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Stability and Oscillatory Behavior of the Solutions on a Class of Coupled Van der Pol-Duffing Equations with Delays

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Abstract: In the present paper, a class of coupled van der Pol-Duffing oscillators with a nonlinear friction of higher polynomial order model which involves time delays is investigated. The coefficients of the highest order of the polynomial determine the boundedness of the solutions. With special attention to the boundedness of the solutions and the instability of the unique equilibrium point of linearized system, some sufficient conditions to guarantee the existence of oscillatory solutions for the model are obtained based on the generalized Chafee's criterion. Convergence of the trivial solution is determined by the negative real part of eigenvalues of the linearized system. Examples are provided to demonstrate the reduced conservativeness for the parameters of the proposed results. The results obtained shown that the passive decay rate in the model affects the oscillatory frequency and amplitude. When a permanent oscillation occurred, time delays affect mainly oscillatory frequency and amplitude slightly.

Keywords: Coupled Van der Pol-Duffing Equation, Delay, Stability, Oscillation

1. Introduction

It is known that the van der Pol (VDP) oscillator could model the typical self-excited or self-sustained oscillation. Various coupled van der Pol or van der Pol-Duffing equations have been applied in physics and engineering. Many good results have appeared [1-12]. For the following three-dimensional autonomous van der Pol-Duffing type oscillator system:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -x(t) + \beta\, x^3(t) + \varepsilon\big(1 - x^2(t)\big)y(t) - kz(t), \\ z'(t) = y(t) - z(t). \end{cases} \tag{1}$$

By analyzing the stability of the equilibrium points, the existence of Hopf bifurcation is established [1]. Barron has considered the stability of a ring of coupled van der Pol oscillators with non-uniform distribution of the coupling parameter as follows:

$$x_i''(t) + a(x_i^2(t) - 1)x_i'(t) + x_i(t) = b_i\,(x_{i-1}(t) - 2x_i(t) + x_{i+1}(t)) \tag{2}$$

where $1 \le i \le n$, b_i are the coupling parameter corresponding to the ith oscillator. For a modified hybrid van der Pol-Duffing-Rayleigh oscillator for modeling the lateral walking force on a rigid floor:

$$x''(t) + \gamma x(t) + w_0^2\, x(t) - ax^2(t) - bx(t)x'(t) + cx^3(t) + dx^2(t)x'(t) = 0 \tag{3}$$

Kumar et al. have studied the stability of the equation (3) by the perturbation and energy balance method [3]. Rompala et al. have considered a system of three van der Pol oscillators [5]. For a ring of four mutually coupled biological systems described by coupled van der Pol oscillators, the stability boundaries and the main dynamical states have been considered on the stability maps by Kadji et al. [6]. A driven van der Pol-like oscillator with a nonlinear friction of higher polynomial order model as follows:

$$x''(t) - \mu\big(1 - x^2(t) + \alpha x^4(t) - \beta x^6(t)\big)x'(t) + x(t) = \theta_0 \sin \omega t \tag{4}$$

The effects of noise correlation on the coherence of a forced van der Pol type birhythmic system has been investigated by Yamapi et al. [7]. It is well known that the time delay is inevitable in many physical and biological phenomena such as manufacturing process, nuclear reactors, rocket motors, mechanical controlling systems, population dynamics, and so on. Naturally the time delay coupled van der Pol equations also have been extensively studied by many researchers [13-24]. For example, Li et al. have studied the coupled van der Pol oscillators with two kinds of delays [13]:

$$\begin{cases} y_1''(t) + w_1^2 y_1(t) - \varepsilon\big(1 - y_1^2(t)\big)y_1'(t) = \varepsilon\alpha\big(y_2(t-\tau) + y_2'(t-\tau)\big), \\ y_2''(t) + w_2^2 y_2(t) - \varepsilon\big(1 - y_2^2(t)\big)y_2'(t) = \varepsilon\alpha\big(y_1(t-\tau) + y_1'(t-\tau)\big). \end{cases} \tag{5}$$

Zhang and Gu used the theory of normal form and central manifold theorem to discuss the following time delay system [14]:

$$\begin{cases} x_1''(t) + \varepsilon(x_1^2(t) - 1)x_1'(t) + x_1(t) = \alpha\big(y_1(t-\tau) - x_1(t)\big), \\ y_1''(t) + \varepsilon(y_1^2(t) - 1)y_1'(t) + y_1(t) = \alpha\big(x_1(t-\tau) - y_1(t)\big). \end{cases} \tag{6}$$

Motivated by the above models, in this paper we consider the following a ring of time delays Duffing-van der Pol-like oscillator with a nonlinear friction of higher polynomial order system:

$$\begin{cases} u_1''(t) + \tilde{\varepsilon}_1\big[\tilde{l}_1 u_1^6(t) - \tilde{k}_1 u_1^4(t) + u_1^2(t) - \tilde{a}_1\big]u_1'(t) + \tilde{c}_1 u_1(t) + \tilde{\beta}_1 u_1^3(t) \\ \qquad = \tilde{b}_1[u_n(t-\tilde{\tau}_n) - 2u_1(t-\tilde{\tau}_1) + u_2(t-\tilde{\tau}_2)], \\ u_2''(t) + \tilde{\varepsilon}_2\big[\tilde{l}_2 u_2^6(t) - \tilde{k}_2 u_2^4(t) + u_2^2(t) - \tilde{a}_2\big]u_2'(t) + \tilde{c}_2 u_2(t) + \tilde{\beta}_2 u_2^3(t) \\ \qquad = \tilde{b}_2[u_1(t-\tilde{\tau}_1) - 2u_2(t-\tilde{\tau}_2) + u_3(t-\tilde{\tau}_3)], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n''(t) + \tilde{\varepsilon}_n\big[\tilde{l}_n u_n^6(t) - \tilde{k}_n u_n^4(t) + u_n^2(t) - \tilde{a}_n\big]u_n'(t) + \tilde{c}_n u_n(t) + \tilde{\beta}_n u_n^3(t) \\ \qquad = \tilde{b}_n[u_{n-1}(t-\tilde{\tau}_{n-1}) - 2u_n(t-\tilde{\tau}_n) + u_1(t-\tilde{\tau}_1)]. \end{cases} \tag{7}$$

