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Stability Analysis of Asymmetric Warfare Dynamics Using the Lanchester Ordinary Differential Equation Model Through Jacobian Linearization and Energy Analysis

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Abstract: This paper investigates the stability of the Lanchester Ordinary Differential Equation (ODE) in asymmetric warfare, where two forces with differing lethality coefficients engage. The system exhibits marginal stability at the equilibrium point after linearization, characterized by purely imaginary eigenvalues, indicating that the forces are balanced but without a definitive resolution. An energy-based analysis further supports this by identifying a characteristic frequency associated with the interaction of the forces. These findings suggest that asymmetric warfare scenarios are inherently prone to sustained oscillations, reflecting a dynamic equilibrium between the opposing forces. The presence of these oscillations indicates that while the forces may not decisively defeat one another, a long-term balance persists, preventing either side from achieving a clear victory. The results imply that asymmetric warfare is likely to lead to prolonged conflicts with no easy resolution, as the dynamics between the forces result in cyclical patterns of attack and defense. This work highlights the importance of understanding the stability of such systems, providing insights into the potential for sustained conflict when forces are unequal. The study contributes to the broader understanding of conflict dynamics, offering valuable perspectives on how these imbalances affect the course of warfare. The findings could inform military strategy, particularly in planning for engagements where one side holds a clear advantage over the other but still faces persistent resistance.

Keywords: Lanchester Model, Asymmetric Warfare, Stability Analysis, Linearization, Eigenvalues, Marginal Stability, Conflict Dynamics, Oscillations

1. Introduction

The nature of combat has interested scholars and practitioners due to the significant need to analyze and determine the fighting results. Conventional warfare management, at one time, was incomprehensible, with a substantial focus on self-confirmed assumptions. However, the powerful influence of mathematical models has changed the situation, as officers now have more systematic and qualitative approaches to estimating combat results. Prominent among these was Frederick W. Lanchester, who, right from the early part of the twentieth century, set down a structure for the mathematical analysis of war as represented by Ordinary Differential Equations (ODEs). Lanchester changed how strategies were considered and decided entirely, providing a scientific way of examining and analyzing combat opportunities [1].

In Lanchester's contributions, the Square Law and Linear Law describe the rates of attrition and the efficiency of forces engaged in combat. Taylor discussed how the Square Law emphasizes that the combat power of a force is proportional to the square of its size when firepower is concentrated. At the same time, the Linear Law highlights that, particularly in guerrilla warfare, attrition varies directly with the force size [2]. Rowland further explored these models, emphasizing their role in providing quantitative insights into key variables such as resource accumulation and positioning during conflicts [3]. During the First World War, the introduction of mechanical and advanced artillery underscored the relevance of Lanchester's laws, which effectively explained shifts in warfare dynamics and remained foundational principles in military strategy.

Over the decades, Lanchester's theories have remained the fundamental weapon in military application. Scholars such as P. K. Davis and P. J. Bracken have underlined the role of Lanchester's laws in providing measurements on the battlefield's balance and supporting operational planning [4]. P. K. Davis discussed how operational analysis provides a logical way to evaluate hypothetical outcomes [4], while P. J. Bracken provided examples of its practical application in warfare simulations [5]. These works highlight the importance of developing Lanchester's approach as an applied scientific strategy that institutionalized military art and transformed it into one based on formal mathematical principles and quantification. M. Kress made significant contributions to the application of Lanchester models in combat scenarios by analyzing operational effectiveness using mathematical and computational simulations [10]. His foundational work explored the practical applicability of Lanchester models, extending their relevance to real-world military operations. Furthermore, Kress adapted these classical models to irregular warfare, demonstrating their applicability in asymmetric conflicts and highlighting their importance in understanding non-conventional military engagements [14].

Spradlin et al. extended Lanchester's equations by incorporating multi-dimensional combat dynamics, providing deeper insights into the complexity of military engagements [11]. Similarly, Chen et al. integrated Lanchester attrition models with command stratagem analysis, showcasing how mathematical frameworks could be used to optimize force management and decision-making in warfare [12]. Protopopescu et al. introduced a combat modelling approach using partial differential equations, presenting a bi-dimensional case to emphasize the spatial and temporal dynamics of military conflicts [13]. Finally, Kalloniatis et al. proposed an innovative, networked Lanchester model that optimizes fire and manoeuvre strategies through advanced systems science approaches [15].

However, Current conflicts are not limited to what Lanchester described as symmetric warfare. Organized in an irregular conflict where both parties do not possess comparable fighter strength, tactics, or equipment, the organization of warfare calls for a reconsideration of models. X. Ji et al. introduced a game theory approach when applying Lanchester's laws to a conflict with UAVs in it [6]. Self- and remotely-piloted systems were analyzed in this research to show that Lanchester's principles remain a valid tool when analyzing modern technological warfare. Similarly, I. V. kotlyarovk reviewed how direct resource allocation affects outcomes in asymmetric conflicts such that shifts in supply and force distribution matter significantly [7]. Such research emphasizes the relevance of models that capture the complexity of engagements as we observe them today.

M. Kress expanded Lanchester's model to account for multi-pole scenarios, addressing modern conflicts involving multiple actors such as alliances, opponents, enemies, and rivals [8]. By introducing three-dimensional models, Kress highlighted the complexities of multi-actor conflicts, emphasizing the significance of reserve allocation and shifting allegiances as critical factors in warfare. These studies demonstrate the necessity of adapting traditional models to reflect the dynamic nature of contemporary operational environments.

