International Journal of Theoretical and Applied Mathematics
2024; 10(4): 51-53
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20241004.11
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

On a Theorem on the Sizes of Conjugacy Classes of a Finite Group

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To cite this article:
Yongcai Ren. (2025). On a Theorem on the Sizes of Conjugacy Classes of a Finite Group. *International Journal of Theoretical and Applied Mathematics, 10*(4), 51-53. https://doi.org/10.11648/j.ijtam.20241004.11

Received: 12 July 2024; **Accepted:** 18 December 2024; **Published:** 13 January 2025

Abstract: In this note, we provide an independent proof of a known result on conjugacy class sizes of a finite group.

Keywords: Finite Group, Conjugacy Class Size, Primary Element, Biprimary Element

1. Introduction and Preliminaries

Throughout this paper, the term group always means a group of finite order. The letter G always denotes a group. For an element x of G, $o(x)$ denotes the order of x, and x^G denotes the conjugacy class of x in G. $|x^G|$ is called the size of the conjugacy class x^G, that is the positive integer $|G : C_G(x)|$. We write $\pi(G)$ to denote the set of the prime divisors of the order $|G|$ of G. For a prime $p \in \pi(G)$, G_p denotes a Sylow p-subgrou of G, and $G_{p'}$ denotes a p-complement of G. The further unexplained notations are standard and can be found in [1].

Let $x \in G$. We say that the element x is primary if $o(x)$ is a power of a prime; We say that the element x is biprimary if $o(x)$ is divisible by exactly two distinct primes. A primary element is also called an element of prime-power order.

Let G be a non-abelian group. We say that G is a Baer-group if the conjugacy class size of every primary element of G is a power of a prime (see [2]).

For any group G, we write

$$T(G) = \{x \in G - Z(G) | x \text{ is primary or biprimary }\}.$$

We say that a non-abelian group G is a B-group if $|x^G|$ is a power of a prime for every $x \in T(G)$.

By using [3, Corollary 2] and some arguments in its proof, we can get the following.

Theorem A. Let G be a non-nilpotent group. Suppose that G is a B-group. Then the following propositions hold:

(1) There exist tow different primes $p, q \in \pi(G)$ such that $G = PQ \times A$, where A is an abelian $\{p, q\}'$-group, P is an abelian Sylow p-subgroup of G, Q is an abelian Sylow q-subgroup of G and $Q \trianglelefteq G$.

(2) $P \cap P^g = O_p(G)$ for each $g \in G - N_G(P)$.

(3) Write $K = PQ$. Then $K/Z(K)$ is a Frobenius group with the Frobenius kernel $QZ(K)/Z(K)$. In addition, the following two statements hold:

(3a) If K has no non-trivial ablian direct factors, then $Z(K) = O_p(G)$.

(3b) Let K_1 be minimal such that $K_1Z(K) = K$. Then, $K_1/Z(K_1)$ is a Frobenius group with the Frobenius kernel $(K_1 \cap Q)Z(K_1)/Z(K_1)$ and $Z(K_1) = O_p(K_1) = K_1 \cap O_p(K)$.

Theorem A is an improvement of [4, Corollary 2.2] and the necessary part of [4, Theorem 2].

The proof of [3, Corollary 2] uses the structure theorem on Baer-groups (see [2] and [3, Theorem 3]). The proof of the structure theorem on Baer groups is long or uses a lot of other results. So, in this note, we provide an independent proof of Theorem A without using the structure theorem on Baer groups in [2] or [3].

By Theorem A, we immediately get the following.

Corollary B. Let G be a non-nilpotent group. Suppose that $|x^G|$ is a prime for every $x \in T(G)$. Then there exist two deferent primes $p, q \in \pi(G)$ such that $G = PQ \times A$, where A is an abelian $\{p, q\}'$-group, P is an abelian Sylow p-subgroup of G, Q is an abelian Sylow q-subgroup of G and $Q \trianglelefteq G$. Furthermore, $G/Z(G)$ is a Frobenius group of order pq.

Below, we list several elementary lemmas which will be used in the sequel. The proofs of these lemmas are easy. The following Lemma 1.1 and Lemma 1.2 are well known.

Lemma 1.1. Let $x \in G$. Assume that $o(x) = p_1^{m_1}...p_n^{m_n}$, where $p_1, ..., p_n$ are distinct primes and $m_1, ..., m_n$ are positive integers. Then $x = x_1...x_n$ with $o(x_i) = p_i^{m_i}$, and $x_r x_s = x_s x_r$ for $s, r = 1, ..., n$. Furthermore, there exist integers k_i such that $x^{k_i} = x_i$ for $i = 1, \cdots, n$.

