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Research Articlek

Minimal Reducing Subspaces of 3-order Slant Toeplitz Operator on Hardy Space over the Disc

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Abstract

The reducing subspace problem and the invariant subspace problem of an operator are two core problems in operator theory. There are lots of works on reducing subspaces and invariant subspaces of Toeplitz operators in recent years. A slant Toeplitz operator is a generalization of Toeplitz operator. In this paper, we study minimal reducing subspaces of the third-order slant Toeplitz operator with the symbol z^N. By classifying N into three cases, we give a complete description of minimal reducing subspaces. Finally, all minimal reducing subspaces of the third-order slant Toeplitz operator with the symbol z^N on the Hardy space of the disc in the complex plane are given. This paper generalizes the relevant results on reducing subspaces of second-order slant Toeplitz Operators, enriches the study of reducing subspaces of slant Toeplitz Operators on Lebesgue spaces, and of the structure of slant Toeplitz Operators.

Keywords

Hardy Space, 3-order Slant Toeplitz Operators, Minimal Reducing Subspace

1. Introduction

Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space. An important theme in operator theory is to study the structure of an operator and the classification of operators. For a bounded operator T on $\mathscr{H}$, an closed subspace $\mathscr{M}$ of $\mathscr{H}$ is called invariant under T if $T\mathscr{M} \subset \mathscr{M}$; $\mathscr{M}$ is reducing under T if it is invariant under T and T^*, $\mathscr{M}$ is nontrivial if it is not $\{0\}$ or $\mathscr{H}$. The famous invariants subspace problem is: does every linear bounded operator on a separable infinite dimensional Hilbert space has a nontrivial invariant subspace? [1, 4, 5, 8, 14-16] A similar important question is the reducing subspace problem [2, 3, 5, 12, 17, 21, 22].

Let $\mathbb{T}$ and $\mathbb{D}$ denote the unit circle and the unit open disc in the complex plane $\mathbb{C}$, $H^2(\mathbb{T})$ denotes the Hardy space on $\mathbb{T}$:

$$H^2(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{C} \,\middle|\, f(z) = \sum_{n\in \mathbb{N}} a_n z^n, a_n \in \mathbb{C}, \sum_{n\in \mathbb{N}} |a_n|^2 < \infty \right\}$$

For any $n \in \mathbb{N}$, $e_n(z) := z^n$, $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal bases of $H^2(\mathbb{T})$. For a function $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ on $H^2(\mathbb{T})$ is defined as $T_\varphi f = P(\varphi f)$, where P is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. It

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φ is analytic, T_φ is called an analytic Toeplitz operator, in this case, it is just a multiplication operator $M_\varphi f=\varphi f$ for $f\in H^2(\mathbb{T})$.

Let $dA(z)$ denotes the area measure on $\mathbb{D}$, $L_a^2(\mathbb{D})$ be the Bergman space on $\mathbb{D}$, that is

$$L_a^2(\mathbb{D})=\left\{f:\mathbb{D}\to £ \left|\int_{\mathbb{D}}\left|f(z)\right|^2 dA(z)<\infty, f \textit{ is anlytic on } \mathbb{D}\right.\right\}$$

One can define Toeplitz operator T_φ on $L_a^2(\mathbb{D})$ similarly for a function $\varphi\in L^\infty(\mathbb{D})$.

In the last several decades, lots of important works on the reducing subspace problem of analytic Toeplitz operators on the Hardy space [2, 19, 20], the Bergman space [22, 6, 7, 13, 10, 9] and other function spaces [5] have been done, for example [3, 4, 8, 14, 15, 18].

For a positive integer $k\geq 2$, the operator W_k on $H^2(\mathbb{T})$ is defined to be

$$W_k\left(z^n\right)=\begin{cases} z^{\frac{n}{k}}, \text{ if } n \text{ can be divided by } k\\ 0, \text{ if } n \text{ cannot be divided by } k\end{cases}$$

The k-order slant Toeplitz operator $B_\varphi^{(k)}$ on $H^2(\mathbb{T})$ is defined by

$$B_\varphi^{(k)}=PW_kM_\varphi=W_kPM_\varphi\Big|_{H^2(T)}=W_kT_\varphi.$$

There are many papers on boundedness, compactness of k-order slant Toeplitz operators. But very recently, Hazarika and Sougata [11] began to study reducing subspaces of 2-order slant Toeplitz operators on the Lebesgue space of the unit circle. In this paper, we study the minimal reducing subspaces of 3-order slant Toeplitz operators on the Hardy space of the unit circle. For simplicity, we use B_{z^N} to denote 3-order slant Toeplitz operators on the Hardy space of the unit circle with symbol z^N.

