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Research Article

Algebraic Representation of Primes by Hybrid Factorization

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Abstract

The representation of integers by prime factorization, proved by Euclid in the Fundamental Theorem of Arithmetic −also referred to as the Prime Factorization Theorem− although universal in scope, does not provide insight into the algebraic structure of primes themselves. No such insight is gained by summative prime factorization either, where a number can be represented as a sum of up to three primes, assuming Goldbach's conjecture is true. In this paper, a third type of factorization is introduced, called hybrid prime factorization, defined as the representation of a number as sum −or difference− of two products of primes with no common factors between them. By using hybrid factorization, primes are expressed as algebraic functions of other primes, and primality is established by a single algebraic condition. Following a hybrid factorization approach, sufficient conditions for the existence of Goldbach pairs are derived, and their values are algebraically evaluated, based on the symmetry exhibited by Goldbach primes around their midpoint. Hybrid prime factorization is an effective way to represent, predict, compute, and analyze primes, expressed as algebraic functions. It is shown that the sequence of primes can be generated through an algebraic process with evolutionary properties. Since prime numbers do not follow any predetermined pattern, proving that they can be represented, computed and analyzed algebraically has important practical and theoretical ramifications.

Keywords

Prime Number, Factorization, Optimization, Encoding, Goldbach Primes

1. Introduction

Prime numbers have been a subject of research for centuries. Mathematicians and scientists from various fields continue to study the numerical, algebraic, geometric, distributional, asymptotic, and other properties of primes. Since the time of Euclid, as the Fundamental Theorem of Arithmetic suggests, primes are linked to some of the most profound, and thus consequential, truths in mathematics and science.

Primes are used to model and better understand several phenomena in nature, such as the lifecycle of the American cicada [1]. Scientists have uncovered a "hyperuniformity" in the distribution of primes, akin to that observed in crystals [2]. The seemingly "random" properties of primes have been used to model the Brownian motion observed in fluid particles [3]. The distribution of primes, and the associated zeroes of the Riemann zeta function, have been linked to the correlation function of random Hermitian matrices [4-6], quantum computing [7], entanglement [8], elementary particle decay [9], and quantum field theory [10].

Since primes can optimally represent any measured quantity to an arbitrary degree of accuracy, a deeper understanding of their mathematical structure, properties and distribution

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enables a more profound understanding of the universe.

The set, P, of primes is a relatively sparse subset of the set, $\mathbb{Z}$, of integers. It has the smallest cardinality and member values capable of optimally encoding all integers through product or sum operations. Moreover, since multiplicative prime factorization represents a number as a product of its (prime) factors, it corresponds to the unique solution of a linear integer optimization problem (ILP), where the factor sum is minimized [11].

Representing a number as a sum of the least number of primes, i.e. through summative factorization, is possible if Goldbach's conjecture (GC), which states that "every even number > 2 can be expressed as the sum of two primes", is true. A consequence of GC is that any number can be written as a sum of up to three primes. Although GC has been computationally verified for extremely large numbers, it has not been proved. This may explain the fact that summative factorization has not been studied as extensively as its multiplicative counterpart [12-17].

In this paper, a hybrid form of factorization is introduced, called hybrid prime factorization (HPF), which represents a number as a sum (or difference) of two products with mutually exclusive prime factors.

In the following section, it is shown that hybrid prime factorization can be used to represent primes algebraically, by direct computation or optimization, and test for primality without the need for conventional, computationally cumbersome primality tests such as sieves or other primality testing algorithms [18].

In Section 3, the properties of hybrid prime factorization are applied to Goldbach pairs. Sufficient conditions are derived for their existence and their values are computed directly by evaluating a simple algebraic formula.

2. HPF Representation of Numbers

A number can be either represented multiplicatively as a product of primes, called prime factors, or summatively, as sum of the least number of primes, up to three, if Goldbach's conjecture is true.

While these two factorizations have different properties, each enables an optimal encoding –i.e. an optimized mathematical representation− of a number [11].

HPF, defined and discussed below, combines the algebraic structure of those two factorizations.

2.1. Definition of Hybrid Prime Factorization (HPF)

Hybrid prime factorization is based on a sum of two products of primes, with no common factors between them. In contrast to the factorizations previously discussed, representing primes by HPF reveals unique properties about their mathematical structure and distribution, i.e. how primes can be generated and analyzed algebraically.

Consider the first n primes, $p_1, p_2, \ldots, p_n$, in ascending order, where $n \geq 2$. HPF is given by an algebraic expression of the form.

$$\mathrm{HPF} = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_n^{a_n} \pm p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_n^{b_n} \tag{1}$$

where the integer exponents a_i, b_i satisfy the following conditions:

$$a_i, b_i \geq 0 \tag{2}$$

$$a_i \cdot b_i = 0 \tag{3}$$

$$a_i + b_i \geq 1 \tag{4}$$

$$\sum_{i=1}^{n} a_i \geq 1 \tag{5}$$

$$\sum_{i=1}^{n} b_i \geq 1 \tag{6}$$

$$\mathrm{HPF} > 1 \tag{7}$$

for i = 1, 2, …, n.

From (1)-(7), an HPF corresponds to a sum (or difference) of two primes, or products of primes, partitioned in a way that they share no common factor, while no prime is excluded from the HPF. The HPF given by (1) is called an HPF of dimensionality n.

The following example illustrates the HPF concept.

