International Journal of Systems Science and Applied Mathematics
2024, Vol. 9, No. 3, pp. 37-43
https://doi.org/10.11648/j.ijssam.20240903.11

Research Article

Bifurcation Analysis and Multiobjective Nonlinear Model Predictive Control of Sustainable Ecosystems

Lakshmi Narayan Sridhar*

Department of Chemical Engineering, University of Puerto Rico, Mayaguez, Puerto Rico

Abstract

Objective: All optimal control work involving ecological models involves single objective optimization. In this work, we perform multiobjective nonlinear model predictive control (MNLMPC) in conjunction with bifurcation analysis on an ecosystem model. *Methods:* Bifurcation analysis was performed using the MATLAB software MATCONT while the multi-objective nonlinear model predictive control was performed by using the optimization language PYOMO. *Results:* Rigorous proof showing the existence of bifurcation (branch) points is presented along with computational validation. It is also demonstrated (both numerically and analytically) that the presence of the branch points was instrumental in obtaining the Utopia solution when the multiobjective nonlinear model prediction calculations were performed. *Conclusions:* The main conclusions of this work are that one can attain the utopia point in MNLMPC calculations because of the branch points that occur in the ecosystem model and the presence of the branch point can be proved analytically. The use of rigorous mathematics to enhance sustainability will be a significant step in encouraging sustainable development. The main practical implication of this work is that the strategies developed here can be used by all researchers involved in maximizing sustainability The future work will involve using these mathematical strategies to other ecosystem models and food chain models which will be a huge step in developing strategies to address problems involving nutrition.

Keywords

Ecosystem, Bifurcation, Optimal Control

1. Introduction

Sustainability is a significant factor to be considered in almost all physical and chemical phenomena. Beneficial activities and situations must be sustained over a considerable amount of time. This is especially true in ecosystem management where the conservation of natural species is essential for ensuring a healthy environment for the long-term well-being of the human population. The issue of sustainability should be implemented in optimization and control studies of ecosystems.

Cabezas and co-workers [1-9] have applied the fisher index [10] as a sustainability criterion for ecosystems. Specifically, the sustainability concept has been applied in the management of ecosystems, by controlling the population of various species.

Shastri and DIwekar [11] and Sorayya et al [12] performed single objective optimal control calculations on ecological models maximizing the fisher index to ensure maximum sustainability.

This work aims to perform bifurcation analysis and mul-

*Corresponding author: lakshmin.sridhar@upr.edu (Lakshmi Narayan Sridhar)

Received: 1 October 2024; **Accepted:** 17 October 2024; **Published:** 11 November 2024

tiobjective nonlinear model predictive control calculations. The bifurcation analysis reveals the existence of branch points. A rigorous mathematical analysis (which is also computationally validated) demonstrating the existence of branch points is presented. It is shown that the presence of the branch points makes the multiobjective nonlinear model predictive control calculations to reach the utopia solution. This demonstrates that one can maximize the conservation of the natural habitat and maintain maximum sustainability.

2. Equations in Ecological Model

The equations are the following

$$\phi_{12} = \frac{a_2 x_2 x_1}{b_2 + x_1}; \qquad \phi_{23} = \frac{a_3 x_3 x_2}{b_3 + x_2} \tag{1}$$

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1 = x_1\left(-\frac{a_2 x_2}{b_2 + x_1} + r\left(1 - \frac{x_1}{K}\right)\right) = r x_1 - \frac{r x_1^2}{k} - \phi_{12} \\
\frac{dx_2}{dt} &= f_2 = x_2\left(-\frac{a_3 x_3}{b_3 + x_2} - d_2 + e_2 \frac{a_2 x_1}{b_2 + x_1}\right) = (e_2 \phi_{12}) - \phi_{23} - d_2 x_2 \\
\frac{dx_3}{dt} &= f_3 = x_3\left(e_3 \frac{a_3 x_2}{b_3 + x_2} - d_3\right) = e_3 \phi_{23} - d_3 x_3
\end{aligned} \tag{2}$$

The base parameter values are

$$k = 710;\ b_2 = 235.50;\ b_3 = 250;$$
$$e_2 = 1.35;\ e_3 = 1.29;\ d_2 = 1.0;$$
$$d3 = 0.04;\ r = 1.2;\ a_2 = 2.0;\ a_3 = 0.1;$$

