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Weak Insertion of an α−Continuous Function

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Abstract: A sufficient condition in terms of lower cut sets are given for the weak α−insertion property and the weak insertion of an α−continuous function between two comparable real-valued functions. Also several insertion theorems are obtained as corollaries of this result.

Keywords: Weak Insertion, Strong Binary Relation, Preopen Set, Semi-Open Set, α−Open Set, Lower Cut Set

1. Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [1].

A subset A of a topological space (X, τ) is called preopen or locally dense or nearly open if A ⊆ Int(Cl(A)). A set A is called preclosed if its complement is preopen or equivalently if Cl(Int(A)) ⊆ A.

The term, preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [2], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael [1].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [3]. A subset A of a topological space (X, τ) is called semi-open [3] if A ⊆ Cl(Int(A)).

A set A is called semi-closed if its complement is semi-open or equivalently if Int(Cl(A)) ⊆ A.

Recall that a subset A of a topological space (X, τ) is called α−open if A is the difference of an open and a nowhere dense subset of X.

A set A is called α−closed if its complement is α−open or equivalently if A is union of a closed and a nowhere dense set.

We have a set is α−open if and only if it is semi-open and preopen.

Recall that a real-valued function f defined on a topological space X is called A−continuous [4] if the preimage of every open subset of R belongs to A, where A is a collection of subset of X.

Most of the definitions of function used throughout this paper are consequences of the definition of A−continuity. However, for unknown concepts the reader may refer to [5, 6].

Hence, a real-valued function f defined on a topological space X is called precontinuous (resp. semi-continuous or α−continuous) if the preimage of every open subset of R is preopen (resp. semi-open or α−open) subset of X.

Precontinuity was called by V. Ptk nearly continuity [7]. Nearly continuity or precontinuity is known also as almost continuity by T. Husain [8].

Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [9].

Results of M. Katˇetov [10, 11] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to F. Brooks [12], are used in order to give a sufficient condition for the insertion of an α−continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write g ≤ f in case g(x) ≤ f(x) for all x in X.

The following definitions are modifications of conditions considered in [13].

A property P defined relative to a real-valued function on a topological space is an α−property provided that any constant function has property P and provided that the sum of a function with property P and any α−continuous function also has property P.

If P1 and P2 are α−property, the following terminology is used:

A space X has the weak α−insertion property for (P1,P2) if and only if for any functions g and f on X such that g ≤ f, g has property P1 and f has property P2, then there exists an α−continuous function h such that g ≤ h ≤ f.

2. The Main Result

Before giving a sufficient condition for insertability of an α−continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all α−open, α−closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). Respectively, we define the α−closure, α−interior, s-closure, s-interior, p-closure and p-interior of a set A, denoted by $\alpha Cl(A)$, $\alpha Int(A)$, $sCl(A)$, $sInt(A)$, $pCl(A)$ and $pInt(A)$ as follows:

$$\alpha Cl(A)= \cap\{F: F \supseteq A, F \in \alpha C(X, \tau)\}, \tag{1}$$

$$\alpha Int(A)= \cup\{O: O \subseteq A, O \in \alpha O(X, \tau)\}, \tag{2}$$

$$sCl(A)= \cap\{F: F \supseteq A, F \in sC(X, \tau)\}, \tag{3}$$

$$sInt(A)= \cup\{O: O \subseteq A, O \in sO(X, \tau)\}, \tag{4}$$

$$pCl(A)= \cap\{F: F \supseteq A, F \in pC(X, \tau)\} \text{ and} \tag{5}$$

$$pInt(A)= \cup\{O: O \subseteq A, O \in pO(X, \tau)\}. \tag{6}$$

Respectively, we have $\alpha Cl(A)$, $sCl(A)$, $pCl(A)$ are α−closed, semi-closed, preclosed and $\alpha Int(A)$, $sInt(A)$, $pInt(A)$ are α−open, semi-open, preopen. The following first two definitions are modifications of conditions considered in [10, 11].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set P (X) of a topological space X is called a strong binary relation in P (X) in case ρ satisfies each of the following conditions:

i). If $A_i \rho B_j$ for any $i \in \{1,...,m\}$ and for any $j \in \{1,...,n\}$, then there exists a set C in P (X) such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1,...,m\}$ and any $j \in \{1,...,n\}$.