2. A Partition of N

For further study, we need a partition of ¥ corresponding to a fixed N in ¥.

Definition 2.1. For a given $N\in$ ¥, let S_N denote

i. $\left\{N+3m, m>-\frac{N}{6}, m\in \quad\right\}\ \left\{N+2+3m, m>-\frac{4+N}{6}, m\in ?\right\}$ *if* $N\equiv 1 \bmod 3$;

ii. $\left\{N+3m, m>-\frac{N}{6}, m\in \quad\right\}\ \left\{N+1+3m, m>-\frac{2+N}{6}, m\in ?\right\}$ *if* $N\equiv 2 \bmod 3$;

iii. $\left\{N+1+3m, m>-\frac{2+N}{6}, m\in \quad\right\}\ \left\{N+2+3m, m>-\frac{4+N}{6}, m\in ?\right\}$ *if* $N\equiv 0 \bmod 3$;

Definition 2.2. For a given $N\in$ ¥, and for any $j\in S_N$, define

$$\Lambda_j^{(N)}:=\left\{\frac{N}{2}+3^t\left(j-\frac{N}{2}\right)\right\}_{t=0}^{\infty}.$$

Let $H_j^{(N)}$ be the closure of the linear span of $\{e_k\}_{k\in\Lambda_j^{(N)}}$. For simplicity, we use $\varepsilon_t^{(j,N)}$ to denote $e_{\frac{N}{2}+3^t\left(j-\frac{N}{2}\right)}$, and so $H_j^{(N)}$ be the closure of the linear span of $\left\{\varepsilon_t^{(j,N)}\right\}_{t\in\mathbb{Z}_+}$. In particular, for $j\in S_N$, $\varepsilon_0^{(j,N)}=\left\{ce_j\left|c\in £\right.\right\}$.

Lemma 2.3. For any $j,k\in S_N\left(j\neq k\right)$, $\Lambda_j^{(N)}$ I $\Lambda_k^{(N)}=\varnothing$.

Proof 2.4. Assume that $x\in\Lambda_j^{(N)}$ I $\Lambda_k^{(N)}$, so there exist $t_1,t_2>0$ such that

$$x=\frac{N}{2}+\left(j-\frac{N}{2}\right)3^{t_1}\in\Lambda_j^{(N)}$$

and

$$x=\frac{N}{2}+\left(k-\frac{N}{2}\right)3^{t_2}\in\Lambda_k^{(N)}$$

It implies that

$$\frac{N}{2}+\left(j-\frac{N}{2}\right)3^{t_1}=\frac{N}{2}+\left(k-\frac{N}{2}\right)3^{t_2}$$

and so

$$\left(j-\frac{N}{2}\right)3^{t_1}=\left(k-\frac{N}{2}\right)3^{t_2}$$

Since $j \neq k$, therefore $t_1 \neq t_2$, without loss of generality, we can assume that $t_1 > t_2$, so

$$(2k-N) = (2j-N)3^{t_1-t_2} \quad (*)$$

i. If $N = 3l+1$, in this case,

$$S_N := \left\{ N+3m, m > -\frac{N}{6}, m \in \quad \right\} \quad \left\{ N+2+3m, m > -\frac{4+N}{6}, m \in ? \right\}.$$

And so,

$$2k-N = \begin{cases} 3l+6m+1, & if \ k = N+3m, \\ 3l+6m+5, & if \ k = N+2+3m. \end{cases}$$

ii. If $N = 3l+2$, in this case,

$$S_N := \left\{ N+3m, m > -\frac{N}{6}, m \in \quad \right\} \quad \left\{ N+1+3m, m > -\frac{2+N}{6}, m \in ? \right\}.$$