3. Computational Procedures Used

Bifurcation analysis

Multiple steady-state solutions are caused by a) Branch Points and b) limit points. At these points the Jacobian matrix of the steady-state equations has a determinant of 0. At a branch point there are 2 distinct tangents while at a limit point, there is only one tangent at the singular point CL_MATCONT [13, 14] is commonly used to detect branch and limit points. Here a continuation procedure implementing the Moore-Penrose matrix pseudo-inverse is used. CL_MATCONT obtains the branches of the solutions starting from the bifurcation points.

For an ODE system

$$\frac{dx}{dt} = f(x, \beta) \tag{3}$$

Where $x \in R^n$ Let the tangent plane at any point x be $[v_1, v_2, v_3, v_4, \ldots. v_{n+1}]$. Consider a matrix A as

$$A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \cdots\cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \beta} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \cdots\cdots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial \beta} \\
\cdots\cdots & & & & & \\
\cdots\cdots & & & & & \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \frac{\partial f_n}{\partial x_4} \cdots\cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \beta}
\end{bmatrix} \tag{4}$$

The matrix A can be written as

$$A = [B \mid \frac{\partial f}{\partial \beta}] \tag{5}$$

The tangent surface must satisfy the equation

$$Av = 0 \tag{6}$$

For limit and branch points the matrix B must be singular. For a limit point (LP) the

n+1 th component of the tangent vector v_{n+1} = 0 and for a branch point (BP) the matrix $\begin{bmatrix} A \\ v^T \end{bmatrix}$ must be singular [15-17].

Multiobjective Nonlinear Model Predictive Control Algorithm

The MNLMPC (multiobjective nonlinear model predictive control) strategy [18, 19] used in this work does not involve the use of weighting functions or impose additional constraints [20]. For a problem that is posed as

$$\begin{aligned}
&\min J(x,u) = (x_1, x_2 \ldots. x_k) \\
&\textit{subject to } \frac{dx}{dt} = F(x,u)
\end{aligned} \tag{7}$$

The MNLMPC method first solves dynamic optimization problems independently minimizing/maximizing each any variable p_i individually. The minimization/maximization of p_i will lead to the values p_i^*. Then the optimization problem that will be solved is

$$\begin{aligned}
&\min \{p_i - p_i^*\}^2 \\
&\textit{subject to } \frac{dx}{dt} = F(x,u)
\end{aligned} \tag{8}$$

This will provide the control values for each time value. The first obtained control value is implemented and the remaining are discarded. This procedure is repeated until the implemented and the first obtained control value are the same.

Pyomo [21], was used for the calculations. Here the differential equations are converted to a Nonlinear Program (NLP) using the orthogonal collocation method [22]. The Lagrange-Radau quadrature with three collocation points is used and 10 finite elements are used to solve the optimal control problems. The resulting nonlinear optimization problem was solved using the solvers IPOPT [23], and confirmed as global solutions with BARON [24] The calculations are repeated until there is no difference between the implemented and the first obtained control values The Utopia point is when $p_i = p_i^*$ for all i.

Effect of singularities (Limit Point (LP) and Branch Point (BP)) on MNLMPC

If the minimization of the variables p_1, p_2 l result in the values M_1 and M_2 the resulting optimization problem will be

$$\begin{aligned} &\min \int_0^{t_f}((p_1(t)dt)-M_1)^2+\int_0^{t_f}((p_2(t)dt)-M_2)^2=\int_0^{t_f}\mathrm{P}(x,t)dt \\ &\textit{subject to } \frac{dx_i}{dt}=g_i(x,u) \end{aligned} \tag{9}$$

Taking λ_i is the lagrangian multiplier., the Euler Lagrange equation(costate equations) will be

$$\frac{d\lambda_i}{dt}=-(\frac{\partial P}{\partial x_i}+\lambda_i g_i) \tag{10}$$

the derivative of the objective function will yield

$$\frac{d}{dx_i}((p_1-M_1)^2+(p_2-M_2)^2)=2(p_1-M_1)\frac{d}{dx_i}(p_1-M_1)+2(p_2-M_2)\frac{d}{dx_i}(p_2-M_2) \tag{11}$$