where $0 < \tilde{l}_i, \tilde{k}_i$ and $0 < \tilde{\varepsilon}_i \ll 1$; $\tilde{c}_i, \tilde{\beta}_i, \tilde{a}_i, \tilde{b}_i \in R$ for each i=1, 2,..., n, $0 \le \tilde{\tau}_i$ are time delays. Our aim is to investigate the dynamical behavior of n coupled oscillators by means of the generalized Chafee's criterion [25, 26].

2. Preliminaries

For convenience, setting $\tilde{\varepsilon}_i = \varepsilon_{2i}, \tilde{l}_i = l_{2i}, \tilde{k}_i = k_{2i}, \tilde{a}_i = a_{2i}, \tilde{c}_i = c_{2i-1}, \tilde{b}_i = b_{2i-1}, \tilde{\beta}_i = \beta_{2i-1}, \tilde{\tau}_i = \tau_{2i-1}$ $(1 \le i \le n)$. Then the coupled system (7) can be written as the following equivalent system:

$$\begin{cases} x_1'(t) = x_2(t), \\ x_2'(t) = -c_1 x_1(t) - \beta_1 x_1^3(t) + b_1[x_{2n-1}(t-\tau_{2n-1}) - 2x_1(t-\tau_1) + x_3(t-\tau_3)] \\ \qquad + \varepsilon_2 a_2 x_2(t) - \varepsilon_2 x_1^2(t)x_2(t) + \varepsilon_2 k_2 x_1^4(t)x_2(t) - \varepsilon_2 l_2 x_1^6(t)x_2(t), \\ x_3'(t) = x_4(t), \\ x_4'(t) = -c_3 x_3(t) - \beta_3 x_3^3(t) + b_3[x_1(t-\tau_1) - 2x_3(t-\tau_3) + x_5(t-\tau_5)] \\ \qquad + \varepsilon_4 a_4 x_4(t) - \varepsilon_4 x_3^2(t)x_4(t) + \varepsilon_4 k_4 x_3^4(t)x_4(t) - \varepsilon_4 l_4 x_3^6(t)x_4(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{2n-3}'(t) = x_{2n-2}(t), \\ x_{2n-2}'(t) = -c_{2n-3}x_{2n-3}(t) - \beta_{2n-3}x_{2n-3}^3(t) + b_{2n-3}[x_{2n-5}(t-\tau_{2n-5}) - 2x_{2n-3}(t-\tau_{2n-3}) \\ \qquad + x_{2n-1}(t-\tau_{2n-1})] + \varepsilon_{2n-2}a_{2n-2}x_{2n-2}(t) - \cdots - \varepsilon_{2n-2}l_{2n-2}x_{2n-3}^6(t)x_{2n-2}(t), \\ x_{2n-1}'(t) = x_{2n}(t), \\ x_{2n}'(t) = -c_{2n-1}x_{2n-1}(t) - \beta_{2n-1}x_{2n-1}^3(t) + b_{2n-1}[x_{2n-3}(t-\tau_{2n-3}) - 2x_{2n-1}(t-\tau_{2n-1}) \\ \qquad + x_1(t-\tau_1)] + \varepsilon_{2n}a_{2n}x_{2n}(t) - \varepsilon_{2n}x_{2n-1}^2(t)x_{2n}(t) + \cdots - \varepsilon_{2n}l_{2n}x_{2n-1}^6(t)x_{2n}(t). \end{cases} \tag{8}$$

The matrix form of system (8) is the following:

$$X'(t) = AX(t) + BX(t-\tau) + g(X) \tag{9}$$

where $X(t) = [x_1(t), x_1(t), \cdots, x_{2n}(t)]^T$,
$X(t-\tau) = [x_1(t-\tau_1), 0, x_3(t-\tau_1), \cdots, x_{2n-1}(t-\tau_{2n-1}), 0]^T$,
$g(X) = [0, -\beta_1 x_1^3(t) - \varepsilon_2 x_1^2(t)x_2(t) + \varepsilon_2 k_2 x_1^4(t)x_2(t) - \varepsilon_2 l_2 x_1^6(t)x_2(t), 0, \cdots, 0, -\beta_{2n-1}x_{2n-1}^3(t) - \varepsilon_{2n}x_{2n-1}^2(t)x_{2n}(t) + \varepsilon_{2n}k_2 x_{2n-1}^4(t)x_{2n}(t) - \varepsilon_{2n}l_{2n}x_{2n-1}^6(t)x_{2n}(t)]^T$. Both $A = (a_{ij})_{2n\times 2n}$ and $B = (b_{ij})_{2n\times 2n}$ are $2n$ by $2n$ matrices as follows:

$$A = (a_{ij})_{2n\times 2n} = \begin{pmatrix} 0 & 1 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ -c_1 & a_{22} & & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & -c_3 & a_{44} & 0 & \cdots & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & & -c_{2n-1} & a_{2n2n} \end{pmatrix},$$

where $a_{22}=\varepsilon_2 a_2$, $a_{44}=\varepsilon_4 a_4$, $a_{2n2n}=\varepsilon_{2n} a_{2n}$.

$$B=(b_{ij})_{2n\times 2n}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -2b_1 & 0 & b_1 & 0 & 0 & \cdots & b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_3 & 0 & -2b_3 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{2n-1} & 0 & 0 & 0 & 0 & \cdots & -2b_{2n-1} & 0 \end{pmatrix}.$$