Lemma 1.2[4, Lemma 1.1]. *Let G be a group, and let N be a normal subgroup of G.*

(1) For every $x \in N$, $|x^N| | |x^G|$.

(2) For every $x \in G$, $|(xN)^{G/N}| | |x^G|$.

Lemma 1.3[3, Lemma J(i)(1i)]. *Let G be a group and $x \in G$. If $K \trianglelefteq G$ and $K \not\leq C_G(x)$, then $p | |x^G|$ for some $p \in \pi(K)$.*

Lemma 1.4[3, Remark 2]. *Suppose that $x_1, \cdots, x_k$ be the system of representatives of the noncentral conjugacy classes of a non-abelian group G. Then $G =< x_1, \cdots, x_k >$.*

2. The Proof of Theorem A

By Lemma 1.1 and Lemma 1.2, the following Lemma 2.1 holds.

Lemma 2.1. Let G be a B-group, and let N be a normal subgroup of G. Then

(1) If N is non-abelian, then N is a B-group.

(2) If G/N is non-abelian, then G/N is a B-group.

Lemma 2.2. Let G be a non-nilpotent B-group. Then G is solvable and $|G : F(G)|$ is a power of a prime, where $F(G)$ is the Fitting subgroup of G.

Proof. For a group K, if there exists a non-central element x of K such that $|x^K|$ is a power of a prime, then K is not a non-abelian simple group (see [1, 15.2 Theorem, p.190]). So, every B-group is not a non-abelian simple group, and thus by Lemma 2.1 we conclude that every B-group is solvable. In particular, G is solvable.

Since G is solvable and non-nilpotent, we have that $1 < F(G) < G$. Since $F(G/\Phi(G)) = F(G)/\Phi(G)$ and $F(G/Z(G)) = F(G)/Z(G)$, by Lemma 2.1 and induction on $|G|$, we can assume that $\Phi(G) = 1$ and $Z(G) = 1$. It follows that $F(G)$ is abelian (see [5, III. 4.5]) and there exists a subgroup A of G such that $G = AF(G)$ and $A \cap F(G) = 1$ (see [5, III. 4.4]). Hence, we have that $|G : F(G)| = |A|$.

Let N be a non-trivial normal subgroup of A. Set $K = NF(G)$. Then $K \trianglelefteq G$ and $F(K) = F(G)$. By Lemma 2.1 and induction on $|G|$, we conclude that $|K : F(K)|$ is a power of a prime, and so $|N|(= |K : F(G)| = |K : F(K)|)$ is a power of a prime. So, we have proved the following statement (*):

(*) Let N be a non-trivial normal subgroup of A. Then $|N|$ is a power of a prime.

Let M be a maximal normal subgroup of A. We have that $|A : M| = p$, where p is a prime. If $M = 1$, then we are done. So, we assume that $M \neq 1$, and thus by the above statement (*) $|M|$ is a power of a prime q. If $q = p$, then $|G : F(G)|(= |A| = p|M|)$ is a power of the prime p, and we are done. Hence, we assume that $q \neq p$ and deduce a contradiction.

Write $M = Q$. We have that $A = PQ$, where P is a p-group, Q is a q-group, $Q \trianglelefteq A$ and $|P| = p$. We have that $P =< x >$ with $o(x) = p$. Then we have that $A = PQ =< x > Q$.

Write $F = F(G)$. Let $C = C_A(F_{q'})$. Clearly, $F_{q'} \neq 1$. We have that $C \trianglelefteq A$. Since $F(G)$ is abelian and $Z(G) = 1$, we have $C < A$. Then, noting that $A = PQ$, by statement (*) we conclude that either $|C| = 1$ or $|C|$ is a power of p or a power of q.

Assume that $|C| \neq 1$ and $|C|$ is a power of p. Then we have that $P = C \lhd A$. Noting that $Q \lhd A$, we have that $A = PQ = P \times Q$. Then by statement (*) we conclude that $|Q| = q$ and $Q =< y >$ with $o(y) = q$. It follows that $A =< x > \times < y >$.

Assume that $|C|$ is a power of q(including $C = 1$). Then $P(=< x >) \not\leq C_G(F_{q'})$, and so by the hypothesis and Lemma 1.3 we conclude that $|x^G|$ is a power of a prime r with $r \neq q$. Noting that $A \cong G/F(G)$, by Lemma 1.2 we have that $|x^A|$ is a power of a prime r with $r \neq q$. Therefore, noting that $Q \trianglelefteq A$, by Lemma 1.3 we conclude that $Q \leq C_A(x)$, and so $A =< x > \times Q$. It follows by statement (*) that $Q =< y >$ with $o(y) = q$ and $A =< x > \times < y >$.