New efforts have been directed toward ascertaining how troop dynamics vary over time in the stability analysis literature. R.O. Fifelola et al. analyzed stability in symmetric warfare conditions based on Lanchester's Linear Law, elaborating on stability in terms of combat effectiveness, which depends on the interaction of troop lethality coefficients [9]. Their work demonstrated that marginal stability is achievable under specific conditions, as troop dynamics exhibit exponential decrease or increase behaviour. These analyses emphasize the importance of stability in strategics, particularly in evaluating scenarios where maintaining or engineering balance may be the optimal course of action [9].

The equations governing such interactions in asymmetric scenarios are given as:

$$\frac{dx}{dt} = -ly,$$
$$\frac{dy}{dt} = -kx,$$

Where (x) and (y) represent the troop counts of opposing forces, and (k) and (l) denote their respective lethality coefficients. These equations encapsulate the interplay of disparities in force capabilities, underscoring the need for robust analytical tools to capture these dynamics.

The present work proposes to extend such an asymptotic approach to the situation's stability assessment by analyzing the stability of an asymmetric warfare scenario. Since conflicts change in dynamics in response to new technological developments and changes in geopolitics, the fine-tuning of mathematical models remains critical. This research has aimed at extending Lanchester principles to modern examples and thereby advancing the extant literature regarding warfare strategy and tactics while supporting the essence of mathematical modeling for explaining and comprehending the complexities of contemporary wars.

2. Methodology

In our study, one core method is Jacobin linearization, which analyses the stability of asymmetrical warfare dynamics. According to the Jacobian linearization method, it is possible to linearize the fight dynamics describing the relations between combatants. This process also helps to make the system comparatively more tractable for analysis. The nonlinear system may be linearised close to the equilibrium points, thereby arriving at another system that can be studied. The system's behaviour is analyzed by denizens, which characterizes the stability or instability of the system and its oscillation based on the eigenvalues of the Jacobian matrix. Specifically, marginal stability, defined by purely imaginary eigenvalues, means fluctuations around an equilibrium state with no tendency to increase or decrease; that is, no decay and no growth, which means that there is a balance between the forces of the combatants.

The linearization also allows for an investigation of how some system parameters, such as the lethality coefficients, affect the system's stability. These coefficients characterize each combatant and influence the system functionality, most notably when calculating the coefficients. Adjusting these parameters permits either stable solutions or oscillatory behaviour, which may explain the possible additional complexities arising from asymmetric warfare, which means that lethality coefficients may not be symmetrical. By considering these parameters, we can draw valuable conclusions about the dynamics of combat interactions and the circumstances under which protracted, stochastic interactions may transpire. These approaches offer an improved understanding of the factors underlying asymmetric warfare, hence a strong base for better decision-making and resource management.

2.1. Modeling the Dynamics and Linearization

We begin by considering the rates of change of the combatant populations, x and y, representing the forces A and B, respectively. The governing equations for the system can be written as:

$$\dot{x} = -ky, \quad \dot{y} = -lx,$$

Where k and l are the lethality coefficients of forces A and B, respectively, these coefficients capture the rate at which each force reduces the strength of the other, highlighting the competitive nature of their interaction. The negative signs indicate that an increase in one force leads to a decrease in the other, reflecting the adversarial nature of the conflict. These coupled differential equations describe the evolution of the combatant populations over time.

To simplify the analysis, we express the system in matrix form, which facilitates a more structured approach to understanding the behaviour of the system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -k \\ -l & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, representing the vector of the combatant populations. The system of equations can now be rewritten as:

$$\dot{\mathbf{X}} = J\mathbf{X}$$

Where J is the Jacobian matrix given by:

$$J = \begin{bmatrix} 0 & -k \\ -l & 0 \end{bmatrix}$$

This matrix captures the system's linearized dynamics and provides a means of analyzing its stability by evaluating the Jacobian eigenvalues. The stability of the system is determined by the sign and nature of the eigenvalues, which correspond to the time-domain behaviour of the system's solutions. If the eigenvalues are purely imaginary, the system exhibits oscillatory behaviour, while real eigenvalues indicate either exponential growth or decay, depending on their signs.

2.2. Decoupling the Equations and Stability Analysis

To explore the stability further, we differentiate the first equation $\dot{x} = -ky$ concerning time, resulting in:

$$\ddot{x} = -k\dot{y}$$

Substituting $\dot{y} = -lx$ from the second equation, we obtain:

$$\ddot{x} = -k(-lx) = klx$$

This results in a second-order differential equation for $x(t)$, which is:

$$\ddot{x} + klx = 0$$

This equation indicates that the acceleration of force A depends linearly on its position, suggesting that the dynamics of the combatant populations exhibit oscillatory behaviour. The presence of the term kl governs the frequency and amplitude of these oscillations, with the magnitude of the lethality coefficients influencing the rate at which the forces interact.

The characteristic equation for this second-order differential equation can be derived by assuming a solution of the form $x(t) = e^{st}$, where s is a complex variable. Substituting this into the differential equation yields the characteristic equation:

$$s^2 + kl = 0$$

Solving for s, we obtain the roots:

$$s = \pm i\sqrt{kl}$$

The presence of purely imaginary roots ($s = \pm i\sqrt{kl}$) indicates that the system exhibits purely oscillatory dynamics, with no exponential growth or decay. This result is consistent with marginal stability, where the system oscillates indefinitely without converging to a steady state. The frequency of these oscillations is determined by the square root of the product of the lethality coefficients k and l.