Obviously, the origin $x_k=0$ ($k=1, 2, \cdots, 2n$) is an equilibrium of system (8). The linearization of system (9) at origin is

$$X'(t)=AX(\mathrm{t})+BX(t-\tau) \tag{10}$$

Let

$$C=(c_{ij})_{n\times n}=\begin{pmatrix} c_{11} & 0 & b_1 & 0 & \cdots & 0 & b_1 \\ b_3 & 0 & c_{23} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{2n-3} \\ b_{2n-1} & 0 & 0 & 0 & \cdots & 0 & c_{nn} \end{pmatrix},$$

where $c_{11}=-2b_1-c_1$, $c_{23}=-2b_3-c_3,\cdots$, $c_{nn}=-2b_{2n-1}-c_{2n-1}$.

Lemma 1 Assume that the matrix C is a nonsingular matrix, then system (8) (or (9)) has a unique equilibrium.

Proof An equilibrium point $x^*=[x_1^*,x_2^*,\cdots,x_{2n}^*]^T$ of system (8) is a constant solution of the following algebraic equation

$$\begin{cases} x_2^*=0, \\ -c_1x_1^*-\beta_1(x_1^*)^3+b_1[x_{2n-1}^*-2x_1^*+x_3^*]+\varepsilon_2a_2x_2^*-\varepsilon_2(x_1^*)^2x_2^*+\varepsilon_2k_2(x_1^*)^4x_2^*-\varepsilon_2l_2(x_1^*)^6x_2^*=0, \\ x_4^*=0, \\ -c_3x_3^*-\beta_3(x_3^*)^3+b_3[x_1^*-2x_3^*+x_5^*]+\varepsilon_4a_4x_4^*-\varepsilon_4(x_3^*)^2x_4^*+\varepsilon_4k_4(x_3^*)^4x_4^*-\varepsilon_4l_4(x_3^*)^6x_4^*=0, \\ \cdots\cdots\cdots \\ x_{2n-2}^*=0, \\ -c_{2n-3}x_{2n-3}^*-\beta_{2n-3}(x_{2n-3}^*)^3+b_{2n-3}[x_{2n-5}^*-2x_{2n-3}^*+x_{2n-1}^*]+\varepsilon_{2n-2}a_{2n-2}x_{2n-2}^* \\ \quad -\varepsilon_{2n}(x_{2n-3}^*)^2x_{2n-2}^*+\varepsilon_{2n-2}k_{2n-2}(x_{2n-3}^*)^4x_{2n-2}^*-\varepsilon_{2n-2}l_{2n-2}(x_{2n-3}^*)^6x_{2n-2}^*=0, \\ x_{2n}^*=0, \\ -c_{2n-1}x_{2n-1}^*-\beta_{2n-1}(x_{2n-1}^*)^3+b_{2n-1}[x_{2n-3}^*-2x_{2n-1}^*+x_1^*]+\varepsilon_{2n}a_{2n}x_{2n}^*-\varepsilon_{2n}(x_{2n-1}^*)^2x_{2n}^* \\ \quad +\varepsilon_{2n}k_{2n}(x_{2n-1}^*)^4x_{2n}^*-\varepsilon_{2n}l_{2n}(x_{2n-1}^*)^6x_{2n}^*=0. \end{cases} \tag{11}$$

Since $x_{2i}^*=0$ $(1\le i\le n)$, from (11) we get

$$\begin{cases} -c_1x_1^*-\beta_1(x_1^*)^3+b_1[x_{2n-1}^*-2x_1^*+x_3^*]=0, \\ -c_3x_3^*-\beta_3(x_3^*)^3+b_3[x_1^*-2x_3^*+x_5^*]=0, \\ \cdots\cdots\cdots \\ -c_{2n-3}x_{2n-3}^*-\beta_{2n-3}(x_{2n-3}^*)^3+b_{2n-3}[x_{2n-5}^*-2x_{2n-3}^*+x_{2n-1}^*]=0, \\ -c_{2n-1}x_{2n-1}^*-\beta_{2n-1}(x_{2n-1}^*)^3+b_{2n-1}[x_{2n-3}^*-2x_{2n-1}^*+x_1^*]=0. \end{cases} \tag{12}$$

System (12) can be written as a matrix form as the following:

$$DX^*=0 \tag{13}$$

where $X^*=[x_1^*,x_3^*,\cdots,x_{2n-1}^*]^T$, and

$$D=(d_{ij})_{n\times n}=\begin{pmatrix} d_{11} & b_1 & 0 & \cdots & b_1 \\ b_3 & d_{22} & b_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_{2n-3} \\ b_{2n-1} & 0 & 0 & \cdots & d_{nn} \end{pmatrix},$$

where $d_{ii}=-2b_{2i-1}-c_{2i-1}-\beta_{2i-1}(x_{2i-1}^*)^2(1\le i\le n)$. According to standard results in linear algebraic, if D is a nonsingular matrix, system (13) has only one solution, namely, the trivial solution. When $x_{2i-1}^*=0$ $((1\le i\le n)$, matrix D changes to C. The proof is completed.

Lemma 2 All solutions of system (8) are uniformly bounded.