And so,

$$2k-N = \begin{cases} 3l+6m+2, & if \ k = N+3m, \\ 3l+6m+4, & if \ k = N+1+3m. \end{cases}$$

iii. If $N = 3l$, in this case,

$$S_N := \left\{ N+1+3m, m > -\frac{2+N}{6}, m \in \quad \right\} \quad \left\{ N+2+3m, m > -\frac{4+N}{6}, m \in ? \right\}.$$

And so,

$$2k-N = \begin{cases} 3l+6m+2, & if \ k = N+1+3m, \\ 3l+6m+4, & if \ k = N+2+3m. \end{cases}$$

(i)-(iii) show that the left side of (*) does not contain a factor of 3, contradiction, so $\Lambda_j^{(N)} \mathrm{I} \ \Lambda_k^{(N)} = \varnothing$.

We have the following corollary by Lemma 1.

Corollary 2.5. For any $j, k \in S_N$ and $j \neq k$,

$$H_j^{(N)} \mathrm{I} \ H_k^{(N)} = \{0\}$$

Lemma 2.6. For a fixed N, any $n > \frac{N}{2}$, there is an unique $j \in S_N$ such that $n \in \Lambda_j^{(N)}$.

Proof 2.7. We prove the case $N = 3k+1$ only, for other two cases $N = 3k$ and $N = 3k+2$, one can prove it similarly. In this case,

The right side of the equation contains a factor of 3, by considering three cases of N and discussing the left of the equation respectively, then we get a proof of the lemma by a contradiction

$$S_N := \left\{ N+3m, m > -\frac{N}{6}, m \in \quad \right\} \quad \left\{ N+2+3m, m > -\frac{4+N}{6}, m \in ? \right\}$$

i. when $n = 3l > \frac{N}{2}$, let $j = n$, $t = 0$, then $j \in S_N$ and so

$$n = \frac{N}{2} + \left(n - \frac{N}{2}\right)3^0 \in \Lambda_j^{(N)}.$$

ii. when $n = 3l+1 > \frac{N}{2}$, let $j = n$, $t = 0$, then $j \in S_N$ and so

$$n = \frac{N}{2} + \left(n - \frac{N}{2}\right)3^0 \in \Lambda_j^{(N)}.$$

iii. when $n = 3l+2 > \frac{N}{2}$, and if $n+N$ is a multiple of 3, assume that

$$\frac{n+N}{3}=k_1,$$

If k_1+N is still a multiple of 3, assume that

$$\frac{k_1+N}{3}=k_2,\ldots\ldots$$

replicate this process until $k_{q-1}+N$ is still a multiple of 3, but k_q+N is not a multiple of 3, i.e.

$$\frac{k_{q-1}+N}{3}=k_q,\ k_q+N\neq 3p$$

So,

$$k_q+N=3p+1 \text{ or } k_q+N=3p+2$$

And let $j=k_q$, $t=q$, then $j\in S_N$ and

$$n=\frac{N}{2}+\left(k_q-\frac{N}{2}\right)3^q$$

And so

$$n\in\Lambda_k^{(N)}.$$

Remark 2.8. The Lemma 2.5 implies that we get a partition of $\mathbb{N}$ corresponding to a fixed N, i.e.

$$=\left\{0,1,\ \ ,\left[\frac{N}{2}\right]\right\}\bigcup_{j\in S_N}\Lambda_j^{(N)},$$

in other words,

$$\left\{\left[\frac{N}{2}\right]+1,\cdots\right\}=\bigcup_{j\in S_N}\Lambda_j^{(N)}.$$

3. Minimal Reducing Subspaces

Theorem 3.1. For a given $N\in\mathbb{N}$, let $\varphi(z)=z^N$. For any $j\in S_N$, $H_j^{(N)}$ is a reducing subspace of B_φ, and the restriction of B_φ on $H_j^{(N)}$ is the backward unilateral shift.

Proof 3.2. For any $j\in S_N$, $\left\{\varepsilon_t^{(j,N)}\right\}_{t\in\mathbb{N}_+}$ is an orthonormal basis of $H_j^{(N)}$.

