N = C_4^4(1;\theta) +[\ C_4^3(1;2) + C_4^3(1;3) + C_4^3(1;4)\ ]\ +\ [\ C_4^2(1;2,2) + C_4^2(1;2,3) + C_4^2(1;2,4) +$$

$$+ C_4^2(1;3,2) + C_4^2(1;3,3) + C_4^2(1;3,4) + C_4^2(1;4,2) + C_4^2(1;4,3) + C_4^2(1;4,4)\ ]+$$

$$+[\ C_4^1(1;2,2,2) + C_4^1(1;2,2,3) + C_4^1(1;2,2,4) + C_4^1(1;2,3,2) + C_4^1(1;2,3,3) +$$

$$+ C_4^1(1;2,3,4) + C_4^1(1;2,4,2) + C_4^1(1;2,4,3) + C_4^1(1;2,4,4) + C_4^1(1;3,2,2) +$$

$$+ C_4^1(1;3,2,3) + C_4^1(1;3,2,4) + C_4^1(1;3,3,2) + C_4^1(1;3,3,3) + C_4^1(1;3,3,4) +$$

$$+ C_4^1(1;3,4,2) ++ C_4^1(1;3,4,3) + C_4^1(1;3,4,4) + C_4^1(1;4,2,2) + C_4^1(1;4,2,3) +$$

$$+ C_4^1(1;4,2,4) ++ C_4^1(1;4,3,2) + C_4^1(1;4,3,3) + C_4^1(1;4,3,4) + C_4^1(1;4,4,2) +$$

$$+ C_4^1(1;4,4,3) + C_4^1(1;4,4,4)\ ]+ C_4^4(2;\theta) +\ [\ C_4^3(2;3) + C_4^3(2;4)\ ]+$$

$$+[\ C_4^2(2;3,3) + C_4^2(2;3,4) + C_4^2(2;4,3) + C_4^2(2;4,4)\ ]+$$

$$+[\ C_4^1(2;3,3,3) + C_4^1(2;3,3,4) + C_4^1(2;3,4,3) + C_4^1(2;3,3,4) + C_4^1(2;4,3,3) +$$

$$+ C_4^1(2;4,3,4) + C_4^1(2;4,4,3) + C_4^1(2;4,4,4)\ ]+ C_4^4(3;\theta) +\ C_4^3(3;4) +$$

$$+ C_4^2(3;4,4) + C_4^1(3;4,4,4) + C_4^4(4;\theta)\ .$$

N =1+(4x3)+(6x9)+(4x27)+1+(4x2)+(6x4)+(4x8)+1+4+6+4+1=256 positions, 256 fuinctions.

5. The Discrete Functions Applied to the Practical Problem

Application 5.1 Construct all electric circuits for 3 electric bulbs and 2 sources. The model is $f : \{a_1, a_2, a_3\} \rightarrow \{b_1, b_2\}$ or $f : \{1,2,3\} \rightarrow \{1,2\}$; $N = 8$.

Solution.

Nr	1	2	3	4	5	6	7	8	(circuit addresses)
$f(1)=$	1	1	1	2	1	2	2	2	
$f(2)=$	1	1	2	1	2	2	1	2	
$f(3)=$	1	2	1	1	2	1	2	2	

The figure 1 shows all the circuits.

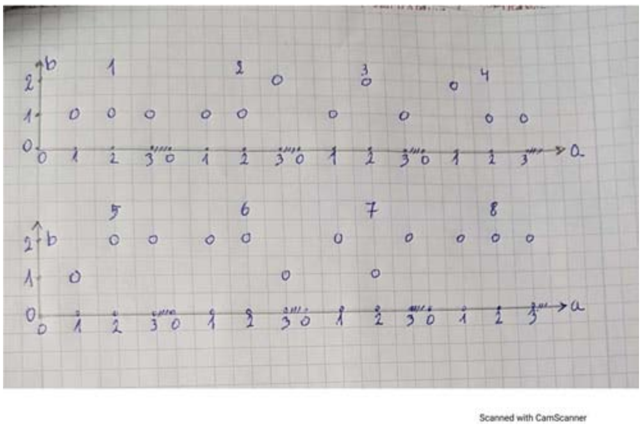

Figure 1. *Electric circuits from application 5.1.*

Application 5.2 Construct all electric circuits for 2 electric bulbs and 3 sources. The model is $f:\ \{a_1,a_2\} \to \{b_1,b_2,b_3\}$ or $f:\{1,2\}\to\{1,2,3\}$; $N=9$.

Solution.

Nr	1	2	3	4	5	6	7	8	9	(circuit addresses)
$f(1)=$	1	1	2	1	3	2	2	3	3	
$f(2)=$	1	2	1	3	1	2	3	2	3	

The figure 2 shows all the circuits from application 5.2, by using the orthogonal axix O a b.

All other cases generates electical circuits like the above circuits.