Proof Construct a Lyapunov function $V(t)=\sum_{i=1}^{2n}\frac{1}{2}x_i^2(t)$. Calculating the derivative of V (t) through system (8) we have

$$V'(t)|_{(8)}=\sum_{i=1}^{2n}x_i(t)\,x_i'(t)$$

$$=x_1(t)x_2(t)+x_2(t)\{-c_1x_1(t)-\beta_1x_1^3(t) + b_1[x_{2n-1}(t-\tau_{2n-1})-2x_1t-\tau_1) + x_3(t-\tau_3)]+\varepsilon_2a_2x_2(t)$$

$$-\varepsilon_2x_1^2(t)x_2(t)+\varepsilon_2k_2x_1^4(t)x_2(t)-\varepsilon_2l_2x_1^6(t)x_2(t)\}$$

$$+\cdots\cdots+x_{2n-1}(t)x_{2n}(t)$$

$$+x_{2n}(t)\{-c_{2n-1}x_{2n-1}(t)-\beta_{2n-1}x_{2n-1}^3(t) + b_{2n-1}[x_{2n-3}(t-\tau_{2n-3})2x_{2n-1}(t - \tau_{2n-1})+x_1(t-\tau_1)]$$

$$+\varepsilon_{2n}a_{2n}x_{2n}(t)-\varepsilon_{2n}x_{2n-1}^2(t)x_{2n}(t) + \varepsilon_{2n}k_{2n}x_{2n-1}^4(t)x_{2n}(t) - \varepsilon_{2n}l_{2n}x_{2n-1}^6(t)x_{2n}(t)\}$$

$$=(1-c_1)x_1(t)x_2(\mathrm{t})-\beta_1x_1^3(t)x_2(\mathrm{t})+b_1x_2(\mathrm{t})[x_{2n-1}(t-\tau_{2n-1})-2x_1t-\tau_1)+$$

$$x_3(t-\tau_3)] + \varepsilon_2 a_2 x_2^2(t) - \varepsilon_2 x_1^2(t)x_2^2(t) + \varepsilon_2 k_2 x_1^4(t)x_2^2(t)$$
$$+\cdots\cdots$$
$$+(1-c_{2n-1})x_{2n-1}(t)x_{2n}(\mathrm{t}) - \beta_{2n-1}x_{2n-1}^3(t)x_{2n}(\mathrm{t}) + b_{2n-1}x_{2n}(\mathrm{t})$$
$$[x_{2n-3}(t-\tau_{2n-3}) - 2x_{2n-1}(t-\tau_{2n-1}) + x_1(t-\tau_1)]$$
$$+\varepsilon_{2n}a_{2n}x_{2n}^2(t) - \varepsilon_{2n}x_{2n-1}^2(t)x_{2n}^2(t) + \varepsilon_{2n}k_{2n}x_{2n-1}^4(t)x_{2n}^2(t)$$
$$-\varepsilon_2 l_2 x_1^6(t)x_2^2(t) - \varepsilon_4 l_4 x_3^6(t)x_4^2(t) - \cdots\cdots - \varepsilon_{2n}l_{2n}x_{2n-1}^6(t)x_{2n}^2(t) \quad (14)$$

Obviously, $x_1^6(t)x_2^2(t), x_3^6(t)x_4^2(t), \cdots\cdots, x_{2n-1}^6(t)x_{2n}^2(t)$ are higher order infinity as $x_i(t)$ $(1 \le i \le n)$ tend to infinity. Since $0 < \varepsilon_{2i,} l_{2i}$ $(1 \le i \le n)$, so there exists a sufficiently large $L > 0$ such that

$V'(t)|_{(8)} < 0$ as $|x_i| > L$ $(1 \le i \le n)$, implying that all solutions of system (8) are bounded.

3. Main Result

First we discuss the stability of the trivial solution of system (8) (or (9)). Noting that $g(X)$ is a higher order infinitesimal in a neighborhood of $\|X\| = 0$. Therefore, the stability of trivial solution of system (10) guarantees the stability of trivial solution of system (8). We consider the following auxiliary system:

$$X'(t) = AX(\mathrm{t}) + BX(t-\tau^*) \quad (15)$$

where $\tau^* = \max\{\tau_1, \tau_3, \cdots, \tau_{2n-1}\}$, $X(t-\tau^*) = [x_1(t-\tau^*), 0, x_3(t-\tau^*), 0, \cdots, x_{2n-1}(t-\tau^*), 0]^T$.

Theorem 1 Assume that system (8) has a unique equilibrium point, for selected parameter values of $a_{2i}, b_{2i-1}, c_{2i-1}$, and ε_{2i} $(1 \le i \le n)$. Let the eigenvalues of matrix R$= A + B$ be $\gamma_i (1 \le i \le 2n)$. If Re (γ_i) $(i = 1, 2, \cdots, 2n)$ $< -r < 0$, then the unique equilibrium point, namely, the trivial solution of system (8) is stable.

Proof Since Re (γ_i) $(i = 1, 2, \cdots, 2n)$ $< -r < 0$, hence there exists a positive constant $K \ge 1$ such that $\|e^{(A+B)t}\| \le Ke^{-rt}$. In (15) for $t \ge \tau^*$ we have

$$X'(\mathrm{t}) = (A+B)X(\mathrm{t}) - B\int_{t-\tau^*}^{t} X'(s)ds$$
$$= (A+B)X(\mathrm{t}) - B\int_{t-\tau^*}^{t}(AX(s) + B(s - tau^*))ds \quad (16)$$

From (16), for $t \ge \tau^*$ we get

$$X(t) = e^{(A+B)(t-\tau^*)}X(\tau^*) - B\int_{\tau^*}^{t} ds \int_{s-\tau^*}^{s} e^{(A+B)(t-s)}(\mathrm{AX(u) + B(u} - tau^*))\mathrm{du} \quad (17)$$