If $t\in\mathbb{N}_+$,

$$B_\varphi\varepsilon_t^{(j,N)}(z)=W_3M_\varphi z^{\frac{N}{2}+\left(j-\frac{N}{2}\right)3^t}$$

$$=W_3z^{\frac{3N}{2}+\left(j-\frac{N}{2}\right)3^t}$$

$$=z^{\frac{N}{2}+\left(j-\frac{N}{2}\right)3^{t-1}}$$

$$=\varepsilon_{t-1}^{(j,N)}.$$

If $t=0$, $j\in S_N$, $\frac{3N}{2}+\left(j-\frac{N}{2}\right)3^t=j+N$. $B_\varphi\varepsilon_t^{(j,N)}(z)=W_3z^{j+N}$, it depends on the value of $j+N$. By considering the value of N, we have

i. when $N=3l+1$,

$$S_N:=\left\{N+3m,m>-\frac{N}{6},m\in\ \ \right\}\ \left\{N+2+3m,m>-\frac{4+N}{6},m\in ?\right\}$$

$$j+N=\begin{cases}6l+3m+2, & if\ j=N+3m,\\ 6l+3m+4, & if\ j=N+2+3m.\end{cases}$$

ii. when $N=3l+2$,

$$S_N:=\left\{N+3m,m>-\frac{N}{6},m\in\ \ \right\}\ \left\{N+1+3m,m>-\frac{2+N}{6},m\in ?\right\}$$

$$j+N=\begin{cases}6l+3m+4, & if\ j=N+3m,\\ 6l+3m+5, & if\ j=N+1+3m.\end{cases}$$

iii. when $N = 3l$,

$$S_N := \left\{ N+1+3m, m > -\frac{2+N}{6}, m \in \mathbb{Z} \right\} \cup \left\{ N+2+3m, m > -\frac{4+N}{6}, m \in \mathbb{Z} \right\}$$

$$j + N = \begin{cases} 6l+3m+1, \ if \ j = N+1+3m, \\ 6l+3m+2, \ if \ j = N+2+3m. \end{cases}$$

From (i) (ii) (iii), we know that $j+N$ cannot be a multiple of 3, and so

$$W_3 z^{j+N} = 0.$$

In summary, we have

$$B_\varphi \varepsilon_t^{(j,N)} = \begin{cases} 0, & t = 0 \\ \varepsilon_{t-1}^{(j,N)}, & t > 0 \end{cases}$$

therefore $H_j^{(N)}$ is invariant under B_φ.

Next we show that $H_j^{(N)}$ is invariant under B_φ^*.

If $k \in \mathbb{N}_+$, we have

$$< B_\varphi \varepsilon_0^{(j,N)}, \varepsilon_k^{(j,N)} > \ = 0 = \ < \varepsilon_0^{(j,N)}, \varepsilon_{k+1}^{(j,N)} >$$

$$< B_\varphi \varepsilon_m^{(j,N)}, \varepsilon_k^{(j,N)} > \ = \ < \varepsilon_{m-1}^{(j,N)}, \varepsilon_k^{(j,N)} > \ = \ < \varepsilon_m^{(j,N)}, \varepsilon_{k+1}^{(j,N)} >.$$

And so for any $f = \sum_i a_i \varepsilon_i^{(j,N)} \in H_j^{(N)}$, we have

$$< B_\varphi f, \varepsilon_k^{(j,N)} > \ = \sum_{i=0}^{\infty} a_i < B_\varphi \varepsilon_i^{(j,N)}, \varepsilon_k^{(j,N)} >$$

$$= \sum_{i=0}^{\infty} a_i < \varepsilon_{i-1}^{(j,N)}, \varepsilon_k^{(j,N)} >$$

$$= \sum_{i=0}^{\infty} a_i < \varepsilon_i^{(j,N)}, \varepsilon_{k+1}^{(j,N)} >$$

$$= \ < f, \varepsilon_{k+1}^{(j,N)} >$$

it implies $B_\varphi^* \varepsilon_k^{(j,N)} = \varepsilon_{k+1}^{(j,N)}$, for any $k \in \mathbb{N}$.

Remark 3.3. From Theorem 3.1, we get all minimal reducing subspaces of B_{z^N} which lie inside the subspace spanned by $\left\{ z^n \middle| n > \left[\frac{N}{2}\right] \right\}$.