Example 1. Consider the first 3 primes {2, 3, 5}. The following expressions satisfy the HPF conditions (2)-(7):

$$2^2 \cdot 3^1 \cdot 5^0 \pm 2^0 \cdot 3^0 \cdot 5^1$$

$$2^3 \cdot 3^0 \cdot 5^1 \pm 2^0 \cdot 3^3 \cdot 5^0$$

$$2^3 \cdot 3^2 \cdot 5^0 \pm 2^0 \cdot 3^0 \cdot 5^1$$

$$2^{10} \cdot 3^0 \cdot 5^1 \pm 2^0 \cdot 3^4 \cdot 5^0.$$

The expressions below violate at least one of (2)-(7) and, therefore, are not considered HPF expressions:

$$2^1 \cdot 3^1 \cdot 5^1 \pm 2^0 \cdot 3^0 \cdot 5^0$$

$$2^3 \cdot 3^0 \cdot 5^1 \pm 2^1 \cdot 3^3 \cdot 5^0$$

$$2^1 \cdot 3^0 \cdot 5^1 \pm 2^0 \cdot 3^3 \cdot 5^0.$$

If conditions (3)-(7) are relaxed, algebraic expressions of the form given by (1), subject to (2), can be shown to represent any number N. For example, if $p_n \leq N < p_{n+1}$, it follows that the quantities $N - 1$ and $N - p_i$, for any $i \leq n$, are always represented by a factorization of $p_1, p_2, \ldots, p_n$. Such representations however, though universal in scope, are mathematically trivial, and do not help us gain any insight into the structural properties of primes. By contrast, as shown in the next section, the structural interdependence of primes can be algebraically established through HPF functions.

Euclid, in his seminal mathematical work, *Elements* [19], constructed the "HPF-like" number

$$1 + p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n \tag{8}$$

to prove the infinitude of primes, i.e. to establish that a prime larger than p_n exists, no matter how large n is. Expression (8) can be generated from (1), by setting $a_i = 0$ and $b_i = 1$ for i = 1, …, n. As this violates condition (5), the expression given by (8) is not considered an HPF.

A special-case HPF, given by $p_2\, p_3 \cdot \ldots \cdot p_n - 2^m$, was introduced in ([11] p. 6) to represent primes, or composites having all prime factors greater than p_n. It can be generated from (1) by setting $a_1 = 0$, $a_i = 1$, $b_1 = m$, $b_j = 0$ for i, j = 2, …, n, with m such that the resulting HPF is non-negative and minimum.

2.2. Fundamental HPF Properties

Proposition 1. Any HPF given by (1), subject to (2)-(7), represents either a prime larger than p_n, or a composite with no prime factor less than or equal to p_n.

Proof. If an HPF equals one of p_1, p_2, …, p_n, or some product of p_1, p_2, …, p_n, then it shares at least one prime factor with one of its two product terms, implying that the other product term also includes this prime factor, something that is impossible, since the prime factors between the two products are, by definition, mutually exclusive. It therefore follows that the HPF either equals a prime larger than p_n, or a composite number having no prime factor less than or equal to p_n.

Example 2. Examine each HPF sum of Example 1, to discern if it represents a prime or a composite number:

$$2^2 \cdot 3^1 \cdot 5^0 + 2^0 \cdot 3^0 \cdot 5^1 = 12 + 5 = 17 \text{ (prime > 5)}$$

$$2^3 \cdot 3^0 \cdot 5^1 + 2^0 \cdot 3^3 \cdot 5^0 = 40 + 27 = 67 \text{ (prime > 5)}$$

$$2^3 \cdot 3^2 \cdot 5^0 \pm 2^0 \cdot 3^0 \cdot 5^1 = 72 + 5 = 77 = 7 \cdot 11 \text{ (composite, all prime factors > 5)}$$

$$2^{10} \cdot 3^0 \cdot 5^1 + 2^0 \cdot 3^4 \cdot 5^0 = 5120 + 81 = 5201 = 7 \cdot 743 \text{ (composite, all prime factors > 5).}$$

The characteristic HPF property to represent a prime greater than p_n, or products of primes greater than p_n, implies that if it takes a value of less than p_{n+1}^2, then it must represent a prime larger than p_n. This is proved in the proposition below.

Proposition 2. If an HPF, given by (1)-(7), satisfies

$$\text{HPF} < p_{n+1}^2 \tag{9}$$

then it represents a prime larger than p_n.

Proof. From Proposition 1, an HPF represents either a prime larger than p_n, or a composite with prime factors larger than p_n. Since the smallest possible such composite is p_{n+1}^2, it follows that if $\text{HPF} < p_{n+1}^2$, then the value of the HPF must equal some prime in the open interval (p_n, p_{n+1}^2).

Proposition 2 implies that any number represented by an HPF, satisfying (1)-(7) and (9), is a priori prime, i.e. is prime regardless of its numerical value. In certain cases, as shown in the following sections, it can be proved that (9) holds without having computed the numerical value of the HPF. To see this, consider the HPF expression given by $2\ 5 \pm 3$. Regardless of its computed value, this algebraic expression represents two primes, since HPF$\in$ [1, 49]. There is no need to compute $2\ 5 \pm 3$, in order to prove that it cannot exceed 7^2. For example, condition (7) requires that $1 \pm 3 < 2\ 5$, and since $\pm 3 < 2\ 5$ it follows that HPF < 20 and thus HPF < 49. This is also true for a parametric family of HPF's with dimensionality 3, e.g. $2^2\ 5 \pm 3$, $2^3\ 5 \pm 3$, $5^2 \pm 2\ 3$, $5^2 \pm 2^2\ 3$ etc. In each case, the structure of the algebraic HPF functions suffices to establish the primeness of its values.

Proposition 2 offers an answer to one of the "open questions" posed by G. H. Hardy ([14] p. 7), since primes in (p_n, p_{n+1}^2) can be represented as algebraic functions of primes in $[2, p_n]$. From Bertrand's postulate ([14] p. 455), it follows that there is always at least one prime in $[p_n+2, p_{n+1}^2)$, since $2 \cdot p_n < p_{n+1}^2$ for $n \geq 1$.