Therefore,

$$\|X(t)\| \le LKe^{-r(t-\tau^*)} + K\|B\| \int_{\tau^*}^{t} ds \int_{s-\tau^*}^{s} e^{-r(t-s)}(\|A\|\|X(u)\| + \|B\|\|X(u-\tau^*)\|)du \quad (18)$$

where $L = \sup_{t\in[-\tau^*, \tau^*]}\|X(t)\|$. We shall prove that there exists a positive constant c ($< r$) such that $\|X(t)\| \le LKe^{-c(t-\tau^*)}$, $t \ge \tau^*$. Indeed, select c ($< r$) such that

$$K\|B\|\left(\|A\| + \|B\|e^{c\tau^*}\right)\left(e^{c\tau^*} - 1\right) = c(r-c)$$

then from (18) we have

$$\|X(t)\| \le LKe^{-r(t-\tau^*)} + K\|B\| \int_{\tau^*}^{t} ds \int_{s-\tau^*}^{s} e^{-r(t-s)}(\|A\|LKe^{-c(u-\tau^*)} + \|B\|LKe^{-c(u-2\tau^*)})du$$
$$= LKe^{-r(t-\tau^*)} + K\|B\| \frac{cK(\|A\|e^{c\tau^*} + \|B\|e^{2c\tau^*})}{-c} \int_{\tau^*}^{t} e^{-r(t-s)}(e^{-cs} - e^{-c(s-\tau^*)})ds$$
$$= LKe^{-r(t-\tau^*)} + K\|B\| \frac{cK(\|A\| + \|B\|e^{c\tau^*})(e^{c\tau^*} - 1)}{-c(r-c)}(e^{-c(s-\tau^*)} - e^{-r(t-\tau^*)})$$
$$= LKe^{-c(t-\tau^*)} \quad (19)$$

This means that $X(t) \to 0$ as $t \to \infty$ in system (15). Since $\tau_{2i-1} \le \tau^*$ $(i = 1, 2, \cdots, n)$, and $g(X)$ is higher infinitesimal as $X(t) \to 0$, based on the property of delayed differential equation, we know that the trivial solution of system (8) (or (9)) is stable. The proof is completed.

Theorem 2 Assume that system (8) has a unique equilibrium point, for selected parameter values of $a_{2i}, b_{2i-1}, c_{2i-1}$, and ε_{2i} $(1 \le i \le n)$. Let the eigenvalues of matrix R$= A + B$ be γ_i $(1 \le i \le 2n)$, the eigenvalues of matrix A be ρ_i $(1 \le i \le 2n)$. If there exists at least one eigenvalue $\rho_k, k \in \{1, 2, \cdots, 2n\}$ such that Re $(\rho_k) > 0$, then the unique equilibrium point of system (8) is unstable, implying that system (8) generates an oscillatory solution.

Proof Obviously, if the trivial solution of system (10) is unstable, then the trivial solution of system (9) is also unstable. Therefore, we only need to consider the stability of the trivial solution of system (10). The characteristic equation corresponding to system (10) is

$$\det\left(\lambda I_{ij} - a_{ij} - b_{ij}e^{-\lambda\tau_j}\right) = 0 \quad (20)$$

Noting that each characteristic value of matrix B is zero. So we have

$$\det\left(\lambda I_{ij} - a_{ij} - b_{ij}e^{-\lambda\tau_j}\right) = \prod_{i=1}^{2n} \lambda_i - \rho_i = 0 \quad (21)$$

By the assumption, there exits at least one k such that Re

$(\lambda_k) = \mathrm{Re}\,(\rho_k) > 0$, this means that the trivial solution of system (10) is unstable, implying that the trivial solution of system (8) (or (9)) is unstable. Since all solutions of system (8) (or (9)) are bounded, and system (8) has a unique unstable equilibrium. Based on the generalized Chafee's criterion, this instability of the unique equilibrium will force system (8) to generate an oscillatory solution.

Theorem 3 Assume that system (8) has a unique equilibrium point, for selected parameter values of $a_{2i}, b_{2i-1}, c_{2i-1}$, and ε_{2i} $(1 \le i \le n)$. If there exists one $\varepsilon_{2k}a_{2k}, k \in \{1,2,\cdots,n\}$ such that

$$c_{2k-1} - \varepsilon_{2k}a_{2k} < 0 \tag{22}$$

then the unique equilibrium point of system (10) is unstable, implying that system (8) generates an oscillatory solution.

Proof As theorem 2, we only need to consider the instability of the trivial solution of system (10). For some $k \in \{1,2,\cdots,n\}$, consider an auxiliary equation

$$\begin{aligned} y_{2k}'(t) &= -c_{2k-1}y_{2k}(t) + b_{2k}[y_{2k}(t-\tau_{2k-3}) \\ &\quad -2y_{2k}(t-\tau_{2k-1}) \\ &\quad + y_{2k}(t-\tau_{2k+1})] + \varepsilon_{2k}a_{2k}y_{2k}(t) \end{aligned} \tag{23}$$

The characteristic equation of (23) is the following

$$\lambda + c_{2k-1} - \varepsilon_{2k}a_{2k} - b_{2k}e^{-\lambda\tau_{2k-3}} + 2b_{2k}e^{-\lambda\tau_{2k-1}} - b_{2k}e^{-\lambda\tau_{2k+1}} = 0 \tag{24}$$

We show that the characteristic equation (24) of the auxiliary equation has a real positive root say $\lambda^*(>0)$. Define a function

$$h(\lambda) = \lambda + c_{2k-1} - \varepsilon_{2k}a_{2k} - b_{2k}e^{-\lambda\tau_{2k-3}} + 2b_{2k}e^{-\lambda\tau_{2k-1}} - b_{2k}e^{-\lambda\tau_{2k+1}} \tag{25}$$