Next, we investigate minimal reducing subspaces which lie inside the subspace spanned by $\left\{ z^n \middle| n = 0, 1, \cdots, \left[\frac{N}{2}\right] \right\}$.

Firstly, if N is even, $\varphi(z) = z^N$, then $B_\varphi e_{\frac{N}{2}} = e_{\frac{N}{2}} = B_\varphi^* e_{\frac{N}{2}}$, and so $H_{\frac{N}{2}}$ is a reducing subspace of B_φ, $\dim H_{\frac{N}{2}} = 1$.

Definition 3.4. Let

$$H_*^{(N)} = \begin{cases} span\{e_0, e_1, \cdots, e_{\left[\frac{N}{2}\right]}\}, \ if \ N \ is \ odd \\ span\{e_0, e_1, \cdots, e_{\left[\frac{N}{2}\right]-1}\}, \ if \ N \ is \ even \end{cases}$$

Now, we investigate minimal reducing subspaces in $H_*^{(N)}$. For that, we give a list of examples.

Example 3.5.

i. When $N = 1$,

$$B_\varphi e_0 = PW_3 M_\varphi e_0 = 0, \ B_\varphi^* e_0 = M_\varphi^* W_3^* P^* e_0 = 0,$$

$\{e_0\}$ is a minimal reducing subspace.

ii. When $N = 4$,

$$B_\varphi e_0 = PW_3 M_\varphi e_0 = 0, \ B_\varphi^* e_0 = M_\varphi^* W_3^* P^* e_0 = 0,$$

$$B_\varphi e_1 = PW_3 M_\varphi e_1 = 0, \ B_\varphi^* e_1 = M_\varphi^* W_3^* P^* e_1 = 0,$$

$\{e_0\}$, $\{e_1\}$ are two minimal reducing subspaces.

iii. When $N = 7$,

$$B_\varphi e_0 = PW_3 M_\varphi e_0 = 0, \ B_\varphi^* e_0 = M_\varphi^* W_3^* P^* e_0 = 0,$$

$$B_\varphi e_1 = PW_3 M_\varphi e_1 = 0, \ B_\varphi^* e_1 = M_\varphi^* W_3^* P^* e_1 = 0,$$

$$B_\varphi e_2 = PW_3 M_\varphi e_2 = e_3, \ B_\varphi^* e_2 = M_\varphi^* W_3^* P^* e_2 = 0,$$

$$B_\varphi e_3 = PW_3 M_\varphi e_3 = 0, \ B_\varphi^* e_3 = M_\varphi^* W_3^* P^* e_3 = e_2,$$

$\{e_0\}$, $\{e_1\}, \{e_2, e_3\}$ are three minimal reducing subspaces.

iv. When $N=10$,

$$B_{\varphi}e_0 = PW_3M_{\varphi}e_0 = 0,\ B_{\varphi}^{*}e_0 = M_{\varphi}^{*}W_3^{*}P^{*}e_0 = 0,$$

$$B_{\varphi}e_1 = PW_3M_{\varphi}e_1 = 0,\ B_{\varphi}^{*}e_1 = M_{\varphi}^{*}W_3^{*}P^{*}e_1 = 0,$$

$$B_{\varphi}e_2 = PW_3M_{\varphi}e_2 = e_4,\ B_{\varphi}^{*}e_2 = M_{\varphi}^{*}W_3^{*}P^{*}e_2 = 0,$$

$$B_{\varphi}e_3 = PW_3M_{\varphi}e_3 = 0,\ B_{\varphi}^{*}e_3 = M_{\varphi}^{*}W_3^{*}P^{*}e_3 = 0,$$

$$B_{\varphi}e_4 = PW_3M_{\varphi}e_4 = 0,\ B_{\varphi}^{*}e_4 = M_{\varphi}^{*}W_3^{*}P^{*}e_4 = e_2,$$

$\{e_0\}$, $\{e_1\}$, $\{e_2,e_4\}$, $\{e_3\}$ are four minimal reducing subspaces.

v. When $N=13$, $\{e_0\}$, $\{e_1\}$, $\{e_2,e_5,e_6\}$, $\{e_3\}$, $\{e_4\}$ are five minimal reducing subspaces.

vi. When $N=16$, $\{e_0\}$, $\{e_1\}$, $\{e_2,e_6\}$, $\{e_3\}$, $\{e_4\}$, $\{e_5,e_7\}$ are six minimal reducing subspaces.

