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Research Article

Non-Homogeneous Binary Cubic Equation a(x-y)^3=8bxy

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Abstract

Polynomial equations an interesting subject in theory of numbers, occupy a pivotal role in the realm of mathematics and have a wealth of historical significance. The theoretical importance of polynomial equations of third degree in two unknowns having integral coefficients is great as they are closely connected with many problems of number theory. Specifically, the third degree polynomial equations having two unknowns in connection with elliptic curves occupy a pivotal role in the region of mathematics. This paper discusses on finding many solutions in integers to a typical third degree equation having two variables expressed as a(x-y)^3=8bxy. The substitution strategy is employed in obtaining successfully different choices of solutions in integers. Some of the special fascinating numbers, namely, Pyramidal numbers, Polygonal numbers, Centered pyramidal numbers, Centered polygonal numbers, Thabit ibn Qurra numbers, Star numbers, Mersenne numbers and Nasty numbers (numbers expressed as product of two numbers in two different ways such that the sum of the factors in one set equals to the difference of factors in another set) are discussed in properties. These special numbers are unique. and have attractive characterization that set them apart from other numbers. The process of formulating second order Ramanujan numbers with base numbers as real integers is illustrated through examples.

Keywords

Non-Homogeneous Cubic, Binary Cubic, Integer Solutions, Ramanujan Numbers

1. Introduction

Number theory is one of the most fascinating and interesting subjects occupying an important place in the history of Mathematics. One of the interesting areas of Number Theory is the subject of Diophantine equations which has fascinated and motivated both Amateurs and Mathematicians alike. It is well-known that Diophantine equation is a polynomial equation in two or more unknowns requiring only integer solutions. The subject of Diophantine equations requiring only the integer solutions is an interesting area in the Theory of Numbers and it is the significant creation of the man-kind.

The beauty of Diophantine equations is that the number of equations is less than the number of unknowns. One can easily understand that the Diophantine problems are rich in variety playing a significant role in the development of Mathematics.

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The theory of Diophantine equations is popular in recent years providing a fertile ground for both Professionals and Amateurs. In addition to known results, this abounds with unsolved problems. Although many of its results can be stated in simple and elegant terms, their proofs are sometimes long and complicated.

There are unlimited varieties of third degree Diophantine equations which contribute in major part of research in this field. [1, 2]. We come across many problems in homogeneous or non-homogeneous cubic Diophantine equations with two or more variables. For the study of different choices of non-homogeneous cubic Diophantine equations with three unknowns by Gopalan et al, one may refer [3-10]. In [11, 12], Vidhyalakshmi et al considered the non-homogeneous third degree equation for obtaining their integer solutions. For varieties of third degree Diophantine equations having four variables by Gopalan et al, look into the references mentioned in [13-16]. In [17], Janaki and saranya considered a special cubic equation with four unknowns for its solutions in integers. Two more interesting cubic equations with four unknowns have been considered by Vidhyalakshmi et al [18, 19]. This paper concerns with getting infinitely many non-zero integral points for third degree Polynomial equation having two variables expressed as $a\,(x-y)^3 = 8b\,x\,y$. Substitution strategy is employed in obtaining successfully different choices of integral points to the above non-homogeneous third degree equation having two variables. The process of formulating second order Ramanujan numbers with base numbers as real integers is illustrated through examples.

Method of analysis

Consider the non-homogeneous third degree Diophantine equation with two unknowns represented by

$$a\,(x-y)^3 = 8\,b\,x\,y \tag{1}$$

Taking

$$x = u + bkv\,,\, y = u - bkv, \quad u \neq bkv \neq 0, k, b > 0 \tag{2}$$

in (1), it leads to

$$u^2 = b^2 k^2 v^2 (1 + akv) \tag{3}$$

Let us write

$$\alpha^2 = 1 + akv \tag{4}$$

which, after some calculations, is satisfied by

$$v = v_0 = akn^2 + 2n, \alpha = \alpha_0 = akn + 1 \tag{5}$$

Assume the second solution to (4) as

$$v_1 = v_0 + h\,, \alpha_1 = h - \alpha_0 \tag{6}$$

where h is an unknown to be determined. Substituting (6) in (4) and simplifying, we have