At the Utopia point both (p_1-M_1) and (p_2-M_2) are zero. Hence

$$\frac{d}{dx_i}((p_1-M_1)^2+(p_2-M_2)^2)=0 \tag{12}$$

The co-state equation in optimal control is

$$\begin{aligned} &\frac{d}{dt}(\lambda_i)=-\frac{d}{dx_i}((p_1-M_1)^2+(p_2-M_2)^2)-g_x\lambda_i \\ &\lambda_i(t_f)=0 \end{aligned} \tag{13}$$

λ_i is the lagrangian multiplier. The first term in this equation is 0 and hence

$$\begin{aligned} &\frac{d}{dt}(\lambda_i)=-g_x\lambda_i \\ &\lambda_i(t_f)=0 \end{aligned} \tag{14}$$

If $\frac{dx}{dt}=g(x,u)$ yields a limit or a branch point, g_x is singular.

This implies that there are two different vectors-values for $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i)>0$ and $\frac{d}{dt}(\lambda_i)<0$. In between there is a vector $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i)=0$. This coupled with the boundary condition $\lambda_i(t_f)=0$ will lead to $[\lambda_i]=0$ which will cause the problem to become unconstrained. The only solution for the unconstrained problem is the Utopia solution. This is illustrated numerically in the next few sections.

4. Results and Discussion

Bifurcation Analysis of Ecological Model

In the first case, d3 was the bifurcation parameter while k was the bifurcation parameter

in the second case. Figures 1 and 2 show the bifurcation diagrams.

In both instances, there are branch points from which two different branches originate

The derivatives of f_1, f_2, f_3 with respect to the variables x_1, x_2, x_3 are

$$\frac{\partial f_1}{\partial x_1} = r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1}; \quad \frac{\partial f_1}{\partial x_2} = -\frac{\partial \phi_{12}}{\partial x_2}; \quad \frac{\partial f_1}{\partial x_3} = 0$$
$$\frac{\partial f_2}{\partial x_1} = e_2 \frac{\partial \phi_{12}}{\partial x_1}; \quad \frac{\partial f_2}{\partial x_2} = e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2; \quad \frac{\partial f_2}{\partial x_3} = -\frac{\partial \phi_{23}}{\partial x_3} \tag{15}$$
$$\frac{\partial f_3}{\partial x_1} = 0; \quad \frac{\partial f_3}{\partial x_2} = e_3 \frac{\partial \phi_{23}}{\partial x_2}; \quad \frac{\partial f_3}{\partial x_3} = e_3 \frac{\partial \phi_{23}}{\partial x_3} - d_3$$

The Jacobian matrix is

$$J = \begin{pmatrix} (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1}) & (-\frac{\partial \phi_{12}}{\partial x_2}) & 0 \\ (e_2 \frac{\partial \phi_{12}}{\partial x_1}) & (e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2) & (-\frac{\partial \phi_{23}}{\partial x_3}) \\ 0 & e_3(\frac{\partial \phi_{23}}{\partial x_2}) & (e_3 \frac{\partial \phi_{23}}{\partial x_3} - d_3) \end{pmatrix} \tag{16}$$

The determinant is given by

$$\det(J) = (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1})(e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2)(e_3 \frac{\partial \phi_{23}}{\partial x_3} - d_3) + e_3(\frac{\partial \phi_{23}}{\partial x_2})(\frac{\partial \phi_{23}}{\partial x_3}) + (\frac{\partial \phi_{12}}{\partial x_2})(e_2 \frac{\partial \phi_{12}}{\partial x_1})e_3(\frac{\partial \phi_{23}}{\partial x_2})$$
$$= (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1})(e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2)(e_3 \frac{\partial \phi_{23}}{\partial x_3} - d_3) + \{e_3((\frac{\partial \phi_{23}}{\partial x_3}) + (\frac{\partial \phi_{12}}{\partial x_2})(e_2 \frac{\partial \phi_{12}}{\partial x_1})e_3\}(\frac{\partial \phi_{23}}{\partial x_2}) \tag{17}$$
$$= (r - \frac{2rx_1}{k} - \frac{\partial \phi_{12}}{\partial x_1})(e_2 \frac{\partial \phi_{12}}{\partial x_2} - \frac{\partial \phi_{23}}{\partial x_2} - d_2)(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) + \{e_3((\frac{\partial \phi_{23}}{\partial x_3}) + (\frac{\partial \phi_{12}}{\partial x_2})(e_2 \frac{\partial \phi_{12}}{\partial x_1})e_3\}(\frac{a_3}{b_3 + x_2} - \frac{a_3 x_2}{b_3 + x_2})x_3$$

