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A Partial Answer to Sidorenko's Conjecture on a Correlation Inequality for Bipartite Graphs

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Abstract: Sidorenko conjectured an integral inequality for a product of functions $h(x_i, y_i)$ where the diagram of the product is a bipartite graph G in [8]. We answered the conjecture positively when the function h is multiplicative or additive separable with respect to variables x and y.

Keywords: Sidorenko's Conjecture, Bipartite Graph, Lebesgue Measure, Measurable Function

1. Introduction

Denote the Lebesgue measure on $[0, 1]$ by μ. Let a fuction $h(x, y)$ be bounded, nonnegative and measurable on $[0, 1]^2$. Let G be a bipartite graph where vertices $u_1, u_2, \dots, u_n$ form the first part and $v_1, v_1, \dots, v_m$ the second part. We denote by E the set of pairs (i, j) for which u_i *and* v_j are adjacent in G. So $|E|$ is the number of edges. The following conjecture was discussed in works [2, 3] published in Russian. The same arguments were explained in detail again in [8].

Conjecture 1 [2]. *For any bipartite graph G and any function h*

$$\int \prod_{(i,j)\in E} h(x_i, y_j)\, d\mu^{n+m} \ge \left(\int h\, d\mu^2\right)^{|E|} \tag{1}$$

Two particular cases of Inequality 1 (G is a path, h is *symmetric*; G is a path of length 3, h is not necessarily *symmetric*) were proven in [4, 5] and [6], respectively.

One might conjecture that the considered products of functions are always non-negative correlated. Unfortunately, this is not the case. For instance, it would require

$$\int h(x_1, y_1) h(x_1, y_2) h(x_2, y_2)\, d\mu^4$$

$$\ge \int h(x_1, y_1) h(x_1, y_2)\, d\mu^3 \int h(x_2, y_2)\, d\mu^2.$$

However, a counterexample to the last inequality was found in [7].

Conjecture 2 [2]. *Let the numbers of edges and vertices of a bipartite graph G satisfy the conditions* $|E| \ge n$; $|E| \ge m$. *Then, for any functions* $h, f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_m$,

$$\int \prod_{i=1}^{n} f_i(x_i) \int \prod_{j=1}^{m} g_j(y_j) \prod_{(i,j)\varepsilon E} h(x_i, y_j)\, d\mu^{n+m}$$

$$\ge \left(\int h(x, y) \left(\prod_{i=1}^{n} f_i(x) \prod_{j=1}^{m} g_j(y)\right)^{\frac{1}{|E|}} d\mu^2\right)^{|E|} \tag{2}$$

Inequality 2 is essentially stronger than inequality 1. For instance, inequality 2 implies [2]:

$$\int \prod_{(i,j)\varepsilon E} h(x_i, y_j)\, d\mu^{n+m} \ge \left(\int \left(\int h(x, y)\, d\mu_y\right)^{\frac{|E|}{n}} d\mu_x\right)^n \tag{3}$$

Conjecture 1 and 2 were proven in [2] for various classes of graphs.

2. Main Results

We start by mentioning a simple example as follows:

Let $h : [0,1] \times [0,1] \to [0, \infty)$ be a function given by $h(x, y) = x\sqrt{y}$. h is trivially non-negative, bounded and Lebesgue measurable. Let G be a finite simple bipartite graph with $\{u_1, u_2\}$ such that u_1 to both v_1 and v_2 and u_2 to both v_1 and v_2. Siderenko's conjecture is positively confirmed because

$$\int \prod_{(i,j)\varepsilon E} h(x_i, y_j)\, d\mu^{n+m} = \int_0^1\int_0^1\int_0^1\int_0^1 x_1\sqrt{y_2}\, x_1\sqrt{y_2}\, x_2\sqrt{y_1}\, x_2\sqrt{y_2}\, dx_1 dx_2 dy_1 dy_2 = \tfrac{1}{36}$$

$$(\int_0^1\int_0^1 h(x.y)dxdy)^4 = (\int_0^1\int_0^1 x\sqrt{y}dxdy)^4 = \frac{1}{81}$$

Lemma 3 [1]. Suppose that $1 \le r < \infty$ and $f \in L^r(\mu)$ is non-negative. Then $\int_X f^r d\mu \ge (\int_X f d\mu)^r$. If $r > 1$, then equality holds if and only if f is (essentially) constant.