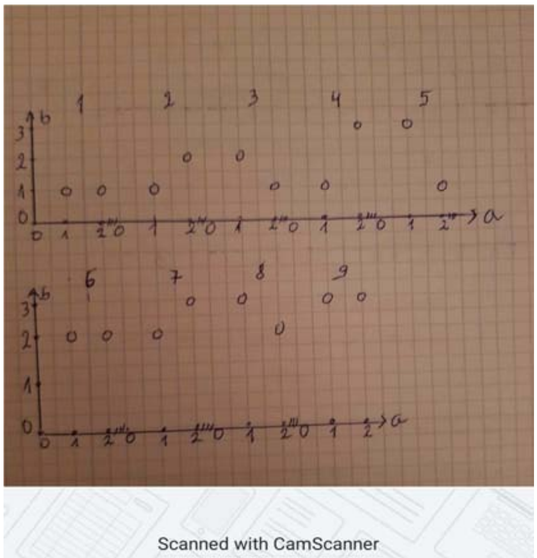

Figure 2. *Electric circuits from application 5.2.*

6. Bijective Functions

We use $k=n$; $f:\ \{a_1,a_2,\cdots,a_n\} \to \{b_1,b_2,\cdots,b_n\}$ $n\geq 1$.

The total number of all bijective functions is $N_1 = n!$.

Application 6.1. $f:1,\ 2,\ 3 \to 1,\ 2,\ 3$; $N_1 = 3! = 6$; $C_3^1(1,2,3) + C_3^1(1,3,2) = 3+3=6$.

Nr	1	2	3	4	5	6	(addresses)
$f(1)=$	1	2	3	1	3	2	
$f(2)=$	2	3	1	3	2	1	
$f(3)=$	3	1	2	2	1	3.	

Application 6.2. $f:1,\ 2,\ 3,\ 4 \to 1,\ 2,\ 3,\ 4$; $N_1 = 4! = 24$ (bijective functions).

$$C_4^1(1,2,3,4) + C_4^1(1,2,4,3) + C_4^1(1,3,2,4) + C_4^1(1,3,4,2) + C_4^1(1,4,2,3) + C_4^1(1,4,3,2) =$$

$$+\ 4+4+4+4+4+4=24;\ C_4^1 = 4.$$

Nr	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$f(1)=$	1	2	3	4	1	2	4	3	1	3	2	4	1	3	4	2	1	4	2	3	1	4	3	2
$f(2)=$	2	3	4	1	2	4	3	1	3	2	4	1	3	4	2	1	4	2	3	1	4	3	2	1
$f(3)=$	3	4	1	2	4	3.	1	2	2	4	1	3	4	2	1	3	2	3	1	3	3	2	1	4
$f(4)=$	4	1	2	3	3	1	2	4	4	1	3	2	2	1	3	4	3	1	4	2	2	1	4	3.

Application 6.3. $f:1,\ 2,\ 3,\ 4,\ 5 \to 1,\ 2,\ 3,\ 4,\ 5$; $N_1 = 5! = 120$ (bijective functions); $C_5^1 = 5$.

$$(\ C_5^1(1,2,3,4,5) + C_5^1(1,2,3,5,4) + C_5^1(1,2,4,3,5) + C_5^1(1,2,4,5,3) + C_5^1(1,2,5,3,4) +$$

$$+\ C_5^1(1,2,5,4,3)\ ) + (\ C_5^1(1,3,2,4,5) + C_5^1(1,3,2,5,4) + C_5^1(1,3,4,2,5) +$$

$$+\ C_5^1(1,3,4,5,2) + C_5^1(1,3,5,2,4) + C_5^1(1,3,5,4,2)\ ) + (\ C_5^1(1,4,2,3,5) + C_5^1(1,4,2,5,3) +$$

$$+\ C_5^1(1,4,3,2,5)+C_5^1(1,4,3,5,2)+C_5^1(1,4,5,2,3)+C_5^1(1,4,5,3,2)\,)+(\,C_5^1(1,5,2,3,4)+$$

$$+C_5^1(1,5,2,4,3)+C_5^1(1,5,3,2,4)+C_5^1(1,5,3,4,2)+C_5^1(1,5,4,2,3)+C_5^1(1,5,4,3,2)\,)=120.$$

Because the construction algorithm has direct access facility, we count all bijective functions, but we illustrate only some of them, related with

$$C_5^1(1,2,3,4,5) \text{ and } C_5^1(1,5,4,3,2).$$

Nr	1	2	3	4	5	······	116	117	118	119	120	(circuit addresses)
$f(1)=$	1	2	3	4	5	······	1	5	4	3	2	
$f(2)=$	2	3	4	5	1	······	5	4	3	2	1	
$f(3)=$	3	4	5	1	2	······	4	3	2	1	5	
$f(4)=$	4	5	1	2	3	······	3	2	1	5	4	
$f(5)=$	5	1	2	3	4	······	2	1	5	4	3.	

7. Conclusions

The work is presented at a level suitable for computer programming.

There are two methods to write a mathematical work. We begin by formulating a practical problem and to seek for the mathematical solution (direct problem). Or, we imagine a mathematucal theory and look for a practical interpretation (inverse problem). Both methods are good. Our work is bassed on direct problem. But, now we can mention several new interpretations of discrete functions. For example, if we associate to each digit one color, we can apply the discrete functions in textile industry. Let us say $f: 1, 2, 3 \to 1$ (red), 2 (green), 3 (yellow).

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