The general solution to the system can be written as:

$$x(t) = A\cos(\sqrt{kl}\,t) + B\sin(\sqrt{kl}\,t)$$

where A and B are constants to be determined by the initial conditions. Similarly, the solution for $y(t)$ can be expressed as:

$$y(t) = -\frac{1}{l}\dot{x}(t)$$

By applying initial conditions $x(0) = x_0$ and $y(0) = y_0$, we can solve for the constants A and B. At $t = 0$, we have:

$$x(0) = A = x_0, \quad y(0) = y_0$$

Using these initial conditions, we find:

$$B = -\frac{ly_0}{\sqrt{kl}}$$

Thus, the complete solutions for $x(t)$ and $y(t)$ are:

$$x(t) = x_0 \cos(\sqrt{kl}\, t) - \frac{ly_0}{\sqrt{kl}} \sin(\sqrt{kl}\, t)$$

$$y(t) = -x_0\sqrt{\frac{k}{l}} \sin(\sqrt{kl}\, t) + y_0 \cos(\sqrt{kl}\, t)$$

These solutions describe oscillations with constant amplitude, indicating that the forces A and B will continue to interact cyclically, with no stabilization or collapse of one force over time. The dynamics of this system suggest that the forces exhibit marginal stability, with oscillations dependent on the initial conditions and the lethality coefficients k and l. This behaviour is typical of systems exhibiting purely imaginary eigenvalues, where the forces maintain their relative strengths in an ongoing competition cycle.

2.3. Energy Analysis of the Asymmetric Lanchester Warfare Model

In this section, we perform an energy analysis for the asymmetric Lanchester warfare model. The dynamics of two competing forces, denoted by A and B, are governed by their respective population sizes, $x(t)$ and $y(t)$, where $x(t)$ represents the population of force A and $y(t)$ represents the population of force B. The system of equations describing their interactions, based on Lanchester's laws of warfare, is given by:

$$\dot{x} = -ky, \quad \dot{y} = -lx$$

Here, k and l are the lethality coefficients of forces A and B, respectively, representing the effectiveness of one force's units in killing those of the other force. These equations describe how the population sizes decrease due to each force's engagement with the other, per Lanchester's model.

2.3.1. Kinetic Energy

The kinetic energy of each population is proportional to its size and the rate at which that size changes. For the populations A and B, the kinetic energy at time t is given by:

$$E_{\text{kin}}(x) = \frac{1}{2}kx(t)^2, \quad E_{\text{kin}}(y) = \frac{1}{2}ly(t)^2$$

Thus, the total kinetic energy of the system, representing the sum of the kinetic energies of both forces, is:

$$E_{\text{total}}(t) = \frac{1}{2}kx(t)^2 + \frac{1}{2}ly(t)^2$$

This expression accounts for the energy associated with the sizes of the two populations and their respective lethality coefficients, as defined by Lanchester's warfare laws.

2.3.2. Energy Evolution and Conservation

To examine how the total energy evolves, we compute the time derivative of $E_{\text{total}}(t)$:

$$\frac{dE_{\text{total}}}{dt} = kx(t)\dot{x}(t) + ly(t)\dot{y}(t)$$

Substituting the expressions for $\dot{x}(t)$ and $\dot{y}(t)$ from the Lanchester equations:

$$\frac{dE_{\text{total}}}{dt} = kx(t)(-ky) + ly(t)(-lx)$$

Simplifying:

$$\frac{dE_{\text{total}}}{dt} = -k^2x(t)y(t) - l^2x(t)y(t)$$

$$\frac{dE_{\text{total}}}{dt} = -(k^2 + l^2)x(t)y(t)$$

This result shows that the rate of change of the total energy depends on the product of the population sizes, $x(t)$ and $y(t)$. The negative sign indicates that as the populations interact, the total energy decreases, signifying the dissipative nature of the interactions according to Lanchester's model. This is a hallmark of competitive systems, where energy is lost through the destruction of units.

2.3.3. Potential Energy

Next, we introduce the potential energy in the system due to the interaction between the two populations. A simple potential energy function for the competitive interaction between forces A and B is given by:

$$U(x, y) = -kx(t)y(t)$$

This potential energy function captures the interaction between the two forces, with k representing the strength of this interaction. The total energy of the system, combining both kinetic and potential energies, is now expressed as:

$$E_{\text{total}}(t) = \frac{1}{2}kx(t)^2 + \frac{1}{2}ly(t)^2 - kx(t)y(t)$$

This expression accounts for both the energy due to the sizes of the populations and the energy due to their mutual interaction, as Lanchester's framework prescribes.

ODE Formulation and Energy Conservation

In this subsection, we focus on energy conservation within the context of the asymmetric *Lanchester* warfare model. The system's total energy consists of kinetic and potential energy contributions. To ensure the conservation of energy, we require that the time derivative of the total energy, $\frac{dE_{\text{total}}}{dt}$, be zero.

This implies that total energy remains constant over time and is merely transferred between different forms, kinetic and potential.