ii). If $A \subseteq B$, then $A \bar{\rho} B$.

iii). If $A \rho B$, then $\alpha Cl(A) \subseteq B$ and $A \subseteq \alpha Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by F. Brooks [12] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if

$$\{x \in X: f(x)<l\} \subseteq A(f, l) \subseteq \{x \in X: f(x) \leq l\} \text{ for a real number } l, \tag{7}$$

then A(f, l) is called a lower indefinite cut set in the domain of f at the level l.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists an α−continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions F and G mapping the rational numbers Q into the power set of X by F (t)= A(f, t) and G(t)= A(g, t). If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then $F(t_1) \bar{\rho} F(t_2)$, $G(t_1) \bar{\rho} G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [11] it follows that there exists a function H mapping Q into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

$$\text{For any x in X, let } h(x) = \inf\{t \in Q: x \in H(t)\}. \tag{8}$$

We first verify that $g \leq h \leq f$: If x is in H(t) then x is in $G(t')$ for any $t' > t$; since x is in $G(t')= A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in H(t), then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t')= A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2)= \alpha Int(H(t_2)) \setminus \alpha Cl(H(t_1))$. Hence $h^{-1}(t_1,t_2)$ is an α−open subset of X, i. e., h is an α−continuous function on X. •

The above proof used the technique of proof of Theorem 1 of [10].

3. Applications

The abbreviations pc and sc are used for precontinuous and semicontinuous, respectively.

Corollary 3.1. If for each pair of disjoint preclosed (resp. semi-closed) sets F_1,F_2 of X , there exist α−open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X has the weak α−insertion property for (pc, pc) (resp. (sc, sc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are pc (resp. sc), and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $pCl(A) \subseteq pInt(B)$ (resp. $sCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X: f(x) \leq t_1\} \subseteq \{x \in X: g(x) < t_2\} \subseteq A(g,t_2); \tag{9}$$

since $\{x \in X: f(x) \leq t_1\}$ is a preclosed (resp. semi-closed) set and since $\{x \in X: g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $pCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (resp. $sCl(A(f, t_1)) \subseteq sInt(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. •

Corollary 3.2. If for each pair of disjoint preclosed (resp. semi-closed) sets F_1,F_2, there exist α−open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then every precontinuous (resp. semi-continuous) function is α−continuous.

Proof. Let f be a real-valued precontinuous (resp. semi-continuous) function defined on the X. Set g = f, then by Corollary 3.1, there exists an α−continuous function h such that $g = h = f$.•

Corollary 3.3. If for each pair of disjoint subsets $F1$,$F2$ of X , such that $F1$ is preclosed and $F2$ is semi-closed, there exist $\alpha-$open subsets $G1$ and $G2$ of X such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ then X have the weak $\alpha-$insertion property for (pc, sc) and (sc, pc).

Proof. Let g and f be real-valued functions defined on the X, such that g is pc (resp. sc) and f is sc (resp. pc), with $g \leq f$.If a binary relation ρ is defined by $A\rho B$ in case $sCl(A) \subseteq pInt(B)$ (resp. $pCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t1 and t2 are any elements of Q with $t1 < t2$, then

$$A(f, t1) \subseteq \{x \in X : f(x) \leq t1\} \subseteq \{x \in X : g(x) < t2\} \subseteq A(g, t2); \quad (10)$$

since $\{x \in X: f(x) \leq t1\}$ is a semi-closed (resp. preclosed) set and since $\{x \in X: g(x) < t2\}$ is a preopen (resp. semi-open) set, it follows that $sCl(A(f, t1)) \subseteq pInt(A(g, t2))$ (resp. $pCl(A(f, t1)) \subseteq sInt(A(g, t2))$). Hence $t1 < t2$ implies that $A(f, t1)\ \rho A(g, t2)$. The proof follows from Theorem 2.1. •

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