Example 3.6.

i. When $N=2$,

$$B_{\varphi}e_0 = PW_3M_{\varphi}e_0 = 0,\ B_{\varphi}^{*}e_0 = M_{\varphi}^{*}W_3^{*}P^{*}e_0 = 0,$$

$\{e_0\}$ is a minimal reducing subspace.

ii. When $N=5$,

$$B_{\varphi}e_0 = PW_3M_{\varphi}e_0 = 0,\ B_{\varphi}^{*}e_0 = M_{\varphi}^{*}W_3^{*}P^{*}e_0 = 0,$$

$$B_{\varphi}e_1 = PW_3M_{\varphi}e_1 = e_2,\ B_{\varphi}^{*}e_1 = M_{\varphi}^{*}W_3^{*}P^{*}e_1 = 0,$$

$$B_{\varphi}e_2 = PW_3M_{\varphi}e_2 = 0,\ B_{\varphi}^{*}e_2 = M_{\varphi}^{*}W_3^{*}P^{*}e_2 = e_1,$$

$\{e_0\}$, $\{e_1,e_2\}$ are two minimal reducing subspaces.

iii. When $N=8$,

$$B_{\varphi}e_0 = PW_3M_{\varphi}e_0 = 0,\ B_{\varphi}^{*}e_0 = M_{\varphi}^{*}W_3^{*}P^{*}e_0 = 0,$$

$$B_{\varphi}e_1 = PW_3M_{\varphi}e_1 = e_3,\ B_{\varphi}^{*}e_1 = M_{\varphi}^{*}W_3^{*}P^{*}e_1 = 0,$$

$$B_{\varphi}e_2 = PW_3M_{\varphi}e_2 = 0,\ B_{\varphi}^{*}e_2 = M_{\varphi}^{*}W_3^{*}P^{*}e_2 = 0,$$

$$B_{\varphi}e_3 = PW_3M_{\varphi}e_3 = 0,\ B_{\varphi}^{*}e_3 = M_{\varphi}^{*}W_3^{*}P^{*}e_3 = e_1,$$

$\{e_0\}$, $\{e_1,e_3\}$, $\{e_2\}$ are three minimal reducing subspaces.

iv. When $N=11$,

$$B_{\varphi}e_0 = PW_3M_{\varphi}e_0 = 0,\ B_{\varphi}^{*}e_0 = M_{\varphi}^{*}W_3^{*}P^{*}e_0 = 0,$$

$$B_{\varphi}e_1 = PW_3M_{\varphi}e_1 = e_4,\ B_{\varphi}^{*}e_1 = M_{\varphi}^{*}W_3^{*}P^{*}e_1 = 0,$$

$$B_{\varphi}e_2 = PW_3M_{\varphi}e_2 = 0,\ B_{\varphi}^{*}e_2 = M_{\varphi}^{*}W_3^{*}P^{*}e_2 = 0,$$

$$B_{\varphi}e_3 = PW_3M_{\varphi}e_3 = 0,\ B_{\varphi}^{*}e_3 = M_{\varphi}^{*}W_3^{*}P^{*}e_3 = 0,$$

$$B_{\varphi}e_4 = PW_3M_{\varphi}e_4 = e_5,\ B_{\varphi}^{*}e_4 = M_{\varphi}^{*}W_3^{*}P^{*}e_4 = e_1,$$

$$B_{\varphi}e_5 = PW_3M_{\varphi}e_5 = 0,\ B_{\varphi}^{*}e_5 = M_{\varphi}^{*}W_3^{*}P^{*}e_5 = e_4,$$

$\{e_0\}$, $\{e_1,e_4,e_5\}$, $\{e_2\}$, $\{e_3\}$ are four minimal reducing subspaces.

v. When $N=14$, $\{e_0\}$, $\{e_1,e_5\}$, $\{e_2\}$, $\{e_3\}$, $\{e_4,e_6\}$, $\{e_7\}$ are five minimal reducing subspaces.