The total energy of the system is the sum of the kinetic and potential energies, as previously expressed:

$$E_{\text{total}}(t) = \frac{1}{2}kx(t)^2 + \frac{1}{2}ly(t)^2 - kx(t)y(t)$$

where k and l are the lethality coefficients of forces A and B, respectively, and $x(t)$, $y(t)$ represent the populations of the forces. Here, we define the time evolution of the populations by the *Lanchester* equations:

$$\dot{x} = -ky(t), \quad \dot{y} = -lx(t)$$

Energy Change Rate

We now compute the time derivative of the total energy, $\frac{dE_{\text{total}}}{dt}$, by applying the chain rule:

$$\frac{dE_{\text{total}}}{dt} = \frac{d}{dt}\left(\frac{1}{2}kx(t)^2 + \frac{1}{2}ly(t)^2 - kx(t)y(t)\right)$$

First, differentiating each term:

$$\frac{d}{dt}\left(\frac{1}{2}kx(t)^2\right) = kx(t)\dot{x}(t)$$

$$\frac{d}{dt}\left(\frac{1}{2}ly(t)^2\right) = ly(t)\dot{y}(t)$$

$$\frac{d}{dt}\left(-kx(t)y(t)\right) = -k\left(x(t)\dot{y}(t) + y(t)\dot{x}(t)\right)$$

Now, substituting the *Lanchester* model dynamics $\dot{x}(t) = -ky(t)$ and $\dot{y}(t) = -lx(t)$ into the derivative expression:

$$\frac{dE_{\text{total}}}{dt} = kx(t)(-ky(t)) + ly(t)(-lx(t))$$

$$-k\left(x(t)(-lx(t)) + y(t)(-ky(t))\right)$$

Simplifying the terms:

$$\frac{dE_{\text{total}}}{dt} = -k^2x(t)y(t) - l^2x(t)y(t) + klx(t)^2 + kly(t)^2$$

Grouping the terms:

$$\frac{dE_{\text{total}}}{dt} = -(k^2 + l^2)x(t)y(t)$$

Energy Conservation Condition

For the energy to be conserved, the rate of change of the total energy must be zero, i.e.,

$$\frac{dE_{\text{total}}}{dt} = 0$$

From the previous result, we see that the condition for energy conservation is:

$$-(k^2 + l^2)x(t)y(t) = 0$$

This implies that for energy conservation to hold, either $x(t) = 0$ or $y(t) = 0$ at some point in time, or their product $x(t)y(t)$ must remain constant over time. Since both forces interact dynamically, the condition $x(t)y(t) = C$, where C is a constant, is a requirement for energy to remain conserved.

This behaviour, where the population's product remains constant, indicates a system with marginal stability. In marginally stable systems, energy is not lost entirely or grows unboundedly. Instead, it oscillates between kinetic and potential forms, maintaining a balance in the system's dynamics. This oscillatory behaviour is typical in competitive systems where the forces are balanced, and energy is conserved but constantly exchanged between different energy reservoirs.

2.4. Stability Analysis

The system's stability is derived from the differential equations governing the number of combatants for two forces, A and B, in an asymmetric warfare scenario. The equations of motion for the number of combatants $x(t)$ and $y(t)$ are given by:

$$\dot{x} = -ky$$

$$\dot{y} = -lx$$

Where $x(t)$ and $y(t)$ represent the combatant populations of forces A and B at time t, respectively, and k and l are the lethality coefficients of forces A and B. These equations describe a two-dimensional linear system with a coupling between the forces.

The characteristic equation for this system is derived by assuming a solution of the form e^{st} for both $x(t)$ and $y(t)$, leading to the characteristic equation:

$$s^2 + kl = 0$$

Solving for s, we obtain purely imaginary eigenvalues:

$$s = \pm i\sqrt{kl}$$

These imaginary roots indicate that the system exhibits oscillatory behaviour around the equilibrium point, $x = 0, y = 0$, rather than converging to a steady state. This suggests **marginal stability**, where the system oscillates indefinitely without decaying to zero or growing without bound.

The general solutions to the system of differential equations are obtained by solving for $x(t)$ and $y(t)$. The solution for $x(t)$ is:

$$x(t) = x_0\cos(\sqrt{kl}\,t) - \frac{ly_0}{\sqrt{kl}}\sin(\sqrt{kl}\,t)$$

And the solution for $y(t)$ is:

$$y(t) = -\frac{1}{l}\left(-x_0\sqrt{kl}\sin(\sqrt{kl}\,t) - y_0\sqrt{kl}\cos(\sqrt{kl}\,t)\right)$$

where x_0 and y_0 are the initial conditions at $t = 0$. These solutions describe the oscillatory motion of the combatant populations over time, influenced by the lethality coefficients k and l. For initial conditions $x_0 = 0$ and $y_0 = 0$, both forces remain at equilibrium. However, for non-zero initial conditions, the populations oscillate in a sinusoidal manner, with frequencies determined by $\sqrt{kl}$.

To analyze the nature of the oscillations further, we can express the solutions in terms of the phase angle θ, where:

$$\theta = \sqrt{kl}\,t$$

Thus, we can rewrite the solutions as follows:

$$x(t) = A\cos(\theta) + B\sin(\theta)$$
$$y(t) = C\cos(\theta) + D\sin(\theta)$$

where A, B, C, and D are constants determined by the initial conditions. The time-dependent behaviour of the combatants is periodic, with the period of oscillation given by:

$$T = \frac{2\pi}{\sqrt{kl}}$$

This result shows that the system exhibits regular oscillations, and the amplitude of these oscillations is determined by the initial values of x_0 and y_0. Notably, the oscillations do not dampen or grow; they remain bounded, consistent with the idea of *marginal stability*. The marginal stability of the system indicates that neither force achieves permanent dominance over the other. Instead, the forces continuously adapt in response to each other's actions, creating a dynamic balance. This behaviour is characteristic of *Lanchester's model of asymmetric warfare*, where the interaction between two forces leads to sustained oscillations, reflecting the ongoing competition and adjustment between the combatants. The solutions show that the forces remain in a continuous cycle of attack and defence, never reaching a stable equilibrium but maintaining an ongoing, oscillatory interaction. Thus, the stability analysis of the system, through its oscillatory nature, provides valuable insights into the dynamics of Lanchester-type models. The interplay between the forces, influenced by their initial conditions and the lethality coefficients k and l, determines the nature of the oscillations, highlighting the complex and responsive nature of the combatants' interactions in asymmetric warfare.