Obviously, $h(\lambda)$ is a continuous function of λ. Under the restrictive condition (22) we have $h(0) = c_{2k-1} - \varepsilon_{2k}a_{2k} - b_{2k} + 2b_{2k} - b_{2k} = c_{2k-1} - \varepsilon_{2k}a_{2k} < 0$. Noting that $e^{-\lambda\tau_{2k-3}} \to 0$ as $\lambda \to \infty$, $e^{-\lambda\tau_{2k-1}} \to 0$ as $\lambda \to \infty$, and $e^{-\lambda\tau_{2k+1}} \to 0$ as $\lambda \to \infty$. Therefore, there exists a suitably large λ say $\lambda_1(>0)$ such that $h(\lambda_1) = \lambda_1 + c_{2k-1} - \varepsilon_{2k}a_{2k} - b_{2k}e^{-\lambda_1\tau_{2k-3}} + 2b_{2k}e^{-\lambda_1\tau_{2k-1}} - b_{2k}e^{-\lambda_1\tau_{2k+1}} > 0$. By means of the Intermediate Value Theorem of continuous function, there exists a $\lambda^* \in (0, \lambda_1)$ such that $h(\lambda^*) = 0$, where λ^* is a positive characteristic root of equation (25). This means that the trivial solution of the auxiliary equation (23) is unstable, implying that the trivial solution of system (10) is also unstable. Based on the generalized Chafee's criterion, this instability of the unique equilibrium will force system (8) to generate an oscillatory solution.

Example 1 Consider the case of $n = 3$ in the following:

$$\begin{cases} x_1'(t) = x_2(t), \\ x_2'(t) = -c_1x_1(t) - \beta_1x_1^3(t) + b_1[x_5(t-\tau_5) - 2x_1t-\tau_1) + x_3(t-\tau_3)] \\ \quad +\varepsilon_2a_2x_2(t) - \varepsilon_2x_1^2(t)x_2(t) + \varepsilon_2k_2x_1^4(t)x_2(t) - \varepsilon_2l_2x_1^6(t)x_2(t), \\ x_3'(t) = x_4(t), \\ x_4'(t) = -c_3x_3(t) - \beta_3x_3^3(t) + b_3[x_1(t-\tau_1) - 2x_3(t-\tau_3) + x_5(t-\tau_5)] \\ \quad +\varepsilon_4a_4x_4(t) - \varepsilon_4x_3^2(t)x_4(t) + \varepsilon_4k_4x_3^4(t)x_4(t) - \varepsilon_4l_4x_3^6(t)x_4(t), \\ x_5'(t) = x_6(t), \\ x_6'(t) = -c_5x_5(t) - \beta_5x_5^3(t) + b_5[x_3(t-\tau_3) - 2x_5(t-\tau_5) + x_1(t-\tau_1)] \\ \quad +\varepsilon_6a_6x_6(t) - \varepsilon_6x_5^2(t)x_6(t) + \varepsilon_6k_6x_5^4(t)x_6(t) - \varepsilon_6l_6x_5^6(t)x_6(t). \end{cases} \tag{26}$$

The parameter values are selected as $c_1 = 5.45, c_3 = 5.65, c_5 = 5.85$; $b_1 = 0.075$, $b_3 = 0.085, b_5 = 0.095$; $a_2 = -1.05$, $a_4 = -0.55$, $a_6 = -0.75$; $\beta_1 = 0.35$, $\beta_3 = 0.25$, $\beta_5 = 0.45$; $l_2 = 0.45$, $l_4 = 0.25$, $l_6 = 0.38$; $\varepsilon_2 = 0.0005$, $\varepsilon_4 = 0.0002, \varepsilon_6 = 0.0004$, and $k_2 = 0.35, k_4 = 0.42$, $k_6 = 0.68$, respectively, the eigenvalues of matrix $R_1 = A_1 + B_1$ are $-0.0002 \pm 7.5168\,i$, $-0.0002 \pm 2.4001\,i$, and $-0.0002 \pm 0.7792\,i$. Obviously, the conditions of Theorem 1 are satisfied. The solutions of system (26) are convergent (see Figure 1). When the parameter values are selected as $c_1 = 5.12$, $c_3 = 5.15$, $c_5 = 5.18$; $b_1 = 0.00175$, $b_3 = 0.00185, b_5 = 0.00165$; $a_2 = 0.95$, $a_4 = 0.55$, $a_6 = 0.75$; $\beta_1 = 0.25$, $\beta_3 = 0.15$, $\beta_5 = 0.45$; $l_2 = 0.15$, $l_4 = 0.24$, $l_6 = 0.18$; $\varepsilon_2 = 0.0005$, $\varepsilon_4 = 0.0002, \varepsilon_6 = 0.0004$, and $k_2 = 0.25, k_4 = 0.32$, $k_6 = 0.16$, respectively. The eigenvalues of matrix A_1 are $0.0024 \pm 2.2803\,i$, $0.0001 \pm 2.2694\,i$, and $0.0002 \pm 2.2760\,i$, the conditions of Theorem 2 are satisfied. System (26) generates an oscillatory solutions (see Figure 2). When the parameter values are selected as $c_1 = 25.12$, $c_3 = 25.15$, $c_5 = 25.18$, the other parameters are the same as in Figure 2, we see that the oscillation of the solutions is maintained. However, the oscillatory amplitude and frequency both are changed (see Figure 3), implying that the values of c_1, c_3 and c_5 affect the oscillatory amplitude and frequency very much of the solutions. When the parameter values are selected as $c_1 = 0.002$, $c_3 = 0.0015$, $c_5 = 0.0018$; $b_1 = 0.00175$, $b_3 = 0.00155, b_5 = 0.00165$; $a_2 = 10.95$, $a_4 = 10.55$, $a_6 = 10.75$; $\beta_1 = 0.25$, $\beta_3 = 0.15$, $\beta_5 = 0.45$; $l_2 = 0.15$, $l_4 = 0.24$, $l_6 = 0.18$; $\varepsilon_2 = 0.0005$, $\varepsilon_4 = 0.0002, \varepsilon_6 = 0.0004$, and $k_2 = 0.25, k_4 = 0.32$, $k_6 = 0.16$, respectively, we have $c_1 - \varepsilon_2a_2 = -0.0035 < 0$, $c_3 - \varepsilon_4a_4 = -0.0006 < 0$, and $c_5 - \varepsilon_6a_6 = -0.0025 < 0$. The conditions of Theorem 3 are satisfied. System (26) generates an oscillatory solutions (see Figure 4).