Theorem 3.7. For a given $N \in$ ¥ , and $j \in S_N$, let

$$\Theta_j^{(N)} = \left\{ \frac{N}{2} + \frac{1}{3^k}\left(j - \frac{N}{2}\right) \in \quad \middle| k = 0,1, \quad ,t_j \right\}$$

where t_j is the maximum integer such that $\frac{N}{2} + \frac{1}{3^{t_j}}\left(j - \frac{N}{2}\right)$ is a natural number. Then $H_{*j}^{(N)} = \{e_n\}_{n \in \Theta_j^{(N)}}$ is a minimal reducing subspace of B_{φ}.

Proof 3.8.

$$B_{\varphi}^{*}e_{\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)} = M_{\varphi}^{*}W_3^{*}P^{*}e_{\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)}$$

$$= M_{\varphi}^{*}e_{\frac{3N}{2}+\frac{1}{3^{t_k-1}}\left(j-\frac{N}{2}\right)}$$

$$= e_{\frac{N}{2}+\frac{1}{3^{t_k-1}}\left(j-\frac{N}{2}\right)}$$

and

$$\left(B_{\varphi}^{*}\right)^{t_k} e_{\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)} = e_j$$

$$\left(B_{\varphi}^{*}\right)^{t_k+1} e_{\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)} = B_{\varphi}^{*}B_j = M_{\varphi}^{*}W_3^{*}P^{*}e_j = M_{\varphi}^{*}e_{3j} = 0$$

since $3j < N$.

On the other side,

$$B_\varphi e_j = PW_3M_\varphi e_j = PW_3e_{N+j} = e_{\frac{N+j}{3}} = e_{\frac{N}{2}+\frac{1}{3}\left(j-\frac{N}{2}\right)}$$

$$\left(B_\varphi\right)^{t_k} e_j = e_{\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)}$$

Because of the definition of $\Theta_j^{(N)}$, 3 divisible $\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)$, but $\frac{N}{2}+\frac{1}{3^{t_k+1}}\left(j-\frac{N}{2}\right)$ cannot be divided by 3,

$$\left(B_\varphi\right)^{t_k+1} e_j = B_\varphi e_{\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)}$$

$$= PW_3M_\varphi e_{\frac{N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)}$$

$$= PW_3 e_{\frac{3N}{2}+\frac{1}{3^{t_k}}\left(j-\frac{N}{2}\right)}$$

$$= 0$$

so that $H_{*j}^{(N)}$ is a minimal reducing subspace of B_φ .

Remark 3.9. $\left\{\Theta_j^{(N)}\right\}$ is a partition of $\left\{0,1,\mathrm{L}\ ,\left[\frac{N}{2}\right]\right\}$ if N is odd; $\left\{\Theta_j^{(N)}\right\}$ and $\left\{\frac{N}{2}\right\}$ is a partition of $\left\{0,1,\mathrm{L}\ ,\left[\frac{N}{2}\right]\right\}$ if N is even. From Theorem 3.6 and the statement after Remark 3.2 we get all minimal reducing subspaces of B_{z^N} which lie inside the subspace spanned by $\left\{z^n \,\middle|\, 0\le n\le\left[\frac{N}{2}\right]\right\}$.

Example 3.10.

i. When $N=3$, Because of the definition of $\Theta_j^{(N)}$,

$$j=0,k=0,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{3}{2}+\frac{1}{3^0}\left(0-\frac{3}{2}\right)=0,$$

$$j=0,k=1,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{3}{2}+\frac{1}{3^1}\left(0-\frac{3}{2}\right)=1,$$

$\Theta_0^{(3)}=\{0,1\}$, $\{e_0,e_1\}$ is a minimal reducing subspace.

$$B_\varphi e_0 = PW_3M_\varphi e_0 = e_1,\ B_\varphi^* e_0 = M_\varphi^* W_3^* P^* e_0 = 0,$$

$$B_\varphi e_1 = PW_3M_\varphi e_1 = 0,\ B_\varphi^* e_1 = M_\varphi^* W_3^* P^* e_1 = e_0,$$

After verification, $\{e_0,e_1\}$ is indeed a minimal reducing subspace.

ii. When $N=6$,

$$j=0,k=0,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{6}{2}+\frac{1}{3^0}\left(0-\frac{6}{2}\right)=0,$$

$$j=0,k=1,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{6}{2}+\frac{1}{3^1}\left(0-\frac{6}{2}\right)=2,$$