Example 3. Given the HPF

$$2^3 \cdot 3^0 \cdot 5^1 - 2^0 \cdot 3^3 \cdot 5^0 = 40 - 27 = 13$$

and since $13 < 7^2 = 49$, by Proposition 2, the HPF represents a prime in the interval (5, 49).

The certainty that the computed value of the above HPF is prime has two salient characteristics:

1) *Structural:* the value is represented by an algebraic expression in the form of (1), and
2) *A priori:* the value's primeness can be established without the need to compute it first.

Proposition 2 provides a sufficient condition for primeness. If the HPF value is larger than or equal to p_{n+1}^2, the HPF may or may not represent a prime. To see this, consider the two HPF expressions:

$$2^3 \cdot 3^0 \cdot 5^1 + 2^0 \cdot 3^3 \cdot 5^0 = 67 > 49$$

$$2^3 \cdot 3^2 \cdot 5^0 + 2^0 \cdot 3^0 \cdot 5^1 = 77 > 49$$

where 67 is prime and 77 is composite with no prime factor less than 5; in both cases, the primality sufficiency condition (9) is not satisfied.

The implication of Proposition 2 is twofold:

1) candidate primes can be generated algebraically by the HPF expression given by (1), and
2) testing if either value of the HPF is prime can be done by evaluating (9) directly, without the need to apply a conventional primality test.

The HPF representation of a prime may also be modeled as a nonlinear integer optimization (NLP) problem, proved in the next proposition.

Proposition 3. Consider the nonlinear integer optimization problem

$$\min_{a_i, b_i}[\text{HPF}] \tag{10}$$

where the objective function HPF is given by (1), subject to the constraints (2)-(6) and

$$1 < \text{HPF} < p_{n+1}^2. \tag{11}$$

The solution to the above NLP (nonlinear minimization problem) is a prime in (p_n, p_{n+1}^2).

Proof. Any solution to (10), subject to (2)-(6) and (11), also satisfies Proposition 2. Therefore, the resulting HPF represents a prime in the interval (p_n, p_{n+1}^2).

If p_{n+1} is not known, the right-hand side of (9) or (11) may be replaced by a tighter upper bound given by $(p_n + 2)^2$, since $(p_n + 2)^2 \leq p_{n+1}^2$.

The value of the obtained minimum solution, given by (10), depends on n. Table 13 of the Appendix shows how this minimum varies for n = 2,…, 10.

It is helpful to note that Propositions 2 and 3 do not imply that one or more solutions to (1)-(6) and (11) always exist, or that HPF expressions represent all primes in $[p_n+2, p_{n+1}^2)$. These statements are revisited in Section 2.4.

An HPF can be used to represent and algebraically compute sequences of primes, by increasing the value of n. In the next example, an HPF based on the first 2 primes is used to algebraically generate and numerically compute the next 7 primes.

Example 4. For n = 2, the HPF, given by (1), takes the form

$$\text{HPF} = 2^{a_1} \cdot 3^{a_2} \pm 2^{b_1} \cdot 3^{b_2}. \tag{12}$$

A sample of computed HPF values, up to the primality sufficiency condition's upper bound $p_3^2 = 25$, given by (9), are tabulated below:

Table 1. *Sum HPF for n = 2.*

a_1	a_2	b_1	b_2	$2^{a_1} \cdot 3^{a_2} + 2^{b_1} \cdot 3^{b_2}$
1	0	0	1	5
2	0	0	1	7
3	0	0	1	11
2	0	0	2	13
3	0	0	2	17
4	0	0	1	19

Table 2. *Difference HPF for n = 2.*

a_1	a_2	b_1	b_2	$2^{a_1} \cdot 3^{a_2} - 2^{b_1} \cdot 3^{b_2}$
3	0	0	1	5
0	2	1	0	7
0	3	4	0	11
4	0	0	1	13
0	4	6	0	17
0	3	3	0	19
5	0	0	2	23
0	3	2	0	23

As shown in Table 2, using the HPF approach described in this section, all primes between 5 and 25 can be algebraically represented by (12), as functions of the first 2 primes. This is notable for the following reasons:

(1) all primes in [5, 25] are parametric expressions of a single HPF function, given by (12);

(2) all of the HPF values obtained are *a priori* known to be prime, because they satisfy a single algebraic condition, i.e. HPF < 25;

(3) The sequence of primes $p_3, p_4,\ldots, p_9$, can be generated parametrically through (12), i.e. an algebraic function of the first two primes, p_1 and p_2.

Since all HPF values such that $\text{HPF} < p_{n+1}^2$ are prime, the HPF approach described in this section, is a useful tool for representing, analyzing, predicting and computing primes.

2.3. Prime Gaps in HPF Expressions

If there are gaps in the sequence of primes used to construct an HPF, Propositions 2 and 3 may be applied iteratively. If the HPF evaluates to a composite number with some (or all) of its prime factors less than p_n, such prime factor(s) correspond to the missing prime(s) in the sequence. Once these primes are included in an augmented version of the HPF the process can be repeated, until there are no gaps in the HPF prime sequence and the HPF represents either a prime larger than p_n or a composite with prime factors larger than p_n.

Even in the case of prime gaps, the "iterative augmentation" of the HPF described here, is more efficient in searching for a larger prime than testing by conventional primality tests. Once there are no prime gaps, computing a larger prime is straightforward, i.e. done in one step, if (2)-(6) and (11) are satisfied.

In Section 3.1, this methodology is used to determine sufficient conditions for the existence of Goldbach pairs, and to compute their values.