$$h = 2\alpha_0 + ak$$

and in view of (6), it is seen that

$$v_1 = v_0 + 2\alpha_0 + ak\,, \alpha_1 = \alpha_0 + ak$$

The repetition of the above process leads to the general solution to (4) as

$$\begin{aligned} v_N &= v_0 + 2N\alpha_0 + akN^2\,, \\ \alpha_N &= \alpha_0 + akN \end{aligned} \tag{7}$$

From (3), we have

$$\begin{aligned} u_N &= bkv_N * \alpha_N \\ &= bk\,(v_0 + 2N\alpha_0 + akN^2)\,(\alpha_0 + akN) \end{aligned} \tag{8}$$

In view of (2), we have

$$\begin{aligned} x_N &= u_N + bkv_N = bkv_N\,(\alpha_N + 1) \\ &= bk\,(v_0 + 2N\alpha_0 + akN^2)\,(\alpha_0 + akN + 1)\,, \\ y_N &= u_N - bkv_N = bkv_N\,(\alpha_N - 1) \\ &= bk\,(v_0 + 2N\alpha_0 + akN^2)\,(\alpha_0 + akN - 1) \end{aligned} \tag{9}$$

Thus, (1) is satisfied by (9).

To characterize the nature of solutions and to obtain varieties of interesting relations among the solutions, one has to go for taking particular values to the parameters in (9).

2. Inspection-I

We take

$a = 11, b = 1$

in (1) and it is written as

$$11\,(x-y)^3 = 8\,xy \tag{10}$$

The substitution of the linear transformations

$$\mathrm{x = u + ky, y = u - ky, u \neq kv \neq 0, k > 0} \tag{11}$$

in (10) leads to

$$\mathrm{u^2 = k^2 v^2 (1 + 11kv)} \tag{12}$$

Let

$$\alpha^2 = 1 + 11kv \tag{13}$$

which, after some calculations, is satisfied by

$$v = v_0 = 11kn^2 + 2n,\ \alpha = \alpha_0 = 11kn + 1 \tag{14}$$

Assume the second solution to (13) as

$$v_1 = v_0 + h,\ \alpha_1 = h - \alpha_0 \tag{15}$$

where h is to be found. Substituting (15) in (13) and simplifying, we have

$$h = 2\alpha_0 + 11k$$

and in view of (15), it is seen that

$$v_1 = v_0 + 2\alpha_0 + 11k,\ \alpha_1 = \alpha_0 + 11k$$

The repetition of the above process leads to the general solution to (13) as

$$v_N = v_0 + 2N\alpha_0 + 11kN^2,\ \alpha_N = \alpha_0 + 11kN \tag{16}$$

From (12), we have

$$\begin{aligned} u_N &= kv_N * \alpha_N \\ &= k(v_0 + 2N\alpha_0 + 11kN^2)(\alpha_0 + 11kN) \end{aligned} \tag{17}$$

In view of (11), we have

$$\begin{aligned} x_N &= u_N + kv_N = kv_N(\alpha_N + 1) \\ &= k(v_0 + 2N\alpha_0 + 11kN^2)(\alpha_0 + 11kN + 1), \\ y_N &= u_N - kv_N = kv_N(\alpha_N - 1) \\ &= k(v_0 + 2N\alpha_0 + 11kN^2)(\alpha_0 + 11kN - 1), \end{aligned} \tag{18}$$

Thus, (10) is satisfied by (18).