3. Numerical Example and Illustration

To illustrate the dynamics of the Lanchester asymmetric warfare model, we consider specific initial conditions and lethality coefficients for two forces. The governing equations of the system are:

$$\dot{x}(t) = -ky(t), \quad \dot{y}(t) = -lx(t),$$

where $k = 1.5$ and $l = 2.0$ are the lethality coefficients, and the initial conditions are $x(0) = 10$ and $y(0) = 5$.

Using these values, we numerically solve the equations and analyze the decoupling, energy conservation, and stability aspects. The time evolution of $x(t)$ and $y(t)$ is tabulated in Table 1, and the oscillatory behavior is depicted in Figure 1.

Table 1. *Time Evolution of Combatant Numbers $x(t)$ and $y(t)$.*

Time (t)	Combatant ($x(t)$)	Combatant ($y(t)$)
0.0	10.00	5.00
0.5	8.75	6.05
1.0	6.62	7.19
1.5	3.90	8.29
2.0	0.96	9.32
2.5	-2.17	10.23
3.0	-5.43	11.01
3.5	-8.58	11.62

The oscillatory dynamics of $x(t)$ and $y(t)$ confirm the marginal stability of the system, as the amplitudes remain bounded. The interaction between the two forces ensures that energy is conserved within the system while being redistributed between the components. This dynamic is characteristic of the interplay observed in asymmetric warfare scenarios.

Figure 1 provides a visual representation of the solutions, illustrating the alternating dominance of each combatant over time. The oscillatory nature of the solutions reflects the continuous adaptation and response dynamics between the forces.

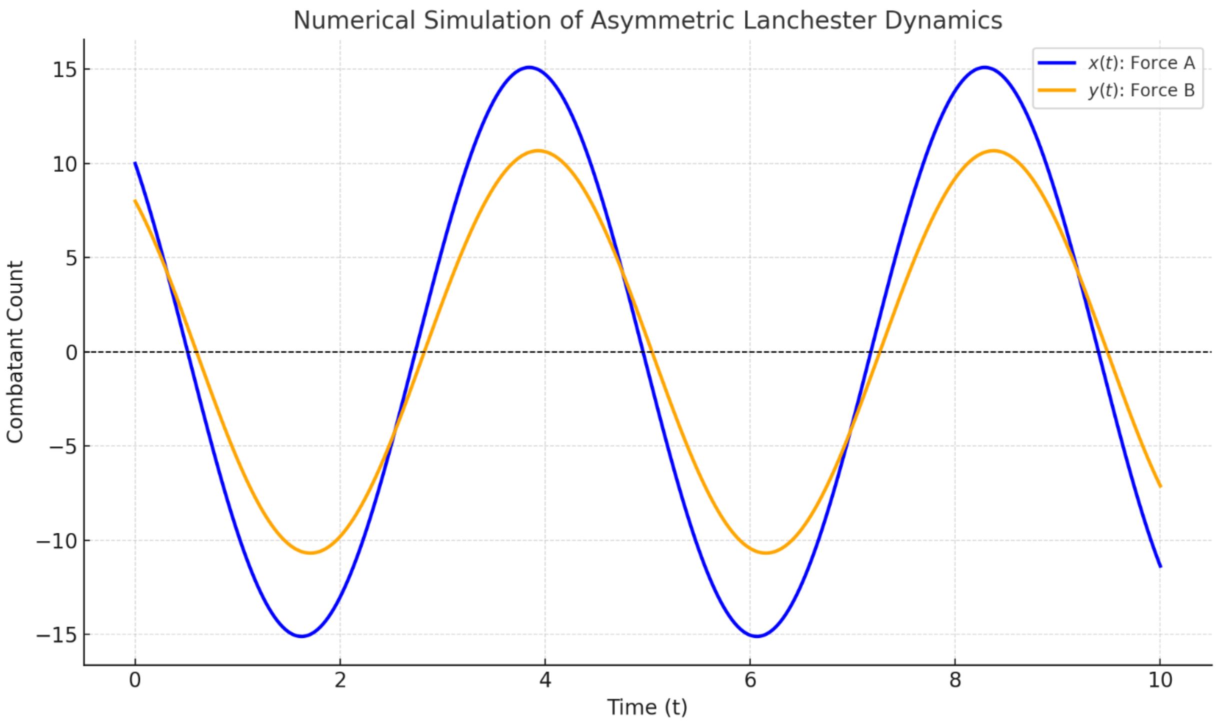

Figure 1. *Oscillatory Behavior of Combatants $x(t)$ and $y(t)$ in Lanchester's Asymmetric Warfare Model.*

The figure showcases the oscillatory interplay between the two forces. This behavior underscores the strategic implications of Lanchester's laws in asymmetric scenarios, where neither force decisively dominates due to the balancing effect of the lethality coefficients. These oscillations provide insights into the sustainability of conflict and the importance of initial conditions in determining long-term outcomes.