For steady-state to be attained $\frac{dx_3}{dt} = f_3 = 0$ This implies that $(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) = 0;$ and/or $x_3 = 0$.

If both these terms are 0 det(J)=0 and the Jacobian matrix is singular. This is the only singular point because $(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) = 0;$ and det (J)=0 t will imply that $x_3 = 0;$ and det (J)=0 and $x_3 = 0;$ will imply that $(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) = 0;$

This singular point will be a branch point with 2 branches that $(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) = 0;$ and $x_3 = 0$.

Computational Validation

Case 1 d_3 bifurcation parameter

At the branch point (singular point) $x_1 = 138.529412; x_2 = 180.631105; x_3 = 0; d_3 = 0.054110$

b_3 = 250; $e_3 = 1.29; a_3 = 0.1$ b3 = 250; the value of $(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) = 0$.

Case 2 K is a bifurcation parameter

At the branch point (singular point) $x_1 = 138.529412;$ $x_2 = 112.359551;$ $x_3 = 0;$ k=277.431490

d_3 = 0.04; b_3 = 250; $e_3 = 1.29; a_3 = 0.1; x_2 = 112.359551$

b3 = 250; the value of $(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) = 0$.

In both cases, at the singular point, $x_3 = 0$ and. $(\frac{e_3 a_3 x_2}{b_3 + x_2} - d_3) = 0$.

Figures 1 and 2 show the bifurcation diagrams when d3 and K are the bifurcation parameters.