2.4. Generalizations and Open Questions

Let L(n) be the set of all HPF values that satisfy (1)-(6) and (11), for $n \geq 2$. It follows that every member of the solution space L(n) is a prime in $[p_n+2, p_{n+1}^2)$. Hence, the following questions are of particular interest:

1) does L(n) contain *at least one* prime in $[p_n+2, p_{n+1}^2)$ for every n?
2) does the union of solution spaces, L(n), for $2 \leq n \leq N$, contain *all primes* in $[5, p_{N+1}^2)$?

Based on these two questions and the results presented so far, two conjectures are proposed for future research:

Weak HPF Conjecture: For any $n \geq 2$, there always exists at least one HPF solution to (2)-(6) and (11) that represents a prime in $[p_n+2, p_{n+1}^2)$.

Strong HPF Conjecture: For any $N \geq 2$, the set $\hat{L}(N)$ defined by

$$\hat{L}(N) = \{2,3\} \cup_{n=2}^{N} L(n) \tag{13}$$

where L(n), $n \leq N$, is the solution space of (1)-(6) and (11), includes every prime in $[2, p_{N+1}^2)$.

In the Appendix, solutions to (1)-(6) and (11) are computed, for n = 2, 3,…, 10. Current research is focused on a better understanding of the evolution and characteristics of solution spaces L(n), more specifically regarding: (1) the minimum value of n for which a prime can be represented by an HPF; (2) conditions under which a prime is represented by multiple HPF expressions, with similar or different dimensionalities.

3. Modeling Goldbach Pairs by Using Hybrid Prime Factorization

Goldbach's conjecture (GC) states that "every even number >2 can be expressed as the sum of two primes". Such primes are called "Goldbach primes" or a "Goldbach pair".

The HPF properties discussed in Section 2 can be used to represent potential Goldbach pairs and subsequently test if they are prime by deriving sufficient conditions for their existence.

3.1. HPF Modeling of Goldbach Pairs

Let s > 2 be an even number. All Goldbach primes that add up to s are, by definition, equidistant from the midpoint s/2. Consequently, if the value of the first product term in the HPF, given by (1), is set to s/2, the second term can be structured so that the HPF satisfies (2)-(6) and (11). When the two HPF expressions are added, the second product terms cancel each other and s is expressed as a sum of two primes, with each HPF corresponding to a Goldbach prime.

This approach is explained in the following example:

Example 5. Let s = 20; the HPF given by (1) with n = 3, is

$$\mathrm{HPF} = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \pm 2^{b_1} \cdot 3^{b_2} \cdot 5^{b_3}. \tag{14}$$

From Proposition 2, if $\mathrm{HPF} < p_4^2 = 7^2 = 49$, the HPF, given by (14), represents a prime. Goldbach primes can always be expressed as a sum or a difference from the midpoint s/2. The first term of the HPF can be set to the prime factorization value of s/2 = 10, and the second term to the distance, $\alpha = 3^{b_2}$, from the midpoint s/2. Therefore, the HPF becomes

$$\mathrm{HPF} = 2^1 \cdot 5^1 \pm 3^{b_2}. \tag{15}$$

Let HPF_1 and HPF_2 be given by

$$\mathrm{HPF}_1 = 10 + 3^{b_2} \text{ and } \mathrm{HPF}_2 = 10 - 3^{b_2}. \tag{16}$$

From Proposition 2, if the exponent b_2 in (16) is such that $\mathrm{HPF}_1 < 49$ and $\mathrm{HPF}_2 < 49$, then HPF_1 and HPF_2 are prime. Also, by adding the above HPF expressions,

$$\mathrm{HPF}_1 + \mathrm{HPF}_2 = 10 + 3^{b_2} + 10 - 3^{b_2} = 20 \tag{17}$$

and thus, s is expressed as the sum of HPF_1 and HPF_2. For $b_2 = 1$, it follows that $\mathrm{HPF}_1 = 13 < 49$ and $\mathrm{HPF}_2 = 7 < 49$, therefore (13,7) is a Goldbach pair.

Note: If the right term in (14) included the factor 7^{b_3}, this would result in a negative value for $\mathrm{HPF}_2 = 2^1 \cdot 5^1 - 3^{b_2} \cdot 7^{b_3}$, since $b_2 \geq 1$ and $b_3 \geq 1$, in order to satisfy (4).

So far, no prime gaps in the HPF have been considered; this, however, is not the case in general, as shown in the next example.

Example 6. Let s = 44, so that s/2 = 22 = 2 11, with Goldbach pairs 22 ± 9 and 22 ± 15, that is, (13, 31) and (7, 37) respectively. An HPF-like expression that can generate these Goldbach pairs is given by

$$\mathrm{HPF} = 2^1 \cdot 11^1 \pm 3^{b_2} \cdot 5^{b_3} \cdot 7^{b_4} \tag{18}$$

where $b_2 = 2$, $b_3 = 0$, $b_4 = 0$ and $b_2 = 1$, $b_3 = 1$, $b_4 = 0$ respectively. Note that these two sets of parameters violate (4).