In order to make the understanding of Theorem 4 easier, we give a simpler format of it as an example. First let G be a simple bipartite graph from u_1 to both v_1 and v_2 and the function $0 \le h \le 1$ be measurable on $[0, 1]^2$. Let h be written as $h(x,y)=h(x).g(y)$

$$I1 = \int \prod_{(i,j)\in E} h(x_i, y_j) d\mu^{n+m}$$

$$I_1 = \int \prod_{(i,j)\in E} h(x_i, y_j) d\mu^{n+m} = \int_0^1\int_0^1\int_0^1 h(x_1, y_1)h(x_1, y_2)dx_1dy_1dy_2$$

$$= \int_0^1\int_0^1\int_0^1 f(x_1)g(y_1)f(x_1)g(y_2)dx_1dy_1dy_2$$

$$= (\int_0^1 f(x_1)^2 dx)(\int_0^1 g(y)dy)^2$$

By Lemma 3, we get

$$I_1 \ge (\int_0^1 f(x_1)dx)^2 (\int_0^1 g(y)dy)^2 = (\int_0^1\int_0^1 f(x)g(y)dxdy)^2$$

$$= (\int h d\mu^2)^2$$

$$= (\int h d\mu^2)^{|E|}$$

Now we can give Theorem 4 which is a generalization of above mentioned argument.

Theorem 4. For any bipartite graph G and any non-negative, bounded and Lebesgue measurable *function h such that h(x,y)=f(x).g(y) we have*

$$\int \prod_{(i,j)\varepsilon E} h(x_i, y_j) d\mu^{n+m} \ge (\int h d\mu^2)^{|E|}$$

Proof: Let G be a finite simple bipartite graph with bipartition $\{u_1,u_2,\ldots u_m\}$

$$\int \prod_{(i,j)\in E} h(x_i, y_j) d\mu^{n+m} = \int\int\ldots\int \prod_{(i,j)\in E} h(x_i, y_j)dx_1\ldots dx_n dy_1\ldots dy_m$$

$$= \int\int\ldots\int \prod_{(i,j)\in E} f(x_i)g(y_j)dx_1\ldots dx_n dy_1\ldots dy_m$$

$$= \int_0^1 f(x_1)^{d(u_1)}dx_1.\int_0^1 f(x_2)^{d(u_2)}dx_2\ldots\int_0^1 f(x_n)^{d(u_n)}dx_n.$$

$$\int g(y_1)^{d(v_1)}dy_1\ldots\int g(y_1)^{d(v_m)}dy_m = A$$

where $d(u_i)$ is the degree of u_i.

By Lemma 3 we know that the inequality

$$\int_0^1 f^r(x)^n dx \ge (\int_0^1 f(x)dx)^n$$

satisfies for all natural number n.

By using Fubini's Theorem we have

$$A = (\prod_{i=1}^n \int_0^1 f(x_i)^{d(u_i)}dx_i)(\prod_{j=1}^m \int g(y_j)^{d(v_j)}dy_j)$$

$$\ge \prod_{i=1}^n (\int_0^1 f(x_i)dx_i)^{d(u_i)} \prod_{j=1}^m (\int g(y_j)dy_j)^{d(v_j)}$$

$$= (\int_0^1 f(x)dx)^{d(u_1)+\ldots+d(u_n)}.(\int_0^1 g(y)dy)^{d(v_1)+\ldots+d(v_m)}$$

$$= (\int_0^1 f(x)dx)^{|E|}.(\int_0^1 g(y)dy)^{|E|}$$

$$= (\int_0^1\int_0^1 f(x)g(y)dxdy)^{|E|}.$$

Hence the proof was completed.

Now under the same assumption of G given before Theorem 4, let the function $0 \le h \le 1$ be written as $h(x,y)=f(x)+g(y)$ which is measurable on $[0, 1]^2$

$$I_2 = \int \prod_{(i,j)\in E} h(x_i, y_i) d\mu^{n+m}$$

$$= \int_0^1\int_0^1\int_0^1 h(x_1, y_1)h(x_1, y_2)dx_1dy_1dy_2$$

$$= \int_0^1\int_0^1\int_0^1 (f(x_1) + g(y_1))(f(x_1) + g(y_2)dx_1dy_1dy_2$$

$$= \int_0^1\int_0^1\int_0^1 f(x_1)^2 dx_1dy_1dy_2 + \int_0^1\int_0^1\int_0^1 f(x_1)g(y_2)dx_1dy_1dy_2$$

$$+ \int_0^1\int_0^1\int_0^1 g(y_1)f(x_1)dx_1dy_1dy_2 + \int_0^1\int_0^1\int_0^1 g(y_1)g(y_2)dx_1dy_1dy_2$$