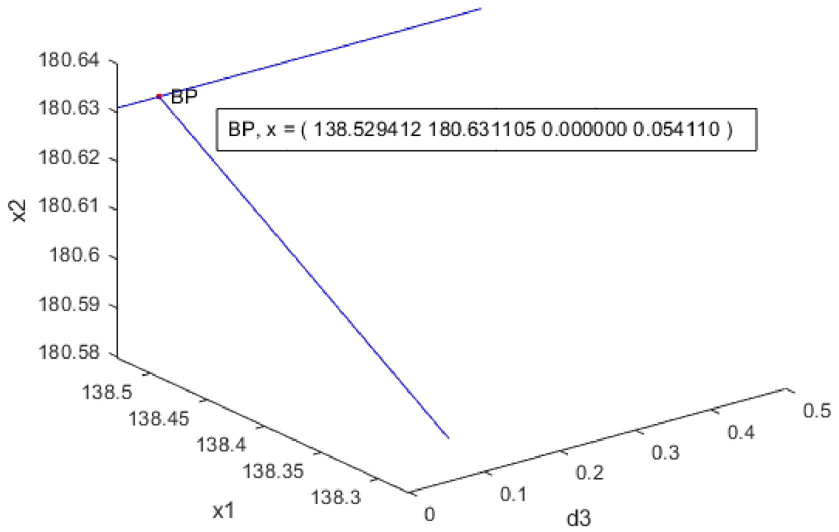

Figure 1. *Bifurcation diagram with d3 as bifurcation parameter.*

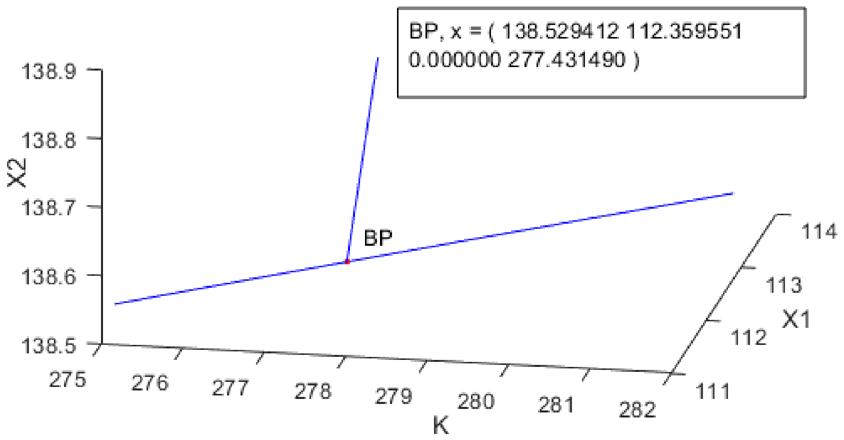

Figure 2. *Bifurcation diagram with K as bifurcation parameter.*

MNLMPC of the ecological model

The averaged fisher index (FI) is given by

$$
\begin{aligned}
FI &= \frac{1}{t_f}\int_0^{t_f} \frac{(a(t))^2}{(v(t))^4} dt \\
v(t) &= \sqrt{\sum_{i=1}^{3} (\frac{dx_i}{dt})^2} = \sqrt{\sum_{i=1}^{3} (f_i)^2} \\
a(t) &= \frac{1}{v(t)} \sum_{i=1}^{3} ((\frac{d^2 x_i}{dt^2})(\frac{dx_i}{dt})) \\
(\frac{d^2 x_i}{dt}) &= \frac{df_i}{dt} = \sum_{j=1}^{3} (\frac{df_i}{dx_j})(f_j) ...i = 1,2,3
\end{aligned}
\tag{18}
$$

The expressions of the functions f_i and the derivatives $\frac{df_i}{dx_j}$ are provided in equation sets 2 and 3. Both d3 and k were used as control variables. Both $\sum_0^{t_f} x_3$ and the Fisher index (FI) were maximized individually. The maximization of $\sum_0^{t_f} x_3$ resulted in a value of 716.534 while the maximization of FI resulted in a value of 3.965e-05. For the multiobjective nonlinear model predictive calculations, the function minimized was $(\sum_0^{t_f} x_3 - 716.534?)^2 + (FI - 3.965e-05)^2$ subject to the equation set 2. The resulting objective function value obtained was the utopia point 0. The multiobjective nonlinear model control variables obtained were d3 = 0.0274 and k 680.00.

Figures 3-6 show the profiles for the MNLMPC calculations.