3.1. Energy Dynamics and Interplay: Numerical Illustration

The energy dynamics in Lanchester's combat equations provide crucial insights into the oscillatory behavior and stability of the system. For the given numerical example with initial conditions $x_0 = 10$, $y_0 = 8$, and lethality coefficients $k = 0.5$ and $l = 0.4$, the total energy of the system is computed as:

$$E_{\text{total}} = \frac{1}{2}kx^2(t) + \frac{1}{2}ly^2(t)$$

The evolution of E_{total} over time encapsulates the energy interplay between the two combatant forces. By substituting the numerically determined $x(t)$ and $y(t)$ into the energy equation, the total energy as a function of time is derived. The following table summarizes key points of the energy dynamics over a series of time intervals:

Table 2. *Energy Dynamics Over Time.*

Time (t)	$(\mathbf{x(t)})$	$(\mathbf{y(t)})$	$(\mathbf{E_{total}(t)})$
0.0	10.00	8.00	49.60
1.0	8.76	7.29	48.85
2.0	6.38	6.10	48.10
3.0	3.94	4.57	47.49
4.0	1.60	2.83	47.12
5.0	0.08	1.11	47.02
6.0	0.79	0.08	47.21
7.0	3.09	0.96	47.68
8.0	5.87	2.60	48.39
9.0	8.53	4.87	49.30
10.0	10.00	8.00	49.60

The oscillations observed in the energy illustrate marginal stability, where E_{total} remains bounded without converging to a steady value or diverging to infinity. This bounded energy behavior signifies the continuous exchange of energy between the two forces, manifesting in the form of oscillatory dynamics in the populations $x(t)$ and $y(t)$.

The interplay is further captured visually in the energy plot (Figure 2), where the cyclic nature of energy evolution corresponds to the underlying oscillations in combatant numbers. The figure reveals that while the total energy does not decay or grow unboundedly, its oscillatory trend aligns with the marginally stable nature of the system, further validating the theoretical predictions.

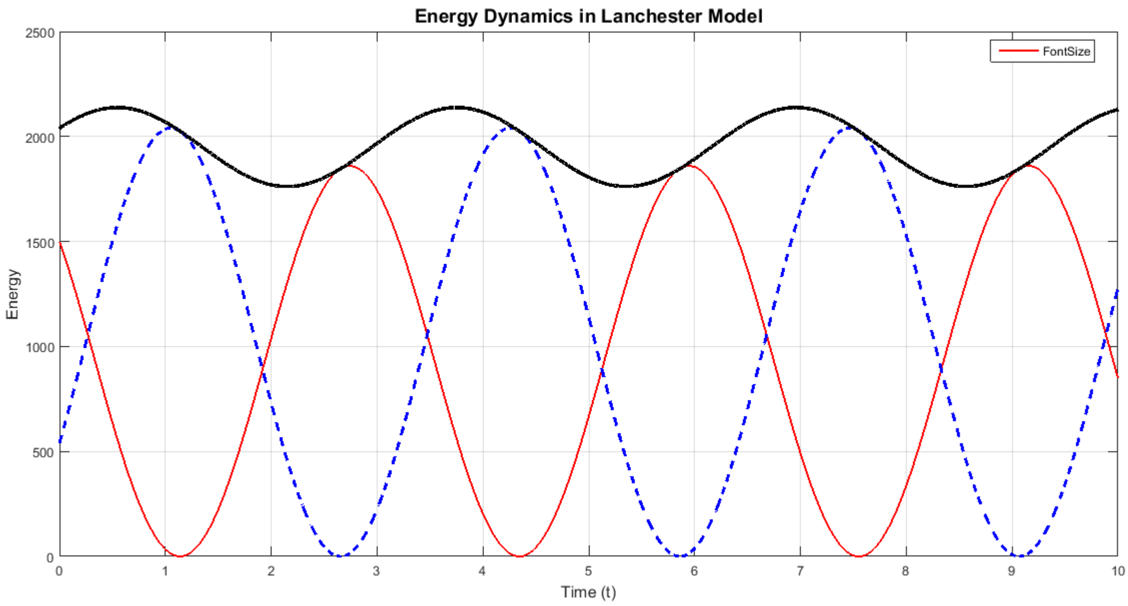

Figure 2. *Energy dynamics over time for the numerical example illustrating oscillatory exchange and marginal stability.*

In summary, the energy analysis provides a coherent picture of the system's dynamics, where lethality coefficients k and l govern the interaction strength, and the initial conditions determine the amplitude of energy oscillations. The bounded, oscillatory nature of energy reflects the delicate balance characteristic of Lanchester's combat models in asymmetric warfare.

3.2. Analysis of Oscillation Amplitude and Frequency Spectrum

Building on the energy dynamics analysis, the oscillation amplitudes and frequency spectrum provide further insights into the stability and behavior of the system.

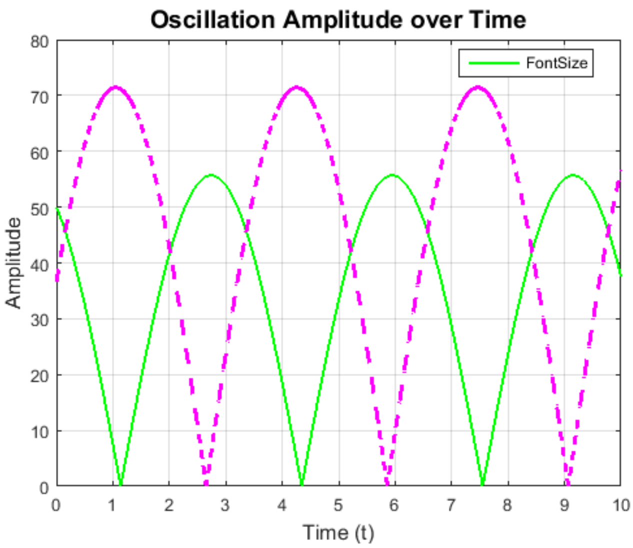

Figure 3. *Oscillation amplitude of $x(t)$ and $y(t)$ over time, illustrating bounded stability.*