Clearly, xy is a biprimary element, that is, $xy \in T(G)$. Then by the hypothesis $|(xy)^G|$ is a power of a prime t. Hence, by Lemma 1.3 we have that $F_{t'} \leq C_G(xy)$, and thus by Lemma 1.1 we have that $F_{t'} \leq C_G(< x > \times < y >) = C_G(A)$. Noting that F is abelian, we conclude that $F_{t'} \leq C_G(AF) = C_G(G)$, and thus $F_{t'} \leq Z(G) = 1$. It follows that $F = F_t$ and $G = AF = AF_t = (< x > \times < y >)F_t$. Since $QF =< y > F_t \trianglelefteq G = AF_t$ and $PF =< x > F_t \trianglelefteq G = AF_t$, we have that $p \neq t \neq q$. Let $z \in F_t = F$. Since F is abelian and $|z^G|$ is a power of a prime by the hypothesis, we conclude that $|z^G|$ is either a power of p or a power of q. It follows that $F = C_F(x) \cup C_F(y)$, and thus either $F = C_F(x)$ or $F = C_F(y)$. Then either $x \in C_G(F)$ or $y \in C_G(F)$. Noting that $C_G(F) \leq F$ (see [5, III. 4.2]), we have that either $x \in F$ or $y \in F$, contradicting the fact that $A \cap F = 1$. This completes the proof.

Lemma 2.3. Let G be a non-nilpotent B-group. Then there exist two distinct primes $p, q \in \pi(G)$ such that $G = PQ \times A$, where A is an abelian $\{p, q\}'$-group, P is an abelian Sylow p-subgroup of G, Q is an abelian Sylow q-subgroup of G and $Q \trianglelefteq G$.

Proof. By Lemma 2.2, $G = PF(G)$, where P is a Sylow p-subgroup of G for some prime $p \in \pi(G)$. Set $F = F(G)$. Then $G = PF = PF_{p'}$.

Step 1. Let $q \in \pi(G) - \{p\}$. If F_q is non-abelian, then P centralises F_q.

We have that $F_q \trianglelefteq G$. Let $x \in F_q - Z(F_q)$. $|x^{F_q}|$ is a power of q, and so by Lemma 1.2 we conclude that $|x^G|$ is a power of q because G is a B-group and $x \in T(G)$. Then, since $G = PF$ and F is nilpotent, there exists an element $t \in F_q$ such that $P \leq C_G(x^t)$, and thus by Lemma 1.4 we conclude that $P \leq C_G(F_q)$.

Step 2. Let $r, q \in \pi(G) - \{p\}$ with $r \neq q$. Suppose that F_r is abelian and F_q is non-abelian. Then $P \leq C_G(F_r)$.

Let $x \in F_r$ and $y \in F_q - Z(F_q)$. We have that $F_r F_q =$

$F_r \times F_q$ and $xy = yx$. Since F_r is abelian and $y \notin Z(F_q)$, $|(xy)^{F_r \times F_q}|$ is a power of q. Clearly, $F_r \times F_q \trianglelefteq G$. Then by Lemma 1.2 we conclude that $|(xy)^G|$ is a power of q because G is a B-group and $xy \in T(G)(xy$ is a biprime element). Then, noting that $P \leq C_G(F_q)$ by Step 1, it is easy to see that $P \leq C_G(x)$. Hence, we have that $P \leq C_G(F_r)$.

Step 3. $F_{p'}$ is abelian.

Suppose that $F_{p'}$ is non-abelian. Then, by Step 1 and Step 2 we conclude that $P \leq C_G(F_{p'})$, and thus $G = PF = PF_{p'} = P \times F_{p'}$. This implies that G is nilpotent, a contradiction.

Step 4. P is abelian.

Suppose that P is non-abelian. $|x^P|$ is power of p for every $x \in P - Z(P)$. Then, since $P \cong G/F_{p'}$, by Lemma 1.2 we conclude that $|x^G|$ is a power of p for every $x \in P - Z(P)$ because $|x^G|$ is a power of a prime by the hypotyhesis. Hence, by Lemma 1.3 we conclude that $F_{p'}$ centralises $P =< P - Z(P) >$, and thus $G = PF_{p'} = P \times F_{p'}$. This implies that G is nilpotent, a contradiction.

Step 5. There exists a primes $q \in \pi(G) - \{p\}$ such that $G = PQ \times A$, where A is an abelian $\{p, q\}'$-group, Q is an abelian Sylow q-subgroup of G and $Q \trianglelefteq G$.

