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The Asymptotic Behavior of Solutions to Quasilinear Elliptic Equation with Hardy Potential and Sobolev Critical Exponent Near Zero

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Abstract: This paper mainly explores the precise asymptotic behavior near zero of positive weak solutions to the quasilinear elliptic equation involving Hardy potential and Sobolev critical exponent, which is expressed as

$$-div\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right)-\frac{\gamma u^{p-1}(x)}{|x|^{(a+1)p}}=\frac{u^{p^*_{a,b}-1}(x)}{|x|^{bp^*_{a,b}}}, x\in\mathbb{R}^N\backslash\{0\}$$

under the conditions that $1<p<N$, $0\le a<\frac{N-p}{p}$, $a\le b<a+1$, $0\le\gamma<(\frac{N-(a+1)p}{p})^p$, and $p^*_{a,b}=\frac{Np}{N-(a+1-b)p}$. The research shows that if $u\in\mathscr{D}^{1,p}(\mathbb{R}^N,\mu_{ap})$ is a positive radial weak solution of this equation, then there exists $\gamma_1\in[0,\frac{N-(a+1)p}{p}]$ such that $\lim_{|x|\to 0}u(x)|x|^{\gamma_1}=C$ $(0<C<\infty)$, where γ_1 is the smallest root of the equation $\Gamma_\gamma(\ell):=\ell^{p-2}[(p-1)\ell^2-(N-(a+1)p)\ell]+\gamma=0$. This result accurately depicts the asymptotic characteristics of positive weak solutions of the equation near zero. Compared with previous relevant studies which only indicate that the solutions are bounded near zero, this study further clarifies the limiting situation of the solutions.

Keywords: Quasilinear Elliptic Equations; Hardy Potential: Critical Sobolev Growth

1. Introduction and Main Results

In this paper, we establish the asymptotic behavior of positive radial weak to the following weighted quasilinear elliptic problem with Hardy potential and critical Sobolev exponent

$$\begin{cases} -\text{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right)-\gamma\frac{u^{p-1}}{|x|^{(a+1)p}}=\frac{u^{p^*_{a,b}-1}}{|x|^{bp^*_{a,b}}}, & x\in\mathbb{R}^N\setminus\{0\},\\ u>0, & x\in\mathbb{R}^N,\\ u\in\mathscr{D}^{1,p}\left(\mathbb{R}^N,\mu_{ap}\right), \end{cases}\tag{1}$$

where $1<p<N$, $0\leqslant a<\frac{N-p}{p}$, $a\leqslant b<a+1$,

$$0\leqslant\gamma<\bar{\gamma}:=\left(\frac{N-(a+1)p}{p}\right)^p,\quad p^*_{a,b}=\frac{Np}{N-(a+1-b)p},\tag{2}$$

and

$$\mathscr{D}^{1,p}_{a,b}\left(\mathbb{R}^N\right) := \left\{u \in L^{p^*_{a,b}}(\mathbb{R}^N, \mu_{bp^*_{a,b}}) : |\nabla u| \in L^p(\mathbb{R}^N, \mu_{ap})\right\}. \tag{3}$$