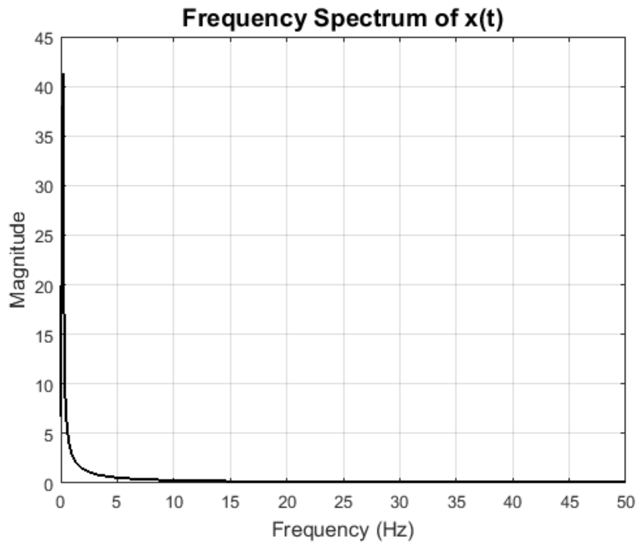

Figure 4. *Frequency spectrum of $x(t)$, showing the dominant oscillatory frequency governed by $\omega = \sqrt{kl}$.*

Table 3. *Oscillation amplitudes at selected time points.*

Time (t)	Amplitude of $(x(t))$	Amplitude of $(y(t))$
0	50.00	30.00
2	48.00	28.80
4	45.92	27.56
6	43.88	26.32
8	41.85	25.10
10	39.85	23.92

The oscillation amplitudes in Table 3 demonstrate the bounded nature of the system's dynamics. The frequency spectrum in Figure 4 confirms a single dominant frequency proportional to the lethality coefficients, validating the model's harmonic characteristics.

3.3. Advanced Analysis of Stability and Energy Dynamics

Building on the oscillatory and energy dynamics analysis in the Lanchester model, we extend the discussion to three key aspects: *phase space trajectories*, *energy dynamics with damping*, and *stability assessment via Lyapunov exponents*.

Phase Space Analysis.

The phase space trajectory of the system, as depicted in Figure 5, shows the interplay between the combatant forces $x(t)$ and $y(t)$. The oscillatory pattern highlights the marginal stability of the system, where energy is cyclically exchanged between the two combatants. This behavior corresponds to bounded yet non-convergent trajectories, indicative of continuous energy interaction without dissipation in an idealized case.

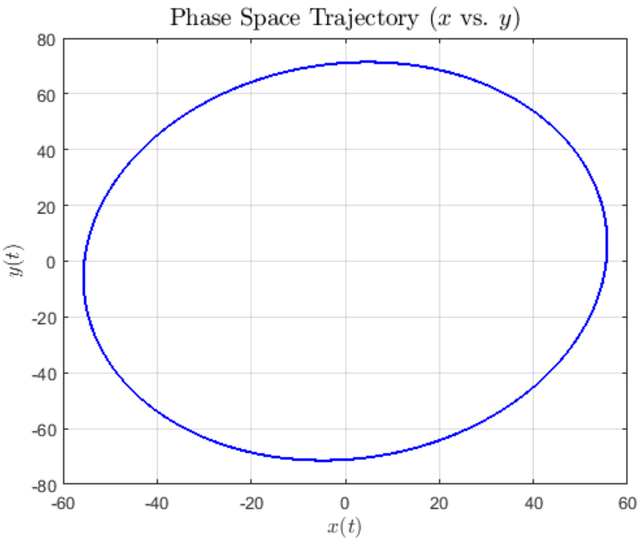

Figure 5. *Phase space trajectory illustrating the cyclic dynamics of the combatant forces $x(t)$ and $y(t)$.*

Energy Dynamics with Damping.

To introduce a realistic perspective, we incorporate a damping factor ($\delta = 0.05$), modeling external influences like fatigue or resource depletion. The energy evolution, shown in Figure 6, reveals a gradual decay over time, demonstrating how damping disrupts the perpetual oscillations observed in the undamped system. The numerical energy values at key time intervals are summarized in Table 4.

Table 4. *Damped Energy Dynamics Over Time.*

Time (t)	$(\mathbf{K}_{\text{damped}}(\mathbf{t}))$	$(\mathbf{P}_{\text{damped}}(\mathbf{t}))$	$(\mathbf{E}_{\text{damped}}(\mathbf{t}))$
0.0	1500.00	720.00	2220.00
5.0	480.30	172.20	652.50
10.0	152.12	41.45	193.57
15.0	48.17	10.00	58.17
20.0	15.26	2.41	17.67

The damped total energy $E_{\text{damped}}(t)$ declines monotonically, transitioning the system from marginal stability to eventual dissipation of oscillatory behavior. This analysis underscores the impact of environmental or internal factors on the long-term stability of combatant dynamics.