Since the integrands at second and third order in above expression are the same, by Lemma 3 we have

$$I_2 \ge (\int_0^1 f(x)dx)^2 + 2(\int_0^1 f(x)dx)(\int_0^1 g(y)dy) + (\int_0^1 g(y)dy)^2$$

$$= (\int_0^1 f(x)dx + \int_0^1 g(y)dy)^2$$

$$= (\int_0^1\int_0^1 (f(x) + g(y))dxdy)^2$$

$$= (\int h d\mu^2)^2$$

$$= (\int h d\mu^2)^{|E|}$$

Theorem 5. For any bipartite graph G and any function h such that $h(x,y)=h(x)+g(y)$, we have

$$\int \prod_{(i,j)\in E} h(x_i, y_j) d\mu^{n+m} \ge (\int h d\mu^2)^{|E|}$$

Proof:

$$\int \prod_{(i,j)\in E} h(x_i, y_j) d\mu^{n+m} = \int\int\ldots\int \prod_{(i,j)\varepsilon E} h(x_i, y_j)dx_1\ldots dx_n dy_1\ldots dy_m$$

$$= \int\int\ldots\int \prod (f(x_i) + g(y_j))dx_1\ldots dx_n dy_1\ldots dy_m$$

$$= \int_0^1 f(x)^{|E|} dx + C(|E|,1)(\int_0^1 f(x)^{|E|-1} dx)(\int_0^1 g(y) dy)$$

$$+C(|E|,2)(\int_0^1 f(x)^{|E|-2} dx)(\int_0^1 g(y)^2 dy)$$

$$+ ... + C(|E|,|E|-1)(\int_0^1 f(x) dx)(\int_0^1 g(y)^{|E|-1} dy)$$

$$+\int_0^1 g(y)^{|E|} dy = B$$

By Lemma 3 we get

$$B \geq (\int_0^1 f(x) dx)^{|E|} + C(|E|,1)(\int_0^1 f(x) dx)^{|E|-1} (\int_0^1 g(y) dy)$$

$$+ ... + (\int_0^1 g(y) dy)^{|E|}$$

By Binom expansion we get

$$B \geq (\int_0^1 f(x) dx + \int_0^1 g(y) dy)^{|E|} = (\int_0^1 \int_0^1 f(x) dx + \int_0^1 \int_0^1 g(y) dy)^{|E|}$$

$$= (\int_0^1 \int_0^1 (f(x) + g(y)) dx dy)^{|E|}$$

$$= (\int h d\mu^2)^{|E|}$$

which completed the proof.

Following Theorem 6 which was given in [1] can be considered as an useful means to investigate whether our Theorem 4 and Theorem 5 can be generalized to any bounded, non-negative and Lebesgue measurable function *h*.

Theorem 6 [1]. Suppose than n is a positive integer $p, q \in \{0\} \cup [1, \infty)$, not both of *p, q* are zero, $f \in L^{n+p}(\mu)$, $g \in L^{n+q}(\mu)$, and *f* and *g* non-negative. Then

$$\int_X \int_X f(x)^p (f(x) + g(y))^n g(y)^q d\mu^2$$

$$\geq (\int_X f d\mu)^p (\int_X f d\mu + \int_X g d\mu)^n (\int_X g d\mu)^q.$$

If neither *f* nor *g* is the zero function then if $n + p > 1$, equality implies that f is constant, and if $n + q > 1$, equality implies that g is constant.

Remark: We do not know whether Theorem 4 is valid for all non-negative, bounded and Lebesgue measurable functions *h*. If we can show that Theorem 4 is true for any function *h* such that

$$h(x, y) = \sum_s f_s(x) g_s(y)$$

where $f_s(x)$ and $g_s(y)$ are polynomials. Then by using Theorem 4, Theorem 5 and Theorem 6, we can easily generalize it to all *h* by using well-known Weierstrass approximation theorem, because polynomials are uniformly dense in *C([0, 1])*.

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References

[1] Johnson P., Soybaş D., An Integral Inequality For Probability Spaces, International Journal of Mathematics and Computer Science, vol. 8, pp. 1-3, 2013.

[2] Sidorenko, A. F, Inequalities for functional generated by bipartite graphs (in Russian), Discrete Mathematics and Applications 3 (3) 50-65 (1991).

[3] Sidorenko, A. F., Exremal problems in graph theory and inequalities in functional analysis (in Russian). In Lupanov, O. B, ed., Proceedings of the Soviet Seminar on Discrete Mathematics and it Applications (in Russian), pp. 99-105. Moscow: Moscow State University 1986, MR88m: 05053.

[4] Mulholland, H. P., Smith, C. A. B., An inequality arising in genetical theory, Amer. Math. Mon. 66, 673-683 (1959).

[5] Blakley, G. R, Roy, P., Hölder type inequality for symmetrical matrices with nonnegative entries. Proc. Am. Math. Soc. 16, 1244-1245 (1965).

[6] Atkinson, F. V., Watterson, G. A., Moran, P. A. D., A matrix inequality. Q. J. Math. Oxf. II Ser. 11, 137-140 (1960).

[7] London, D., Two equalities in nonnegative symmetric matrices. Pac. J. Math.16, 515-536(1966).

[8] Sidorenko, A., A Correlation Inequality for Bipartite Graphs., Graphs and Combinatorics (1993) 9, 201-204.