Quasilinear elliptic equations are a class of partial differential equations that arise in various fields of mathematics and physics, including fluid dynamics, elasticity, and differential geometry. These equations often involve the p-Laplacian operator and may include singular terms such as Hardy potentials or critical Sobolev exponents. In the study of quasilinear elliptic equations with weights, such as the Caffarelli-Kohn-Nirenberg inequality, gradient estimates play a pivotal role in understanding the asymptotic behavior of solutions. Shakerian and Vétois [1] investigated the asymptotic behavior of solutions to a class of weighted quasilinear elliptic equations. They obtained sharp pointwise estimates, extending previous results in the unweighted case. The authors used a Kelvin-type transformation to reduce the problem at infinity to another elliptic-type problem near the origin. This approach allowed them to refine the asymptotic expansion and obtain Holder-type estimates. Their results are significant for understanding the behavior of solutions in the presence of weights and for developing methods to handle non-radial solutions. Li and Zhao [2] studied the exponential decay properties of ground states for quasilinear elliptic equations. They provided an explicit formula for the decay properties of ground states in the whole space. The authors focused on the quasilinear elliptic equation involving the degenerate m-Laplace operator and derived asymptotic estimates for radial ground states. Their results are particularly useful for applications in nonlinear scalar field equations and for understanding the long-term behavior of solutions. He and Xiang [3] explored the asymptotic behaviors of solutions to quasilinear elliptic equations with Hardy potential. They obtained optimal estimates for both positive radial and general weak solutions at the origin and at infinity. The authors used a combination of comparison principles, Harnack inequalities, and weak Lebesgue space embeddings to derive their results. Their work provides a comprehensive understanding of how the Hardy potential affects the asymptotic behavior of solutions and extends previous results to a broader class of equations. Dutta [4] established sharp decay estimates for solutions to the Euler-Lagrange equation corresponding to the Hardy-Sobolev-Maz'ya inequality. The author used a combination of rescaling techniques, weak Harnack inequalities, and Poincaré-Sobolev inequalities to derive the decay estimates. The results are significant for understanding the behavior of solutions near the origin and at infinity and have applications in the study of Brézis-Nirenberg problems involving lower-order perturbations of Hardy-Sobolev equations. Pu et al. [5] investigated the asymptotic behaviors of positive weak solutions to quasilinear elliptic equations with Hardy potential and critical Sobolev exponent. They obtained optimal estimates for the asymptotic behavior of solutions at the origin and at infinity. The authors used comparison principles and auxiliary results to derive their estimates. Their work extends previous results and provides a deeper understanding of the interplay between the Hardy potential and the critical Sobolev exponent in determining the asymptotic behavior of solutions. For other relevant results on the gradient estimates of solutions to quasilinear elliptic equations, please refer to Reference [6–9] and its cited literature.

Based on the above research achievements, this paper mainly focuses on the precise asymptotic behavior of the solutions to equations (1). The main results of this paper is

Theorem 1.1. Assume that $0 \leqslant \gamma < \left(\frac{N-(a+1)p}{p}\right)^p$. Let $u \in \mathscr{D}^{1,p}\left(\mathbb{R}^N, \mu_{ap}\right)$ be a positive radial weak solution of equation (1). Then there exists $\gamma_1 \in [0, \frac{N-(a+1)p}{p}]$ such that

$$\lim_{|x|\to 0} u(x)|x|^{\gamma_1} = C \tag{4}$$

for a constant $0 < C < \infty$, where γ_1 is smallest roots of

$$\Gamma_\gamma(\ell) := \ell^{p-2}\left[(p-1)\ell^2 - (N-(a+1)p)\ell\right] + \gamma = 0. \tag{5}$$

Remark 1.1. Pu et al. [5, Theorem 1.1] show that there exist positive constants C, c depending on N, p, γ and the solution u such that

$$c|x|^{-\gamma_1} \leqslant u(x) \leqslant C|x|^{-\gamma_1} \quad \text{for } |x| < r_0. \tag{6}$$

It is well know that (6) only shows that $u(x)|x|^{\gamma_1}$ is bounded near zero. (4) shows that it is not only bounded but also has a limit.

2. Proof of Theorem 1.1

Assuming $u(x) = u(r)$, where $r = |x|$. Then $u(r)$ satisfiles

$$-\frac{d}{dr}\left(r^{N-1-ap}|u'(r)|^{p-2}u'(r)\right) = r^{N-1}\left(\gamma\frac{u^{p-1}(r)}{r^{(a+1)p}} + \frac{u^{p^*_{a,b}-1}(r)}{r^{bp^*_{a,b}}}\right). \tag{7}$$

Lemma 2.1. Suppose that $u(r)$ is the solution to problem (7). Then for sufficiently small r,

$$u'(r) < 0. \tag{8}$$

Proof According to (6), we know that

$$\lim_{r\to 0} r^{\frac{N-(a+1)p}{p}} u(r) = 0,$$

which implies that

$$\lim_{r\to 0} u^{p^*_{a,b}-p} r^{(a+1)p-bp^*_{a,b}} = 0.$$

This fact together with (7) leads to, for small r,

$$\frac{d}{dr}\left(r^{N-1-ap}|u'(r)|^{p-2}u'(r)\right) < 0, \tag{9}$$

which shows that $\{r^{N-1-ap}|u'(r)|^{p-2}u'(r)\}$ is a monotonic decreasing convergent series. Without loss of generality, we assume that