Example 2 Consider the case of $n = 4$ in the following:

$$\begin{cases} x_1'(t) = x_2(t), \\ x_2'(t) = -c_1x_1(t) - \beta_1x_1^3(t) + b_1[x_7(t-\tau_7) - 2x_1t-\tau_1) + x_3(t-\tau_3)] \\ \quad +\varepsilon_2a_2x_2(t) - \varepsilon_2x_1^2(t)x_2(t) + \varepsilon_2k_2x_1^4(t)x_2(t) - \varepsilon_2l_2x_1^6(t)x_2(t), \\ x_3'(t) = x_4(t), \\ x_4'(t) = -c_3x_3(t) - \beta_3x_3^3(t) + b_3[x_1(t-\tau_1) - 2x_3(t-\tau_3) + x_5(t-\tau_5)] \\ \quad +\varepsilon_4a_4x_4(t) - \varepsilon_4x_3^2(t)x_4(t) + \varepsilon_4k_4x_3^4(t)x_4(t) - \varepsilon_4l_4x_3^6(t)x_4(t), \\ x_5'(t) = x_6(t), \\ x_6'(t) = -c_5x_5(t) - \beta_5x_5^3(t) + b_5[x_3(t-\tau_3) - 2x_5(t-\tau_5) + x_7(t-\tau_7)] \\ \quad +\varepsilon_6a_6x_6(t) - \varepsilon_6x_5^2(t)x_6(t) + \varepsilon_6k_6x_5^4(t)x_6(t) - \varepsilon_6l_6x_5^6(t)x_6(t), \\ x_7'(t) = x_8(t), \\ x_8'(t) = -c_7x_7(t) - \beta_7x_7^3(t) + b_7[x_5(t-\tau_5) - 2x_7(t-\tau_7) + x_1(t-\tau_1)] \\ \quad +\varepsilon_8a_8x_8(t) - \varepsilon_8x_7^2(t)x_8(t) + \varepsilon_8k_8x_7^4(t)x_8(t) - \varepsilon_8l_8x_7^6(t)x_8(t). \end{cases} \tag{27}$$

The parameter values are selected as $c_1 = 0.0025$, $c_3 = 0.0024$, $c_5 = 0.0022$ $c_7 = 0.0028$; $b_1 = 0.00135$, $b_3 = 0.00125$, $b_5 = 0.00115$, $b_7 = 0.00125$; $a_2 = 12.85$, $a_4 = 12.65$, $a_6 = 12.75$, $a_8 = 12.45$; $\beta_1 = 0.28$, $\beta_3 = 0.16$, $\beta_5 = 0.22$, $\beta_7 = 0.25$; $l_2 = 0.16$, $l_4 = 0.24$, $l_6 = 0.18$, $l_8 = 0.22$; $\varepsilon_2 = 0.00024$, $\varepsilon_4 = 0.00035$, $\varepsilon_6 = 0.00032$, $\varepsilon_8 = 0.00036$, and $k_2 = 0.35$, $k_4 = 0.32$, $k_6 = 0.26$, $k_8 = 0.36$, respectively. We have $c_1 - \varepsilon_2a_2 = -0.0005 < 0$, $c_3 - \varepsilon_4a_4 = -0.002 < 0$, $c_5 - \varepsilon_6a_6 = -0.0019 < 0$, and $c_7 - \varepsilon_8a_8 = -0.0019 < 0$. The conditions of Theorem 3 are satisfied. System (27) generates an oscillatory solutions (see Figures 5, 6 and Figure 7).