To obtain the relations among the solutions, one has to go for taking particular values to the parameters. For simplicity and brevity, we consider the integer solutions to (10) taking

$$k = 1, N = 0, v_0 = 11n^2 + 2n, \alpha_0 = 11n + 1$$

in (18) and they are given by

$$\begin{aligned} x_0 &= x_0(n) = (11n^2 + 2n)(11n + 2), \\ y_0 &= y_0(n) = (11n^2 + 2n)(11n) \end{aligned} \tag{19}$$

A few numerical values for the obtained solutions (19) to equation (10) are shown in Table 1 as follows:

Table 1. *Numerical values.*

n	$X_0(n)$	$y_0(n)$
1	13^2	11*13
2	$2*24^2$	44*24
3	$3*35^2$	99*35
4	$4*46^2$	176*46
5	$5*57^2$	275*57

From the above Table 1, it is seen that both the values of $x_0(n), y_0(n)$ are alternatively odd and even.

A few interesting relations among the integer solutions are presented below:

1. $121\ x_0(n)\ y_0(n)$ is a cubical integer
2. $x_0(n) - y_0(n) - Ct_{20,,n} - Ct_{24,n} \cong 0 \pmod 2$
3. $x_0(2^n) - y_0(2^n) = 7Th_{2n} + Th_n + 2\ t_{3,2^k} + 8$
4. $6\{n\ x_0(n) - n\ y_0(n) - 6\ CP_n^{22} + 16\}$ is a nasty number
5. $3\sum_{k=1}^{n}[x(k) - y(k)] = 44\,P_n^5 + 34\ t_{3,n}$
6. $3\sum_{k=1}^{n}[x(k) - y(k)] = 48\ P_n^3 + 28\ P_n^5 + 2t_{3,n}$
7. $x_0(n) - y_0(n) - S_n - 2n + 2$ is a perfect square
8. $\{x_0(n+2) - y_0(n+2)\} - 2\{x_0(n+1) - y_0(n+1)\} + \{x_0(n) - y_0(n)\} = 44$

Procedure to obtain Ramanujan numbers of order Two:

The process of obtaining Ramanujan numbers of order Two from (10) is illustrated.

Illustration 1

Consider

$$\begin{aligned} y(n) &= 11n^2\,(11n + 2) \\ &= 11n^2 * (11n + 2) = (11n^2 + 2n) * 11n \\ &= A * B = C * D \ \textit{say} \end{aligned}$$

From the above relation, one may observe that

$$\begin{aligned} (A+B)^2 + (C-D)^2 &= (A-B)^2 + (C+D)^2 \\ &= A^2 + B^2 + C^2 + D^2 \\ (11n^2 + 11n + 2)^2 + (11n^2 - 9n)^2 & \\ &= (11n^2 - 11n - 2)^2 + (11n^2 + 13n)^2 \\ &= 242n^4 + 44n^3 + 246n^2 + 44n + 4 \end{aligned}$$

Thus, $242n^4 + 44n^3 + 246n^2 + 44n + 4$ represents the second order Ramanujan number.

Illustration 2

Consider

$$\begin{aligned} x(n) &= n(11n+2)^2 \\ &= (11n^2+2n)*(11n+2) = (11n+2)^2 * n \\ &= G*H = E*F \ \ say \end{aligned}$$

In this case, the corresponding Second order Ramanujan number is found to be

$$14762\ n^4 + 10692\ n^3 + 3030\ k^2 + 396\ n + 20$$

Illustration 3

Consider

$$\begin{aligned} x(n) - y(n) &= 2\ n(11n+2) \\ &= (11n+2)*(2n) = 2\ (11n+2)*n \\ &= P*Q = R*S \ \ say \end{aligned}$$

For this choice, the corresponding Second order Ramanujan number is found to be

$$610n^2 + 220n + 20$$

Remark

It is worth mentioning that, apart from (14), other choice of integer solutions for (13) is given as

$$v = v_0 = 11kn^2 - 2n, \alpha = \alpha_0 = 11kn - 1$$

and taking

$$k = 1, N = 0, v_0 = 11n^2 - 2n, \alpha_0 = 11n - 1$$

in (18), we have

$$\begin{aligned} x_0 &= x(n) = (11n^2 - 2n)(11n), \\ y_0 &= y(n) = (11n-2)^2 (n) \end{aligned}$$