Let $x \in P$. Since P is abelian by Step 4, by the hypothesis $|x^G|$ is a power of a prime q with $q \neq p$. Let $y, z \in P$ with $y \neq z$. Then $|y^G|$ is a power of a prime r with $r \neq p$, and $|z^G|$ is a power of a prime q with $q \neq p$. Suppose that $r \neq q$. Then by Lemma 1.3 we conclude that $F_q \leq C_G(y)$ and $F_r \leq C_G(z)$. It follows that $F_r \not\leq C_G(yz)$; otherwise $F_r \leq C_G(y)$ and $|y^G|$ is not a power of r, a contradiction. Hence, by Lemma 1.3 we conclude that $|(yz)^G|$ is is a power of r, and thus $F_q \leq C_G(yz)$. It follows that $F_q \leq C_G(z)$ and $|z^G|$ is not a power of q, a contradiction. So, for every non-identity element x of P, $|x^G|$ is a power of a same prime q, and thus by Lemma 1.3 we have that $F_{q'} \leq C_G(P)$. Noting that $F_{p'}$ is abelian by Step 3, we have that $G = PF = PF_{p'} = PF_q \times F_{\{p,q\}'}$. Let $Q = F_q$ and $A = F_{\{p,q\}'}$. Then $G = PQ \times A$, where A is an abelian $\{p, q\}'$- group, P is an abelian Sylow p-subgroup of G, Q is an abelian Sylow q-subgroup of G and $Q \trianglelefteq G$. This completes the proof.

Lemma 2.4. Let G be a non-nilpotent B-group. Assume that $G = PQ$, where P is an abelian Sylow p-subgroup of G, Q is an abelian Sylow q-subgroup of G and $Q \trianglelefteq G$(p and q are two distinct primes). Then $G/Z(G)$ is a Frobenius group with the Frobenius kernel $QZ(G)/Z(G)$. In addition, the following two propositions hold:

(i) If G has no non-trivial abelian direct factors, then $Z(G) = O_p(G)$.

(ii) Let K be minimal such that $KZ(G) = G$. Then, $K/Z(K)$ is a Frobenius group with the Frobenius kernel $(K \cap Q)Z(K)/Z(K)$ and $Z(K) = O_p(K) = K \cap O_p(G)$.

Proof. Let $x \in G$. We have $x = zy = yz$, where z is a p-element and y is a q-element. Since G is a B-group and $\pi(G) = \{p, q\}$, we have that $|x^G|$ is either a power of p or a power of q. If $|x^G|$ is a power of p, then $|z^G|$ is also a power of p (see Lemma 1.1). Then, since z is a p-element and Sylow p-subgroups of G are abelian, we have that $|z^G| = 1$, and thus $z \in Z(G)$. By the same argument we conclude that, if $|x^G|$ is a power of q, then $y \in Z(G)$. So, either $z \in Z(G)$ or $y \in Z(G)$. It follows that the order of every element of $G/Z(G)$ is a power of a prime. Then, noting that Q is normal in G, $G/Z(G)$ is a Frobenius group with the Frobenius kernel $QZ(G)/Z(G)$ (see [6, p.121, Problems (7.1)]).

Next, we prove proposition (i). By Fitting's Lemma (see [7, Theorem 2.3, p.177]), we have that $Q = [P, Q] \times C_Q(P)$. Clearly, $Z(G)_q = C_Q(P)$. Then, since G has non non-trivial abelian direct factors, we have that $Z(G)_q = 1$, and thus $Z(G) = Z(G)_p$. Clearly, $Z(G)_p = O_p(G)$. So, we have that $Z(G) = Z(G)_p = O_p(G)$. This completes the proof of proposition (i).

Finally, we prove proposition (ii). Since $KZ(G) = G$, we have that $Z(K) = K \cap Z(G)$ and $G/Z(G) = KZ(G)/Z(G) \cong K/(K \cap Z(G)) = K/Z(K)$. Then, since $G/Z(G)$ is a Frobenius group with the Frobenius kernel $QZ(G)/Z(G)$ (see proposition (i)), $K/Z(K)$ is a Frobenius group with the Frobenius kernel $(K \cap Q)Z(K)/Z(K)$. By the minimal property of K and by Fitting's Lemma we conclude that $Z(K)_q = 1$ and $Z(K) = O_p(K) = K \cap O_p(G)$, completing the proof of proposition (ii). The proof is finished.

Proof of Theorem A. By Lemma 2.3 and Lemma 2.4, statement (1) and statement (3) of Theorem A are true. By statement (1) and statement (3) of Theorem A we conclude that statement (2) of Theorem A holds (see [6,(7.1), p.99] and [1,16.1, p.196]). This completes the proof of Theorem A.

Conflicts of Interest

I have no competing intersts.

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