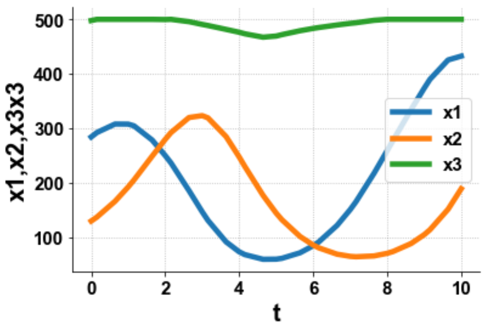

Figure 3. *X1, X2, X3 profiles for MNLMPC calculations.*

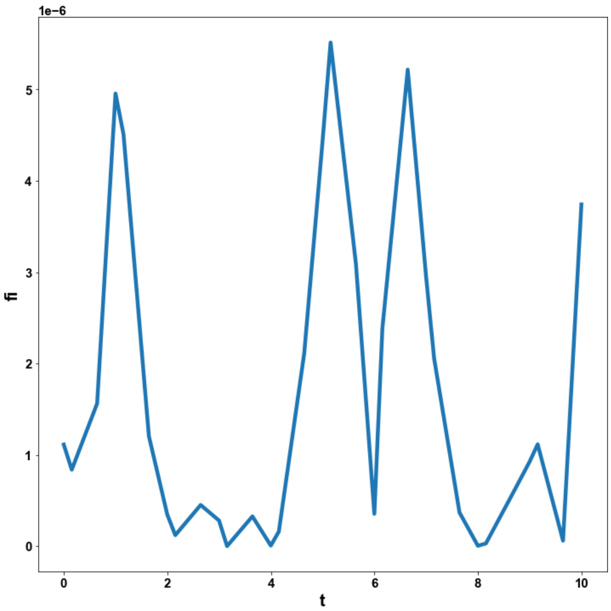

Figure 4. *FI versus t.*

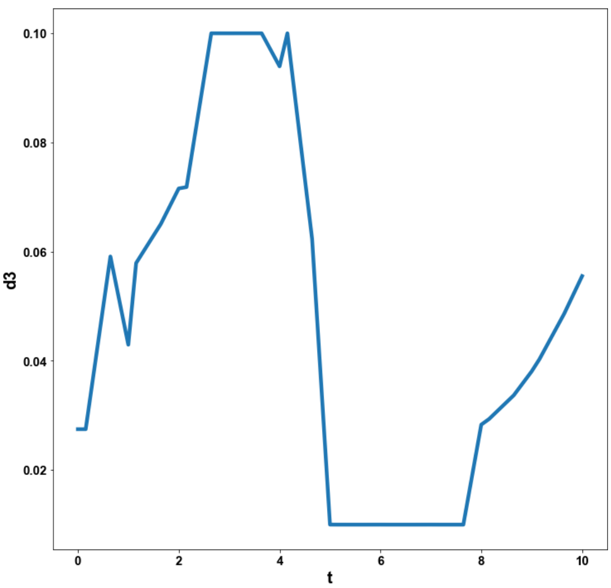

Figure 5. *d3 versus t.*

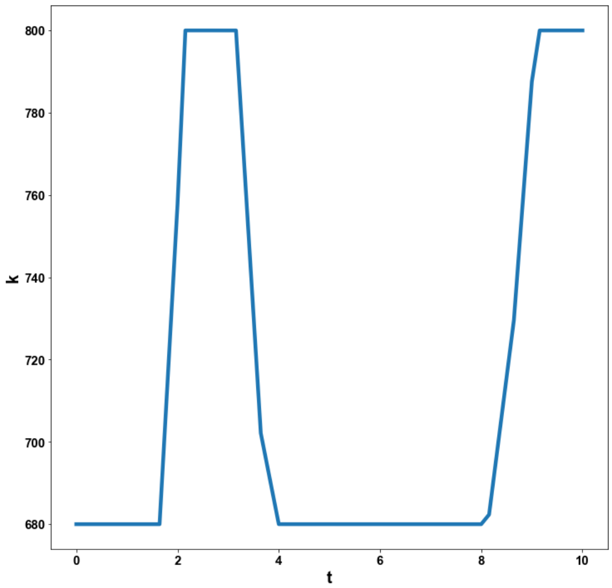

Figure 6. *K versus t.*

5. Conclusions

The main conclusions of this work are that one can attain the utopia point in MNLMPC calculations because of the branch points that occur in the ecosystem model and the prese nce of the branch point can be proved analytically. The use of rigorous mathematics to enhance sustainability will be a significant step in encouraging sustainable development. The main practical implication of this work is that the strategies developed here can be used by all researchers involved in maximizing sustainability The future work will involve using these mathematical strategies to other ecosystem models and food chain models which will be a huge step in developing strategies to address problems involving nutrition.

Nomenclature

x_1	Prey Population
x_2	Predator Population
x_3	Super Predator Population
r	Prey Growrh Rate
K	Predator Growth rate
a_2, a_3	The Maximum Predation rate of Predator and Super Predator
b_2, b_3	Half Saturation Constant of Predator and Super Predator
d_2, d_3	Death of Predator and Super Predator
FI	Fisher Index
BP	Branch Point
LP	Limit Point
MNLMPC	Multi-objective Nonlinear Model Predictive Control

Author Contributions

Lakshmi Narayan Sridhar is the sole author. The author read and approved the final manuscript.

Data Availability

All data used is presented in the paper.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Ahmad, N., Derrible, S., Eason, T., Cabezas, H., 2016. Using fisher information to track stability in multivariate systems. R. Soc. Open Sci. 3 160582.

[2] Cabezas, H., Fath, B. D., 2002. Towards a theory of sustainable systems. Fluid Phase Equilib. 194–197, 3–14.

[3] Cabezas, H., Pawlowski, C. W., Mayer, A. L., Hoagland, N. T., 2003. Sustainability: Ecological, social, economic, technological, and systems perspectives. Clean Tech. Environ. Policy 5, 167–180.