For each set of the above exponents, it can be proved that

$$\mathrm{HPF}_1 = 2^1 \cdot 11^1 - 3^{b_2} \cdot 5^{b_3} \cdot 7^{b_4} \tag{19}$$

is prime, since it is less than the smallest possible composite with prime factors not in HPF_1. More specifically, for $b_2 = 2$, $b_3 = 0$, $b_4 = 0$, this composite is 5 5 = 25 and for $b_2 = 1$, $b_3 = 1$ $b_4 = 0$, it is 7 7 = 49. In both cases, although prime gaps exist, the applicable sufficiency inequalities are satisfied, i.e.

$$\mathrm{HPF}_1 = 2^1 \cdot 11^1 - 3^2 = 13 < 25 \tag{20}$$

$$\mathrm{HPF}_1 = 2^1 \cdot 11^1 - 3^1 \cdot 5^1 = 7 < 49 \tag{21}$$

and, in both cases, HPF_1 is prime. The first summation term

$$\mathrm{HPF}_2 = 2^1 \cdot 11^1 + 3^2 = 31 > 25 \quad (22)$$

$$\mathrm{HPF}_2 = 2^1 \cdot 11^1 + 3^1 \cdot 5^1 = 37 < 49 \quad (23)$$

violates the respective inequality condition, and so there is no guarantee that, by adding (20) and (22), a Goldbach pair is obtained. HPF_2 satisfies the primality sufficiency condition (23), therefore 22 ±15 is a Goldbach pair.

As shown in the previous example, a more general result can be established if the “no prime gap” condition, given by (4), is relaxed, i.e. if both exponents α_i, b_i, in (1), are allowed to be zero~~,~~ for some i. In this case, the primality sufficiency condition (11) should also be adjusted. This is proved in the next proposition.

Proposition 4. Consider an even number s ≥ 6 and the HPF given by

$$\mathrm{HPF} = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_m^{a_m} \pm p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_m^{b_m} \quad (24)$$

for m ≥ 2, where p_1, p_2, …, p_m are the first m primes in ascending order, with $p_{m+1} \geq s/2$, subject to

$$a_i, b_i \geq 0 \quad (25)$$

$$a_i \cdot b_i = 0 \quad (26)$$

$$\sum_{i=1}^{m} a_i \geq 1 \quad (27)$$

$$\sum_{i=1}^{m} b_i \geq 1 \quad (28)$$

$$1 < \mathrm{HPF} < p_k^2 \quad (29)$$

where p_k, k ∈ [2, m] corresponds to the smallest prime that is not a factor of either product term in (24). Assuming that a solution to (24)-(29) always exists, there is an HPF such that its values correspond to a Goldbach pair.

Proof. Let s ≥ 6 be an even number and q_1, q_2, …, q_m be the primes in [2, s/2], listed in ascending order. Consider the HPF expression given by

$$\mathrm{HPF} = s/2 \pm q_1^{b_1} \cdot q_2^{b_2} \cdot \ldots \cdot q_m^{b_m} \quad (30)$$

where the first term is equal to the prime factorization of s/2, that is, either equals one of the primes q_i or a product of more than one q_i, and the second term is equal to the product of some, or all, of the remaining primes that are not prime factors of s/2. Let q_k be the smallest prime factor not included in (30). Since, by assumption, there is a solution that satisfies (30) and (25)-(29), then there exist b_1,…, b_m such that, both values of (30) are prime, since (29) is also satisfied, with $p_k = q_k$, i.e. the values of HPF, given by (30), are less than the smallest feasible composite number; thus the proposition is proved.

This method is applied in Section 3.2.2 (see Example 8).

3.2. Partially Satisfied Sufficiency Conditions

There are even numbers for which (29) does not hold and therefore, Proposition 4 cannot be applied. For example, consider s = 88, hence, s/2 = 44 = 2^2 ·11. The Goldbach pairs associated with s = 88 are generated by the following HPF values: 44±3, 44±15, 44±27 and 44±39. For each HPF, the respective sufficiency condition is violated for one or both terms: 44+3 > 25 (false), 44+15 > 49 (false), 44+27 > 25 (false) and 44+39 > 25 (false). Therefore, Proposition 4 cannot be directly applied here, to a priori guarantee that both HPF values are prime. In such cases, one of the three approaches described in the sections below can be followed.

3.2.1. Capacity-Based Approach

One way to circumvent (29) is to use a “capacity-type” argument, based on the count differential between primes and composites in an interval.

Let r, m be the number of odd composites and primes in [1, s] respectively, where s is even. Since any number >1 is either prime or composite, and the number of even composites is $\frac{s}{2} - 1$, it follows that

$$s = 1 + m + r + (\tfrac{s}{2} - 1) \quad (31)$$

or

$$r = \tfrac{s}{2} - m. \quad (32)$$

A sufficient condition for the existence of a Goldbach pair is $r < m$, since, if true, the odd composites are fewer than the residuals $R_i = s - p_i$ of the Goldbach partition, and thus, at least one such residual R_j is prime. Hence, by using (32), the sufficiency condition $r < m$ can be expressed as

$$r = \frac{s}{2} - m < m$$

or equivalently

$$m > \tfrac{s}{4} \quad (33)$$

Example 7. For s = 88, it follows that m = 23 and (33) is satisfied, since $23 > 88/4 = 22$. Therefore, at least one Goldbach pair exists.

By using this method, the existence of at least one Goldbach pair has been proved, subject to (33) being true. However, in contrast to the HPF approach described in Section 3.1, there is no formula for computing the values of Goldbach primes. There are limitations to this approach, discussed below.

Inequality (33) holds up to s = 126; beyond that value, the number of odd composites exceeds that of primes, and the capacity-based approach is not applicable.

Testing for the existence of Goldbach pairs when (29) and/or (33) are violated is the subject of the next section.

3.2.2. Composite Elimination Approach

Consider the $\overline{\mathrm{HPF}}$ given by

$$\overline{\mathrm{HPF}}_{\pm} = w \pm q_1^{b_1} \cdot q_2^{b_2} \cdot \ldots \cdot q_m^{b_m} \quad (34)$$

where $w \geq 6$ is even, and $q_1, q_2, \ldots, q_m$ are the remaining primes, in ascending order, that are not prime factors of w, with $q_i < w$ for i = 1,…, m.