3. Inspection-II

We take

$$a = 24, b = 1$$

in (1) and it is written as

$$3(x-y)^3 = xy \tag{20}$$

Procedure 1

Taking

$$x = u + v, y = u - v, u \neq v \neq 0 \tag{21}$$

in (20), we have

$$u^2 = v^2(1 + 24v) \tag{22}$$

which is satisfied by

$$\begin{aligned} v &= \frac{n(3n+1)}{2}, \\ u &= \frac{n(3n+1)(6n+1)}{2} \end{aligned} \tag{23}$$

Substituting (23) in (21), one has

$$\begin{aligned} x &= x(n) = n(3n+1)^2, \\ y &= y(n) = 3n^2(3n+1) \end{aligned} \tag{24}$$

Observe that (24) satisfies (20).

Some numerical examples for (20) are exhibited below in Table 2:

Table 2. *Numerical examples.*

n	x(n)	y(n)
1	16	12
2	98	84
3	300	270
4	676	624
5	1280	1200

Relations among the solutions:

1. $2y(n) - x(n) - 6P_{n-1}^3$ is a cubical integer
2. $2y(n) - x(n) - 6CP_n^9$ is a multiple of 2
3. $2x(n) - y(n) - 18P_n^5$ is a multiple of 2
4. $x(n) + y(n) - 18P_n^5 - 6CP_n^9$ is a multiple of 4
5. $2[y(n+1) - y(n) + 3n]$ represents the area of Pythagorean triangle $(3(3n+2), 4(3n+2), 5(3n+2))$
6. $3y(n)$ is a perfect square for $n = 3k^2 \pm 2k$
7. $x(n) - y(n) - 3n = t_{8,n}$
8. $[x(n+1) - x(n)] - [y(n+1) - y(n)]$ is a perfect square for $n = 6k^2 \pm 4k$
9. $\sum_{n=1}^{N} (x(n) - y(n)) = 6P_N^3 - 2t_{3,N}$
10. $\sum_{n=1}^{N} y(n) = 3P_N^4 + (3t_{3,N})^2$

11. $9xy$ is a cubical integer
12. $9n^3 x(n) = (y(n))^2$
13. $16\sum_{n=1}^{N} x(n) - 136P_N^5 - 48t_{3,N}$ is written as the difference of two squares
14. $\sum_{n=1}^{N}[x(n)-y(n)]^2 = 6(t_{3,N})^2 + \frac{P_N^4(9Ct_{6,N}-4)}{5}$
15. $x(2^n) - y(2^n) = Th_{2n} + M_n + 2$
16. $x(n) - y(n)$ is a perfect square when n takes the values

$$n = n_s = \frac{(7+2\sqrt{12})^{s+1} + (7-2\sqrt{12})^{s+1} - 2}{12}, s = 0,1,2,...$$

17. $[x(n+2)-y(n+2)] - 2[x(n+1)-y(n+1)] + [x(n)-y(n)] = 6$
18. $[x(n+4)-y(n+4)] - 2[x(n+3)-y(n+3)] + 2[x(n+2)-y(n+2)]$ $-2[x(n+1)-y(n+1)] + [x(n)-y(n)] = 12$
19. $[x(n+k)-y(n+k)] - [x(n+k-1)-y(n+k-1)] = 2(3(n+k)-1)$

Procedure to obtain Ramanujan numbers of order Two:

The process of obtaining Ramanujan numbers of order Two from (20) is illustrated.

Illustration 4

$$\begin{aligned} x(n) &= n(3n+1)^2 \\ &= n(3n+1)*(3n+1) = (3n+1)^2 * n \\ &= A*B = C*D \ say \end{aligned}$$

From the above relation, one may observe that

$$(A+B)^2 + (C-D)^2 = (A-B)^2 + (C+D)^2 = A^2 + B^2 + C^2 + D^2$$

$$\begin{aligned} (3n^2+4n+1)^2 + (9n^2+5n+1)^2 \\ &= (3n^2-2n-1)^2 + (9n^2+7n+1)^2 \\ &= 90n^4 + 114n^3 + 65n^2 + 18n + 2 \end{aligned}$$

Thus, $90n^4 + 114n^3 + 65n^2 + 18n + 2$ represents the second order Ramanujan number.