$$\lim_{r\to 0} r^{N-1-ap}|u'(r)|^{p-2}u'(r) = m.$$

Now we show that $m = 0$. Otherwise, there exist constants C and r_0 such that

$$|u'(r)| \geq Cr^{-\frac{N-1-ap}{p-1}}$$

for any $0 < r < r_0$. Therfore,

$$\int_0^{r_0} |u'(r)|^p r^{N-1-ap} \geq C\int_0^{r_0} r^{-\frac{N-1-ap}{p-1}} = \infty,$$

since $a < \frac{N-p}{p}$. While the fact contradict to $u \in \mathscr{D}^{1,p}\left(\mathbb{R}^N, \mu_{ap}\right)$. Thus $m = 0$. This fact combined with (9) shows that (8) holds.

Lemma 2.2. Define

$$w(r) = -\frac{r^{p-1}\left|u'(r)\right|^{p-2}u'(r)}{u^{p-1}(r)}.$$

Then

$$\lim_{r\to 0} w(r) = \gamma_1^{p-1}. \tag{10}$$

Proof By (8), rewrite $w(r)$ as

$$w(r) = r^{p-1}\frac{(-u'(r))^{p-1}}{u^{p-1}(r)}.$$

Thus

$$\begin{aligned}
&w'(r)\\
=&\left(r^{p-1}\frac{(-u'(r))^{p-1}}{u^{p-1}(r)}\right)'\\
=&(p-1)r^{p-2}\frac{(-u'(r))^{p-1}}{u^{p-1}(r)} + r^{p-1}\frac{d}{dr}\left(\frac{(-u'(r))^{p-1}}{u^{p-1}(r)}\right)\\
=&(p-1)r^{p-2}\frac{(-u'(r))^{p-1}}{u^{p-1}(r)} + \frac{(p-1)(-u'(r))^{p-2}(-u''(r))u^{p-1}(r) + (-u'(r))^{p-1}(p-1)u^{p-2}(r)u'(r)}{r^{1-p}u^{2(p-1)}(r)}\\
=&(p-1)r^{p-2}\frac{(-u'(r))^{p-1}}{u^{p-1}(r)} - (p-1)\frac{(-u'(r))^{p-2}u''(r)}{r^{1-p}u^{p-1}(r)} + (p-1)\frac{(-u'(r))^{p}}{r^{1-p}u^{p}(r)}\\
=&(p-1)\frac{w}{r} - \frac{(p-1)u''(r)w}{-u'(r)} + \frac{p-1}{r}w^{\frac{p}{p-1}}.
\end{aligned} \tag{11}$$

By (7), we have

$$\begin{aligned}
\frac{(p-1)u''}{-u'} =& \frac{(N-1-ap)}{r} + \gamma\frac{(-u')^{1-p}u^{p-1}}{r^p} + \frac{(-u')^{1-p}u^{p^*_{a,b}-1}}{r^{bp^*_{a,b}-ap}}\\
=&\frac{(N-1-ap)}{r} + \frac{\gamma}{rw} + \frac{u^{p^*_{a,b}-p}}{r^{bp^*_{a,b}-(a+1)p+1}w}.
\end{aligned}$$

This fact together with (11), leads to

$$\begin{aligned}
&w'(r)\\
=&(p-1)\frac{w}{r}+\frac{(p-1)u''(r)w}{-u'(r)}+(p-1)\frac{-u'(r)w}{u}\\
=&(p-1)\frac{w}{r}-\frac{(N-1-ap)}{r}w+\frac{\gamma}{r}-\frac{u^{p^*_{a,b}-p}}{r^{bp^*_{a,b}-(a+1)p+1}}+\frac{p-1}{r}w^{\frac{p}{p-1}}\\
=&-\frac{N-(a+1)p}{r}w+\frac{\gamma}{r}-\frac{u^{p^*_{a,b}-p}}{r^{bp^*_{a,b}-(a+1)p+1}}+\frac{p-1}{r}w^{\frac{p}{p-1}}\\
=&\frac{1}{r}\Gamma_\gamma\left(w^{\frac{1}{p-1}}(r)\right)-\frac{u^{p^*_{a,b}-p}}{r^{bp^*_{a,b}-(a+1)p+1}}
\end{aligned}\tag{12}$$

where

$$\Gamma_\gamma(\ell)=\ell^{p-2}\left[(p-1)\ell^2-(N-(a+1)p)\ell\right]+\gamma.$$