Assume the exponent values b_i are set so that

$$\overline{\mathrm{HPF}}_{-} = w - q_1^{b_1} \cdot q_2^{b_2} \cdot \ldots \cdot q_m^{b_m} \quad (35)$$

satisfies (29), i.e.

$$1 < \overline{\mathrm{HPF}}_{-} < q_k^2 \quad (36)$$

where q_k is the smallest prime not included in (38). From (35)-(36), it follows that $\overline{\mathrm{HPF}}_{-}$ is prime. It remains to be investigated if

$$\overline{\mathrm{HPF}}_{+} = w + q_1^{b_1} \cdot q_2^{b_2} \cdot \ldots \cdot q_m^{b_m} < q_k^2 \quad (37)$$

is satisfied. If (37) is not satisfied, the next step is to test if $\overline{\mathrm{HPF}}_{+}$ equals any of the composites in range. If this is not true, $\overline{\mathrm{HPF}}_{+}$ is prime.

The following example illustrates this method.

Example 8. Let w = 128 = 2^7 and write (34) as

$$\overline{\mathrm{HPF}}_{\pm} = 128 \pm 3^{b_1} \cdot 5^{b_2} \cdot 7^{b_3} \quad (38)$$

so that the $\overline{\mathrm{HPF}}$ primality sufficiency conditions are

$$1 < 128 \pm 15 < 49 \quad (39)$$

$$1 < 128 \pm 105 < 121. \quad (40)$$

The only primality condition satisfied is

$$1 < 128 - 105 < 121 \quad (41)$$

which implies that $128 - 105 = 23$ is prime.

Hence, a potential candidate for a Goldbach pair is (23, 128+105), provided that 128+105 = 233 is also prime. In Section 2 it was shown that 233 is either prime or composite with prime factors greater than 7, i.e. no prime factor less than 11. Given that $\lfloor 233/11 \rfloor = 21$, and assuming that 233 is composite, the only possible prime factors of 233 could be: 11, 13, 17 or 19. However, since 11^2, 13^2, 11·13, 11·17, 11·19, 13·17 do not equal 233, and any other product of these prime factors is greater than 233, it follows that 233 is prime. Thus, the pair (23,233) is a Goldbach pair associated with s = 2 w = 256.

3.2.3. Dimensional Augmentation Method

In the previous example, it may be directly established that 233 is prime by adjusting the dimensionality of the HPF.

Given that $233 < 17^2$, consider the HPF given by (1) and n = 6

$$\overline{\overline{\mathrm{HPF}}} = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4} \cdot 11^{a_5} \cdot 13^{a_6} \pm$$

$$\pm 2^{b_1} \cdot 3^{b_2} \cdot 5^{b_3} \cdot 7^{b_4} \cdot 11^{b_5} \cdot 13^{b_6}. \quad (42)$$

Table 3 shows how 233 can be algebraically represented by the 6-dimensional $\overline{\overline{\mathrm{HPF}}}$, given by (42). Since $233 < 17^2$, the sufficiency primality condition (9) of Proposition 2 is satisfied and therefore, $\overline{\overline{\mathrm{HPF}}} = 233$ is prime.

Table 3. *Algebraic representations of "233" using a 6-dimensional HPF with n = 6, given by (45).*

a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	$\overline{\overline{\mathrm{HPF}}}$
0	0	0	3	3	0	2	3	2	0	0	2	233
0	2	0	2	0	1	2	0	3	0	1	0	233
3	0	0	1	0	1	0	2	1	0	1	0	233

Dimensional augmentation can be used to test if a number is prime, by encoding it using a dimensionally higher HPF, i.e. with a higher value of n, in (1), than that originally used to generate it. In the previous example, 233 was first generated by (38) as follows

$$233 = 2^7 + 3^1 \cdot 5^1 \cdot 7^1 = \mathrm{HPF}(2,3,5,7)$$

and subsequently by (42), whereby using the exponent values of the first row of Table 3.

$$233 = 7^3 \cdot 11^3 - 2^2 \cdot 3^3 \cdot 5^2 \cdot 13^2 = \overline{\overline{\mathrm{HPF}}}(2,3,5,7,11,13)$$

in order to apply the corresponding primality sufficiency condition, given by $233 < 17^2$, to prove that it is prime.

The above approach is a computationally efficient way to conduct a targeted search for Goldbach primes, when the original primality sufficiency conditions fail.

4. Discussion

The HPF expression, given by (1), can be written as

$$\mathrm{HPF}(m) = G(p_m, \bar{a}) \pm G(p_m, \bar{b}) \tag{43}$$

where

$$G(p_m, \bar{a}) = G_m(\bar{p}, \bar{a}) = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_m^{a_m}$$

$$G(p_m, \bar{b}) = G_m(\bar{p}, \bar{b}) = p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_m^{b_m}$$

and $\bar{p}, \bar{a}, \bar{b}$ are the m-dimensional vectors of the first m primes, $p_1, p_2, \ldots, p_m$, and the exponents a_i, b_i, for i = 1,…, m, respectively. Note that the parametric vectors $\bar{a}$ and $\bar{b}$ may be considered as "exponential weights" on the elements of vector $\bar{p}$.

The primality sufficiency condition

$$1 < \mathrm{HPF}(m) < p_{m+1}^2 \tag{44}$$

implies that there exist vectors $\bar{a}$, $\bar{b}$ such that the sum and difference of $G_m(\bar{p}, \bar{a})$, $G_m(\bar{p}, \bar{b})$ are prime, i.e.

$$G_m(\bar{p}, \bar{a}) \pm G_m(\bar{p}, \bar{b}) = p_{m+j} \tag{45}$$

where $j \geq 1$ and the primes $p_{m+j} \in [p_m+2, p_{m+1}^2)$.