Illustration 5

Consider

$$\begin{aligned} x(n) &= n(3n+1)^2 \\ &= n(3n+1)*(3n+1) = n(3n+1)^2 * 1 \\ &= A*B = E*F \ say \end{aligned}$$

In this case, the corresponding Second order Ramanujan number is found to be

$$81n^6 + 108n^5 + 63n^4 + 18n^3 + 11n^2 + 6n + 2$$

Illustration 6

Consider

$$\begin{aligned} y(n) &= 3n^2(3n+1) \\ &= 3n^2 * (3n+1) = (3n^2+n)*3n \\ &= A*B = C*D \ say \end{aligned}$$

From the above relation, one may observe that

$$\begin{aligned} (A+B)^2 + (C-D)^2 &= (A-B)^2 + (C+D)^2 \\ &= A^2 + B^2 + C^2 + D^2 \end{aligned}$$

$$\begin{aligned} (3n^2+3n+1)^2 + (3n^2-2n)^2 \\ &= (3n^2-3n-1)^2 + (3n^2+4n)^2 \\ &= 18n^4 + 6n^3 + 19n^2 + 6n + 1 \end{aligned}$$

Thus, $18n^4 + 6n^3 + 19n^2 + 6n + 1$ represents the second order Ramanujan number.

Remark:

In addition to the solutions (24), we have an another set of solutions to (20) given by

$$x = x(n) = 3n^2(3n-1), y = y(n) = n(3n-1)^2$$

4. Conclusions

In this article, the substitution strategy is utilized to obtain successfully integer solutions for third degree Polynomial equation having two variables. The readers may search for different approaches to analyze third degree equations with two unknowns. Further, they may search for varieties of relations through the obtained integer solutions.

Notations

$$P_n^5 = \frac{n^2(n+1)}{2}$$

$$P_n^3=\frac{n(n+1)(n+2)}{6}$$

$$t_{m,n}=\frac{n[2+(n-1)(m-2)]}{2}$$

$$Ct_{20,n}=10n^2+10n+1$$

$$Ct_{24,n}=12n^2+12n+1$$

$$Th_n=3*2^n-1$$

$$S_n=6n(n-1)+1$$

$$CP_k^{22}=\frac{22k^3-16k}{6}$$

$$M_n=2^n-1$$

Author Contributions

Nagarajan Thiruniraiselvi: Data curation, Investigation, Methodology, Resources, Visualization, Writing – review & editing

Sharadha Kumar: Formal Analysis, Methodology, Project administration, Validation, Writing – original draft

Mayilrangam Ambravaneswaran Gopalan: Conceptualization, Methodology, Resources, Supervision, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] L. E. Dickson, History of Theory of Numbers, Vol. 2, Chelsea Publishing Company, NewYork, 1952.

[2] L. J. Mordell, Diophantine equations, Academic press, New York, 1969.

[3] M. A. Gopalan, G. Sangeetha, "On the ternary cubic Diophantine equation $y^2=Dx^2+z^3$", Archimedes J. Math 1(1), 2011, 7-14.

[4] M. A. Gopalan, B. Sivakami, "Integral solutions of the ternary cubic equation $4x^2-4xy+6y^2=((k+1)^2+5)w^3$", Impact J. Sci. Tech, 6(1), 2012, 15-22.

[5] M. A. Gopalan, B. Sivakami, "On the ternary cubic Diophantine equation $2xz=y^2(x+z)$", Bessel J. Math, 2(3), 2012, 171-177.

[6] M. A. Gopalan, K. Geetha, "On the ternary cubic Diophantine equation $x^2+y^2-xy=z^3$", Bessel J. Math., 3(2), 2013, 119-123.