$\Theta_0^{(6)}=\{0,2\}$, $\{e_0,e_2\}$ is a minimal reducing subspace.

$$j=1,k=0,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{6}{2}+\frac{1}{3^0}\left(1-\frac{6}{2}\right)=1,$$

$\Theta_0^{(6)}=\{1\}$, $\{e_1\}$ is a minimal reducing subspace.

$$B_\varphi e_0 = PW_3M_\varphi e_0 = e_2,\ B_\varphi^* e_0 = M_\varphi^* W_3^* P^* e_0 = 0,$$

$$B_\varphi e_1 = PW_3M_\varphi e_1 = 0,\ B_\varphi^* e_1 = M_\varphi^* W_3^* P^* e_1 = 0,$$

$$B_\varphi e_2 = PW_3M_\varphi e_2 = 0,\ B_\varphi^* e_2 = M_\varphi^* W_3^* P^* e_2 = e_0,$$

$\{e_0,e_2\}$, $\{e_1\}$ are two minimal reducing subspaces.

iii. When $N=9$,

$$j=0,k=0,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{9}{2}+\frac{1}{3^0}\left(0-\frac{9}{2}\right)=0,$$

$$j=0,k=1,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{9}{2}+\frac{1}{3^1}\left(0-\frac{9}{2}\right)=3,$$

$$j=0,k=2,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{9}{2}+\frac{1}{3^2}\left(0-\frac{9}{2}\right)=4,$$

$\Theta_0^{(9)}=\{0,3,4\}$, $\{e_0,e_3,e_4\}$ is a minimal reducing subspace.

$$j=1,k=0,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{9}{2}+\frac{1}{3^0}\left(1-\frac{9}{2}\right)=1,$$

$\Theta_0^{(9)}=\{1\}$, $\{e_1\}$ is a minimal reducing subspace.

$$j=2,k=0,\ \frac{N}{2}+\frac{1}{3^k}\left(j-\frac{N}{2}\right)=\frac{9}{2}+\frac{1}{3^0}\left(2-\frac{9}{2}\right)=2,$$

$\Theta_0^{(9)}=\{2\}$, $\{e_2\}$ is a minimal reducing subspace.

$$B_\varphi e_0 = PW_3M_\varphi e_0 = e_3,\ B_\varphi^* e_0 = M_\varphi^* W_3^* P^* e_0 = 0,$$

$$B_\varphi e_1 = PW_3M_\varphi e_1 = 0,\ B_\varphi^* e_1 = M_\varphi^* W_3^* P^* e_1 = 0,$$

$$B_\varphi e_2 = PW_3M_\varphi e_2 = 0,\ B_\varphi^* e_2 = M_\varphi^* W_3^* P^* e_2 = 0,$$

$$B_\varphi e_3 = PW_3M_\varphi e_3 = e_4,\ B_\varphi^* e_3 = M_\varphi^* W_3^* P^* e_3 = e_0,$$

$$B_\varphi e_4 = PW_3M_\varphi e_4 = 0,\ B_\varphi^* e_4 = M_\varphi^* W_3^* P^* e_4 = e_3,$$

$\{e_0, e_3, e_4\}$, $\{e_1\}$, $\{e_2\}$ are three minimal reducing subspaces.

iv. When $N = 12$,

$\{e_0, e_4\}$, $\{e_1\}$, $\{e_2\}$, $\{e_3, e_5\}$ are four minimal reducing subspaces.

v. When $N = 15$,

$\{e_0, e_5\}$, $\{e_1\}$, $\{e_2\}$, $\{e_3, e_6, e_7\}$, $\{e_4\}$ are five minimal reducing subspaces.

Remark 3.11. From Theorem 3.1 and 3.6, all minimal reducing subspaces of a 3-order slant Toeplitz operator B_{z^N} on the Hardy space of the unit circle are given explicitly. For a k-order slant Toeplitz operator $B_{z^N}^{(k)}$ on the Hardy space of the unit circle, one can get a similar result.

Author Contributions

Yang Zou is the sole author. The author read and approved the final manuscript.

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Conflicts of Interest

The authors declare no conflicts of interest.

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