Primes in $[p_m+2, p_{m+1}^2)$ are considered "novel" since they cannot be mathematically represented by either factorization term $G_m(\bar{p}, \bar{a})$ or $G_m(\bar{p}, \bar{b})$. However, as (45) implies, they can be generated by adding or subtracting two optimally partitioned, exponentially weighted factorization terms of primes, up to and including p_m.

The expression (45) can be viewed as an "evolutionary" algebraic prime number generator: it can be used iteratively in a feedback-loop configuration, to augment the dimensionality, m, of the HPF given by (43), by including the novel primes generated, subject to (44).

This type of "prime generation mechanism" closely resembles a natural process, where the new primes, p_{m+j}, are outcomes ("offsprings") of an evolutionary interaction between two structurally partitioned, jointly optimized entities: $G_m(\bar{p}, \bar{a})$ and $G_m(\bar{p}, \bar{b})$.

5. Conclusions

Hybrid prime factorization (HPF) represents a number as a sum of two product terms with no common prime factors. HPF expressions are an effective tool to algebraically represent primes and test for primality directly, thus bypassing computational primality tests.

HPF expressions can be applied to model, analyze, predict, algebraically represent and numerically compute primes, including Goldbach primes.

The HPF-based representation of primes exhibits predictive capabilities, in the sense that the first n primes, $p_1, p_2, \ldots, p_n$, can be used to algebraically represent and compute primes in the interval $[p_n+2, p_{n+1}^2)$. This process is illustrated in the Appendix, where solutions to (1)-(7) and (9) are computed for n = 2, 3,…, 10.

It is well known that primes do not follow any pattern; proving that primes can be represented algebraically −and therefore studied analytically on that foundational basis− presents interesting challenges for future research.

Abbreviations

ILP: Integer Linear (Optimization) Problem
GC: Goldbach's Conjecture
HPF: Hybrid Prime Factorization

Conflicts of Interest

The authors declare no conflicts of interest.

Appendix: Programmatic Computation of HPF Solutions

HPF solutions to (1)-(6) and (9), are programmatically computed, for n = 2, 3,…, 10. The maximum exponent values are limited by a fixed upper bound, to ensure that runtimes, especially for larger values of n, remain reasonable. Due to space limitations, only some of the computed solutions are provided here.

In the following tables, for each n, the values of the exponents α_i, b_i and the HPF are provided. Only the smaller term in (1) is evaluated~~,~~ since it generates more primes, as it is less likely to violate (9).

The HPF is given by

$$\mathrm{HPF} = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_n^{a_n} - p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_n^{b_n} \tag{46}$$

subject to (2)-(6) and (11), for n = 2, 3,…, 10.

For n = 2, the HPF expression (46) becomes

$$\mathrm{HPF} = 2^{a_1} \cdot 3^{a_2} - 2^{b_1} \cdot 3^{b_2}$$

and from (9) it follows that the "cutoff" HPF value is $5^2 = 25$. Table 4 shows all computed HPF values. All primes in [5, 25) are generated using this HPF.

Table 4. HPF solutions for n = 2.

a_1	a_2	b_1	b_2	HPF
0	2	2	0	5
3	0	0	1	5
5	0	0	3	5
0	2	1	0	7
4	0	0	2	7
0	3	4	0	11
4	0	0	1	13
8	0	0	5	13
0	4	6	0	17
0	3	3	0	19
0	3	2	0	23
5	0	0	2	23

For n = 3, Table 5 shows all solutions for HPF = 7, HPF = 11 and only the first computed solution for all other HPF values. All primes in [7,49) are generated by this HPF.

Table 5. HPF solutions for n = 3.

a_1	a_2	a_3	b_1	b_2	b_3	HPF
0	0	2	1	2	0	7
0	3	0	2	0	1	7
0	1	1	3	0	0	7
0	3	1	7	0	0	7
1	0	1	0	1	0	7
1	0	3	0	5	0	7
2	1	0	0	0	1	7
0	1	1	2	0	0	11
0	1	2	6	0	0	11
2	0	1	0	2	0	11
2	2	0	0	0	2	11
0	1	1	1	0	0	13
0	3	0	1	0	1	17
0	0	2	1	1	0	19
1	0	2	0	3	0	23
0	2	1	4	0	0	29
0	4	0	1	0	2	31
0	2	1	3	0	0	37

a_1	a_2	a_3	b_1	b_2	b_3	HPF
0	2	1	2	0	0	41
0	2	1	1	0	0	43
1	0	2	0	1	0	47

For n = 4, Table 6 includes one solution for each prime HPF value computed. All primes in [11,121) are generated by this HPF.

Table 6. *HPF solutions for n = 4.*

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	HPF
0	1	0	1	1	0	1	0	11
0	0	2	1	1	4	0	0	13
0	0	1	1	1	2	0	0	17
0	0	0	2	1	1	1	0	19
0	0	1	1	2	1	0	0	23
0	0	1	1	1	1	0	0	29
0	2	1	0	1	0	0	1	31
0	3	1	0	1	0	0	2	37
0	0	3	0	2	1	0	1	41
0	0	0	3	2	1	2	0	43
0	1	0	2	2	0	2	0	47
0	2	0	1	1	0	1	0	53
0	2	1	1	8	0	0	0	59
0	1	2	0	1	0	0	1	61
0	0	2	1	2	3	0	0	67
3	1	1	0	0	0	0	2	71
0	0	0	3	1	3	1	0	73
0	3	1	0	3	0	0	1	79
0	0	3	0	1	3	1	0	83
0	3	0	1	2	0	2	0	89
0	1	0	2	1	0	2	0	97
0	1	1	1	2	0	0	0	101
0	3	1	0	4	0	1	0	107
0	3	0	1	4	0	1	0	109
0	2	2	0	4	0	0	1	113

Table 7 includes only one computed HPF solution for each prime value obtained, for n=5. All primes in [13,169) are generated by this HPF.