[7] M. A. Gopalan, S. Vidhyalakshmi, A. Kavitha "Observations on the ternary cubic equation $x^2+y^2+xy=12z^3$", Antartica J. Math, 10(5), 2013, 453-460.

[8] M. A. Gopalan, S. Vidhyalakshmi, K. Lakshmi, "Lattice points on the non-homogeneous cubic equation $x^3+y^3+z^3+(x+y+z)=0$", Impact J. Sci. Tech, 7(1), 2013, 21-25.

[9] M. A. Gopalan, S. Vidhyalakshmi, K. Lakshmi "Lattice points on the non-homogeneous cubic equation $x^3+y^3+z^3-(x+y+z)=0$", Impact J. Sci. Tech, 7(1), 2013, 51-55.

[10] M. A. Gopalan, S. Vidhyalakshmi, S. Mallika, "On the ternary non-homogenous cubic equation $x^3+y^3-3(x+y)=2(3k^2-2)z^3$", Impact J. Sci. Tech, 7(1), 2013, 41-45.

[11] S. Vidyalakshmi, T. R. Usharani, M. A. Gopalan, "Integral solutions of non-homogeneous ternary cubic equation $ax^2+by^2=(a+b)z^3$", Diophantus J. Math 2(1), 2013, 31-38.

[12] S. Vidhyalakshmi, M. A. Gopalan, S. Aarthy Thangam, "On the ternary cubic Diophantine equation $4(x^2+x)+5(y^2+2y)=-6+14z^3$" International Journal of Innovative Research and Review (JIRR), 2(3), July-Sep 2014, 34-39.

[13] M. A. Gopalan, S. Vidhyalakshmi, G. Sumathi, "On the homogeneous cubic equation with four unknowns $X^3+Y^3=14Z^3-3W^2(X+Y)$", Discovery, 2(4), 2012, 17-19.

[14] M. A. Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi, "On homogeneous cubic equation with four unknowns $(x+y+z)^3=z(xy+31w^2)$", Cayley J. Math, 2(2), 2013, 163-168.

[15] M. A. Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi, "On homogeneous cubic equation with four unknowns $x^3+y^3=21zw^2$", Review of Information Engineering and Applications, 1(4), 2014, 93-101.

[16] M. A. Gopalan, S. Vidhyalakshmi, E. Premalatha, C. Nithya, "On the cubic equation with four unknowns $x^3+y^3=31(k^2+3s^2)zw^2$", IJSIMR, 2(11), Nov-2014, 923-926.

[17] G. Janaki, C. Saranya, "Integral solutions of the ternary cubic equation $3(x^2+y^2)-4xy+2(x+y+1)=972z^3$", IRJET, 4(3), March 2017, 665-669.

[18] S. Vidhyalakshmi, M. A. Gopalan, A. Kavitha, "Observation on homogeneous cubic equation with four unknowns $X^3+Y^3=7^{2n}ZW^2$", IJMER, 3(3), May-June 2013, 1487-1492.

[19] S. Vidhyalakshmi, T. Mahalakshmi and M. A. Gopalan, "A Search On The Integer Solutions of Cubic Diophantine Equation with Four Unknowns $x^2+y^2+4(35z^2-4-w^2)=6xyz$", International Journal of Grid and Distributed Computing, 13(2021), 2, 2581-2584.

Biography

Nagarajan Thiruniraiselvi is working as Assistant Professor in Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Trichy-621 112 and is teaching Mathematics since 06 years. She is interested in finding solutions to different types of Diophantine Equation and Number patterns. She has published more than 75 papers in National and International journals. She, along with her colleagues, has published 03 books in the area of Diophantine equations and presented 15 papers in various International Conferences. She is a reviewer for “Journal of Experimental Agriculture International” and “Asian Research Journal of Mathematics”.

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Research Field

Nagarajan Thiruniraiselvi: Number Theory
Sharadha Kumar: Number Theory
Mayilrangam Ambravaneswaran Gopalan: Number Theory