In order to show that (10) holds, we firstly show $\lim_{r\to 0} w(r)$ exists. We prove this by contradiction that

$$\beta\equiv\limsup_{r\to 0} w>\liminf_{r\to 0} w\equiv\alpha.$$

Then there exist a local maximum $\{\xi_i\}$ and a local minimum $\{\eta_i\}$ such that $\xi_i\to 0$, $\eta_i\to 0$ and $\eta_i>\xi_i>\eta_{i+1}$ for all $i=1,2,\cdots$. That is

$$\lim_{i\to\infty} w(\xi_i)=\beta,\ \lim_{i\to\infty} w(\eta_i)=\alpha.$$

This fact together with (12) implies that

$$\lim_{i\to\infty}\Gamma_\gamma\left(w^{\frac{1}{p-1}}(\xi_i)\right)=\lim_{i\to\infty}\Gamma_\gamma\left(w^{\frac{1}{p-1}}(\eta_i)\right)=0.$$

Thus

$$\Gamma_\mu\left(\beta^{\frac{1}{p-1}}\right)=\Gamma_\mu\left(\alpha^{\frac{1}{p-1}}\right)=0.$$

Therefore

$$\beta=\gamma_2^{p-1},\ \ \alpha=\gamma_1^{p-1}.$$

Note that $\gamma_1<(N-(a+1)p)/p<\gamma_2$. So there exists $\zeta_i\in(\eta_{i+1},\xi_i)$) such that

$$w(\eta_{i+1})<w(\zeta_i)=\left(\frac{N-(a+1)p}{p}\right)^{p-1}<w(\xi_i)$$

for i enough. Then by (12), we obtain that

$$\zeta_i w'(\zeta_i)=\Gamma_\mu\left(\frac{N-(a+1)p}{p}\right)+\frac{u^{p^*_{a,b}-p}}{r^{bp^*_{a,b}-(a+1)p}}=-(\overline{\mu}-\mu)+o(1)<0$$

for i large enough. Hence $w'(\zeta_i)<0$ for i large enough. Therefore w strictly decreasing in a neighborhood of ζ_i. Since $\zeta_i<\xi_i$ and $w(\zeta_i)<w(\xi_i)$, there exists $\zeta_i<\zeta_i'<\xi_i$ such that $w(r)\geq w(\zeta_i)$ for $\zeta_i<r<\zeta_i'$ and $w(\zeta_i')=w(\zeta_i)$. Thus $w'(\zeta_i')\geq 0$. However, $w'(\zeta_i')<0$. We reach a contradiction. Therefore $\lim_{r\to 0} w(r)$ exists.

Without loss of generality, we may assume $k^{p-1}=\lim_{r\to 0} w(r)$, We prove that $k=\gamma_1$. By (12), we have that

$$\lim_{r\to 0} rw'(r)=\Gamma_\mu(k).$$

We claim that $\Gamma_\mu(k)=0$. Otherwise, suppose that $\Gamma_\mu(k)\neq 0$. Note that for any $0<s<s_0$, we have

$$w(s_0)=w(s)+\int_s^{s_0} w'.$$

Then $\Gamma_\mu(k) \neq 0$ implies that $\lim_{s\to 0} |\int_s^{s_0} w'| = \infty$ if s_0 is small enough. This contradicts to the existence of $\lim_{r\to 0} w(r)$. Hence $\Gamma_\mu(k) = 0$. Recall that $\Gamma_\mu(\gamma) = 0$ if and only if $\gamma = \gamma_1$ or $\gamma = \gamma_2$. Thus we have either $k = \gamma_1$ or $k = \gamma_2$. Then we can deduce that $k = \gamma_1$. This proves (10) .

Proof of Theorem 1.1 (4) is a direct conclusion of (10).

Conflicts of Interest

The authors declare no conflicts of interest.

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