Table 7. HPF solutions for n = 5.

a_1	a_2	a_3	a_4	a_5	b_1	b_2	b_3	b_4	b_5	HPF
0	0	0	3	0	1	1	1	0	1	13
0	4	0	1	0	1	0	2	0	1	17
0	0	4	1	0	2	2	0	0	2	19
0	0	0	3	1	1	1	4	0	0	23
0	2	0	0	1	1	0	1	1	0	29
0	1	0	1	1	3	0	2	0	0	31
0	0	0	1	2	1	4	1	0	0	37
0	1	0	1	2	2	0	4	0	0	41
0	0	2	1	0	2	1	0	0	1	43
0	0	0	1	1	1	1	1	0	0	47
0	2	2	0	4	2	0	0	7	0	53
0	3	2	0	0	3	0	0	1	1	59
0	0	1	1	1	2	4	0	0	0	61
0	1	1	0	1	1	0	0	2	0	67
0	2	2	0	0	1	0	0	1	1	71
0	2	1	1	0	1	0	0	0	2	73
0	3	0	1	0	1	0	1	0	1	79
0	1	1	1	0	1	0	0	0	1	83
0	0	0	2	1	1	2	2	0	0	89
0	0	0	1	2	1	1	3	0	0	97
0	0	1	0	2	3	2	0	1	0	101
0	2	1	0	1	3	0	0	2	0	103
0	0	2	0	1	3	1	0	1	0	107
0	0	2	1	0	1	1	0	0	1	109
0	3	0	2	0	1	0	1	0	2	113
0	4	0	1	0	3	0	1	0	1	127
0	1	0	1	1	2	0	2	0	0	131
0	5	1	0	0	1	0	0	2	1	137
0	1	2	0	1	1	0	0	3	0	139
0	0	3	1	0	1	1	0	0	2	149
0	1	1	0	1	1	0	0	1	0	151
0	3	0	0	1	2	0	1	1	0	157
0	0	4	0	0	1	1	0	1	1	163
0	2	2	1	0	7	0	0	0	1	167

In Tables 8-12, only five randomly selected solutions are shown, for n = 6, …, 10, respectively.

***Table 8**. Sample HPF solutions for n = 6.*

a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	HPF
0	0	3	1	0	0	1	1	0	0	1	1	17
1	0	1	1	0	1	0	4	0	0	1	0	19
0	4	2	0	0	0	1	0	0	1	1	1	23
0	5	2	0	0	1	10	0	0	1	1	0	127
0	2	1	1	1	0	8	0	0	0	0	1	137

***Table 9**. Sample HPF solutions for n = 7.*

a_1	a_2	a_3	a_4	a_5	a_6	a_7	b_1	b_2	b_3	b_4	b_5	b_6	b_7	HPF
0	1	1	3	1	0	0	8	0	0	0	0	1	1	19
0	4	1	0	0	1	0	2	0	0	1	1	0	1	29
0	10	0	1	0	0	0	1	0	1	0	1	1	2	73
0	0	1	2	1	0	2	9	2	0	0	0	2	0	103
10	1	2	1	0	0	0	0	0	0	0	1	2	2	349

***Table 10**. Sample HPF solutions for n = 8.*

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	HPF
0	0	0	1	1	0	2	0	1	2	1	0	0	1	0	1	23
2	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	31
1	3	1	0	1	1	0	0	0	0	0	1	0	0	2	1	173
2	3	2	0	1	0	0	0	0	0	0	1	0	1	1	1	307
2	0	0	1	2	1	0	0	0	3	1	0	0	0	1	1	439

***Table 11**. Sample HPF solutions for n = 9.*

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	HPF
3	0	0	1	1	0	1	0	1	0	1	2	0	0	2	0	1	0	31
0	3	0	1	2	1	0	0	0	3	0	1	0	0	0	1	1	1	137
3	1	1	0	1	1	0	1	0	0	0	0	2	0	0	2	0	1	337
0	0	0	1	1	0	0	1	1	1	1	2	0	0	1	1	0	0	499
0	1	0	0	0	1	1	0	1	1	0	1	1	1	0	0	1	0	619

Table 12. *Sample HPF solutions for n = 10.*

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	HPF
0	3	2	1	0	0	1	1	0	0	4	0	0	0	1	1	0	0	1	1	79
4	2	0	1	0	0	1	0	1	0	0	0	1	0	1	1	0	1	0	1	163
0	0	0	0	0	1	1	2	2	0	2	3	2	2	1	0	0	0	0	1	449
1	2	0	0	0	2	0	1	1	0	0	0	1	2	1	0	1	0	0	1	719
1	2	0	0	0	1	2	1	0	0	0	0	2	1	1	0	0	0	1	1	919

The following table shows how the solution to the NLP minimization problem of Proposition 3 varies with n.

For n = 2,…, 8, the minimum HPF values coincide with the sequence of primes. For $n \geq 9$, gaps start to exist. This may be caused by the fact that an upper bound of 15 has been set for all HPF exponent parameters. Relaxing this bound may enable the computation of lower minimum HPF values. However, this comes at a significant computational cost, since the computed values of the two exponential products can easily exceed the numerical processing capabilities of a desktop computer system.

Table 13. *min[HPF] values for n = 2,…,10.*

n	min[HPF]
2	5
3	7
4	11
5	13
6	17
7	19
8	23
9	31
10	79

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