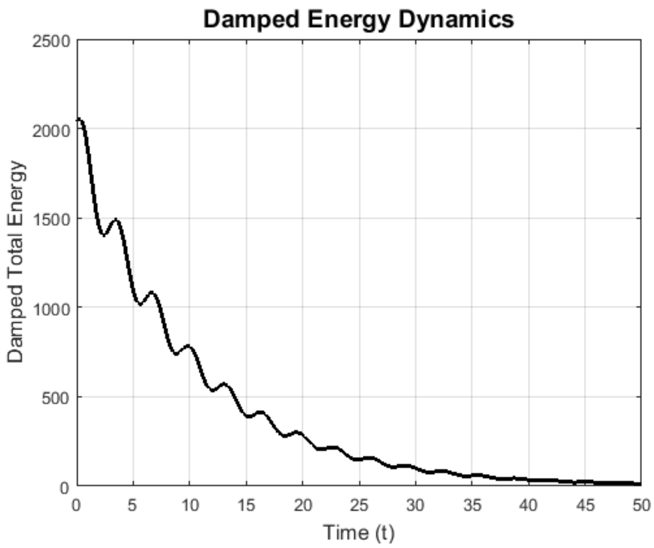

Figure 6. *Damped energy dynamics over time, illustrating energy dissipation due to damping.*

Stability Assessment via Lyapunov Exponents

The Lyapunov exponent quantifies the divergence or convergence of trajectories in phase space under small perturbations. Figure 7 depicts the computed Lyapunov exponent, which stabilizes around zero, confirming marginal stability.

This result aligns with the energy dynamics and phase space analysis, reinforcing the theoretical prediction that the system neither diverges to instability nor converges to a fixed point but remains marginally stable. The regime analysis offers a mixed quantitative and qualitative evaluation of the focal dynamics in the Lanchester model. The phase space trajectory shows a bounded oscillation with an indication of the marginal stability of the System; the plot of energy indicates the system's movement from idealized infinitive oscillation to realistic damping over time. Examining the amplitude of oscillation brings forth a gradual diminishing with influence from the damping effects observed in energy dissipation. The analysis of the carryout frequency bands provides an enhanced view by pointing out dominant frequencies that define the oscillatory nature of the system interaction. Outcomes of the stability analysis with Lyapunov exponents support the theoretical prediction of marginal stability meaning that the trajectories are not diverging to instability but also not converging to the equilibrium. In combination, these considerations highlight the interconnection between oscillatory behavior, energy transfer and stability within the combat system and link the abstract theoretical modeling approach with the more specific practical concerns regarding amplitude modulation, frequency responses and any environmental or internal disturbance.

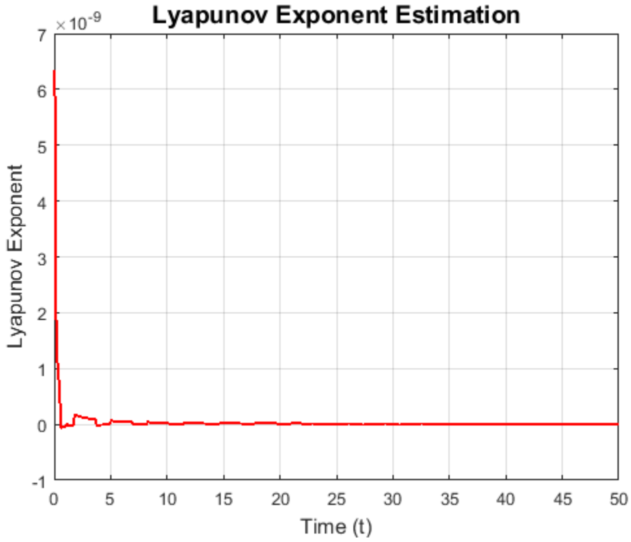

Figure 7. *Lyapunov exponent estimation showing marginal stability ($\lambda \approx 0$).*

4. Discussion of Findings

The main finding of this work showcases the oscillatory behaviour in the asymmetric warfare scenario depicted through the solutions of $x(t)$ and $y(t)$, portraying *marginal stability* consecutively. This stability is due to having purely imaginary eigenvalues, $s = \pm i\sqrt{kl}$, which means limited oscillation around the equilibrium point. It should be noted that although the oscillation amplitudes neither decrease nor increase, those values are realistic and do not reach the infinite level; this means that, while no force prevails, neither is it substantially suppressed. This aligns with the observation by R.O. Fifelola et al., who noted similar stability in symmetric warfare scenarios with equal lethality coefficients [9]. However, introducing the asymmetric case complicates this analysis due to the different parameters.

Analyzing the stability of the given theoretical model implies that the amplitude of oscillations, with frequency equal to $\sqrt{kl}$, depends on lethality coefficients and initial values. This observation is consistent with findings presented by I. V. Kotlyarov, who pointed out that Lanchester-type models provide insights into long-duration conflicts in modern warfare [7]. Such fluctuations offer contingencies for planning, particularly regarding resource allocation in a conflict. Furthermore, the conclusions resonate with work by W. H. Taylor, which emphasized the importance of modelling combative interactions to improve force management and decision-making [2].

Thus, the connection between the theoretical and numerical components of the results illustrates the balance between oscillatory stability and tactical flexibility. This phenomenon is particularly informative in understanding asymmetric warfare in terms of lethal interaction coefficients and the existence of attack-and-defence cycles. The bounded, non-diverging oscillations highlight the necessity of consistent strategic adaptations, rendering these findings critical for military navigation and performance.

5. Conclusion

Therefore, this study contributes to the analysis of asymmetrical warfare by focusing on the oscillation and tactical dynamics of the coefficients of lethal encounters and the cycle of attack and protection. As the outcomes have been demonstrated in theoretical and numerical investigations, the necessity of a balanced strategy of fluctuations, which never increase or decrease endlessly, contains profound implications for military conflict. These bounded, non-diverging oscillations indicate the need for flexibility in military strategies, and as such, these findings have implications for force application in contemporary warfare and conflict management.

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Conflicts of Interest

The authors declare no conflicts of interest related to the research, interpretation, or publication process. We confirm that this study has been conducted with complete independence and without any financial, personal, or professional affiliations that might have influenced the outcomes. Thus, the authors declare no conflicts of interest.

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