[4] Cabezas, H., Pawlowski, C. W., Mayer, A. L., Hoagland, N. T., 2005. Simulated experiments with complex sustainable systems: Ecology and technology. Resour. Conserv. Recycl. 44, 279–291.

[5] Cabezas, H., Pawlowski, C. W., Mayer, A. L., Hoagland, N. T., 2005. Sustainable systems theory: Ecological and other aspects. J. Cleaner Prod. 13, 455–467.

[6] Cabezas, H., Whitmore, H. W., Pawlowski, C. W., Mayer, A. L., 2007. On the sustainability of an integral model system with industrial, ecological, and macroeconomic components. Resour. Conserv. Recycl. 50, 122–129.

[7] Doshi, R., Diwekar, U., Benavides, P. T., Yenkie, K. M., Cabezas, H., 2015. Maximizing sustainability of ecosystem model through socio-economic policies derived from multivariable optimal control theory. Clean Techn. Environ. Policy 17, 1573–1583.

[8] Fath, B. D., Cabezas, H., 2004. Exergy and fisher information as ecological indices. Ecol. Modell. 174, 25–35.

[9] Fath, B. D., Cabezas, H., Pawlowski, C. W., 2003. Regime changes in ecological systems: An information theory approach. J. Theor. Biol. 222, 517–530.

[10] Fisher, R. A., 1922. On the mathematical foundations of theoretical statistics. Philos. Trans. R. Soc. A 222, 309–368.

[11] Shastri, Y., U. Diwekar, Sustainable ecosystem management using optimal control theory: Part 1 (deterministic systems), Journal of Theoretical Biology, Volume 241, Issue 3, 2006, Pages 506-521, ISSN 0022-5193 https://doi.org/10.1016/j.jtbi.2005.12.014

[12] Soraya Rawlings, E., J. C. Barrera-Martinez, Vicente Rico-Ramirez, Fisher information calculation in a complex ecological model: An optimal control-based approach, Ecological Modelling, Volume 416, 2020, 108845, ISSN 0304-3800, https://doi.org/10.1016/j.ecolmodel.2019.108845

[13] Dhooge, A., Govearts, W., and Kuznetsov, A. Y., MATCONT: A Matlab package for numerical bifurcation analysis of ODEs, ACM transactions on Mathematical software 29(2) pp. 141-164, 2003.

[14] Dhooge, A., W. Govaerts; Y. A. Kuznetsov, W. Mestrom, and A. M. Riet, CL_MATCONT; A continuation toolbox in Matlab, 2004.

[15] Kuznetsov, Y. A. Elements of applied bifurcation theory. Springer, NY, 1998.

[16] Kuznetsov, Y. A.(2009). Five lectures on numerical bifurcation analysis, Utrecht University, NL., 2009.

[17] Govaerts, w. J. F., Numerical Methods for Bifurcations of Dynamical Equilibria, SIAM, 2000.

[18] Flores-Tlacuahuac, A. Pilar Morales and Martin Riveral Toledo; Multiobjective Nonlinear model predictive control of a class of chemical reactors. I & EC research; 5891-5899, 2012.

[19] Sridhar, Lakshmi N., (2019) Multiobjective optimization and nonlinear model predictive control of the continuous fermentation process involving Saccharomyces Cerevisiae, Biofuels, https://doi.org/10.1080/17597269.2019.1674000

[20] Miettinen, Kaisa, M., Nonlinear Multiobjective Optimization; Kluwers international series, 1999.

[21] Hart, William E., Carl D. Laird, Jean-Paul Watson, David L. Woodruff, Gabriel A. Hackebeil, Bethany L. Nicholson, and John D. Siirola. *Pyomo – Optimization Modeling in Python*. Second Edition. Vol. 67. Springer, 2017.

[22] Biegler, L. T. An overview of simultaneous strategies for dynamic optimization. *Chem. Eng. Process. Process Intensif.* 46, 1043–105 (2007).

[23] Wächter, A., Biegler, L. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math. Program.* 106, 25–57 (2006). https://doi.org/10.1007/s10107-004-0559-y

[24] Tawarmalani, M. and N. V. Sahinidis, A polyhedral branch-and-cut approach to global optimization, Mathematical Programming, 103(2), 225-249, 200.