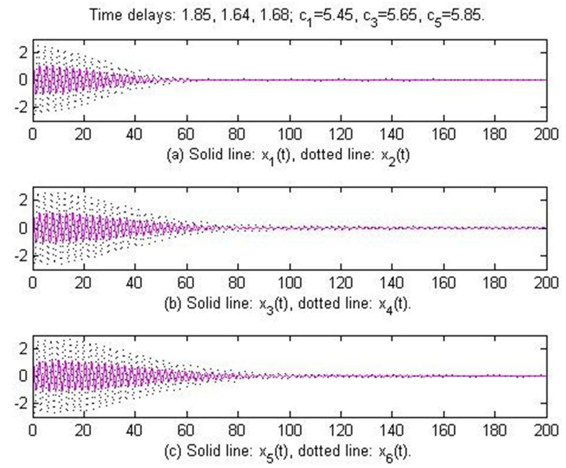

Figure 1. *Convergence of the solutions.*

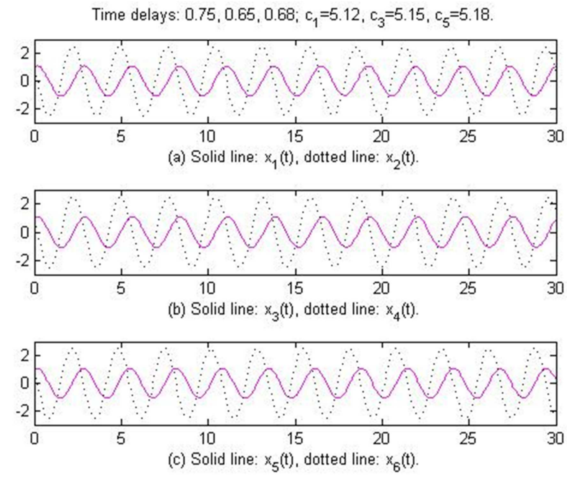

Figure 2. *Oscillation of the solutions with c_1 =5.12, c_2 =5.15, c_3 =5.18.*

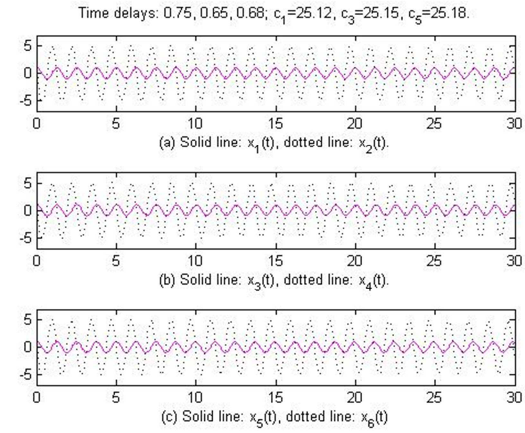

Figure 3. *Oscillation of the solutions with c_1 =25.12, c_2 =25.15, c_3 =25.18.*

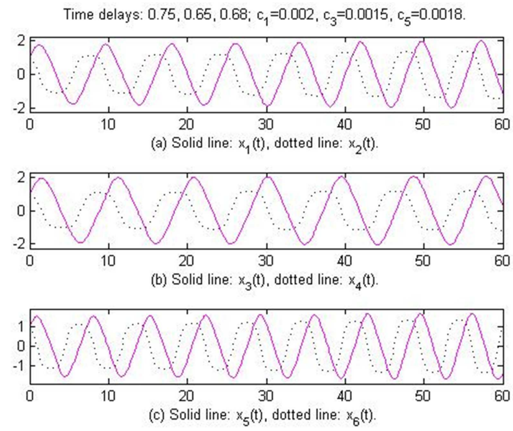

Figure 4. *Oscillation of the solutions.*

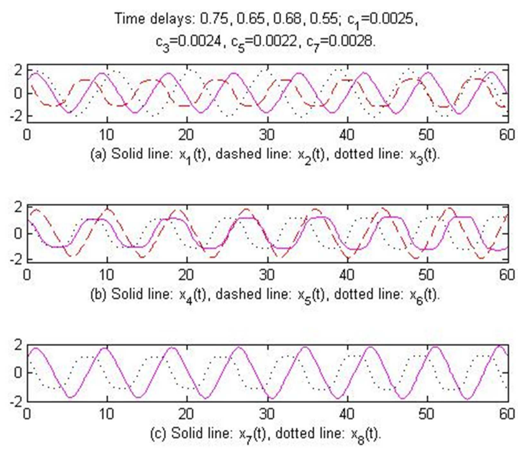

Figure 5. *Oscillation of the solutions with delays: 0.75, 0.65, 0.68, 0.55.*

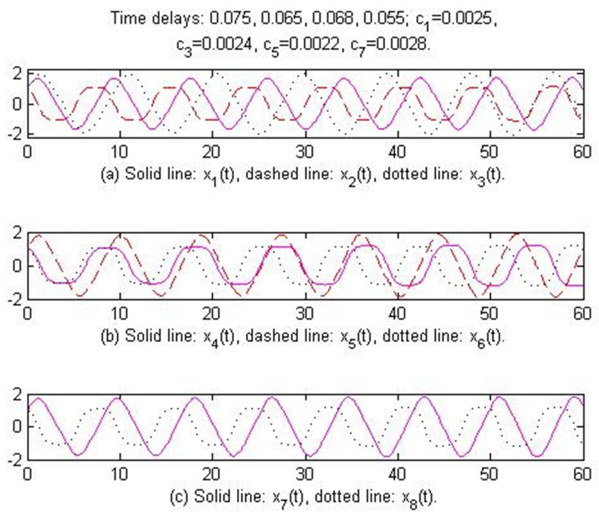

Figure 6. *Oscillation of the solutions with delays: 0.075, 0.065, 0.068, 0.055.*

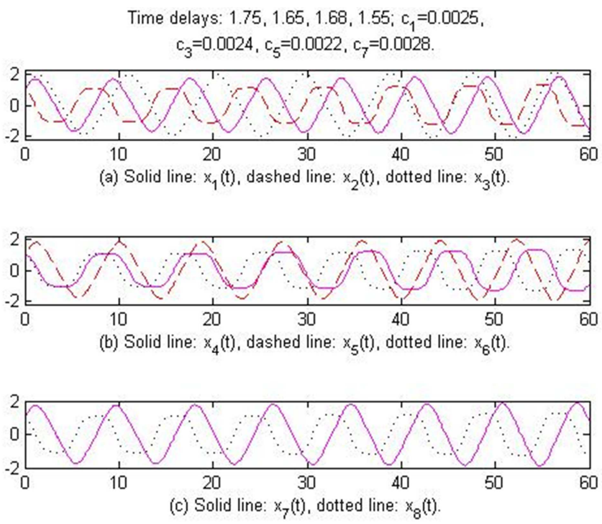

Figure 7. *Oscillation of the solutions with delays: 1.75, 1.65, 1.68, 1.55.*

4. Discussion

When system generates an oscillatory solution, from the Figures we know that time delay affects the oscillatory amplitude and frequency not too much. However, the positive parameter values c_1, c_3, c_5, c_7 affect the stability and oscillation of the system. The oscillatory frequency changes too much when different values of c_1, c_3, c_5, and c_7 are selected.

5. Conclusion

In this paper, we have discussed the oscillatory behavior of the solutions on a class of coupled van der Pol-Duffing equations with delays. Based on the generalized Chafee's theory, a simple criterion to guarantee the existence of permanent oscillations, which is easy to check, as compared to predicting the regions of bifurcation have been proposed. In this network, the passive decay rate affects the oscillatory frequency and amplitude. When these time delay systems generate a permanent oscillation, the delays affect oscillatory frequency and amplitude slightly.

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