International Journal of Statistical Distributions and Applications
2024; 10(2): 25-37
http://www.sciencepublishinggroup.com/j/ijsda
doi: 10.11648/j.ijsd.20241002.12
ISSN: 2472-3487 (Print); ISSN: 2472-3509 (Online)

Three-parameters Gumbel Distribution: Formulation and Parameter Estimation

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To cite this article:
Otieno Kevin Okumu, Omondi Joseph Ouno, Anthony Nyutu Karanjah, Samuel Nganga Muthiga. (2024). Three-parameters Gumbel Distribution: Formulation and Parameter Estimation. *International Journal of Statistical Distributions and Applications, 10*(2), 25-37. https://doi.org/10.11648/j.ijsd.20241002.12

Received: 6 March 2024; **Accepted:** 3 April 2024; **Published:** 20 June 2024

Abstract: Modeling extreme value theories is really gaining interest in the world with scientist working to improve the flexibility of the distributions by adding parameter(s). Extreme value distributions are always described to include families of Gumbel, Weibull and Frechet distributions. Of the three distributions, Gumbel distribution is the most commonly used in the extreme value theory analysis. Existing literature has shown that the addition of parameter to a distribution makes it robust and/or more flexible hence the study intends to improve the existing two parameters Gumbel distribution using the Marshall and Olkin proposed method for introducing a new estimator/parameter to an existing distribution. The developed distribution will be important to the applications in some life time studies like high temperature, earthquakes, network designs, horse racing, queues in supermarket, insurance, winds, risk management, ozone concentration, flood, engineering and financial concepts. The parameters for the introduced distribution was estimated using Maximum Likelihood Estimation method. The introduced three parameters Gumbel distribution is a probability distribution function which can be used in modelling statistical data. The maximum likelihood estimates for the three parameters namely shape, location and dispersion are efficient, sufficient and consistent and this makes the function more flexible and better for application. The three parameters Gumbel distribution can be used in modeling and analysis of normal data, skewed data and extreme data since it will provide efficient, sufficient and consistent estimates.

Keywords: Gumbel Distribution, Maximum Likelihood Estimation, Marshall Olkins Method, Parameters, Three Parameters Gumbel Distribution

1. Introduction

Extreme value analysis is a branch of statistics dealing with the extreme deviations from the centre of probability distribution and it focuses on limiting distributions which are distinct from normal distribution. Extreme value studies originated majorly from the experts in astronomy who focused on analyzing the data observed from astronomical objects like comets, planets, moons, stars etc. The early papers on the extreme value theories focused both on methods of statistical analysis and on the application of the formulated extreme value distributions [3, 15].

Over past years, extreme value theory has indicated that the world is gaining a better understanding of the statistical modeling and analysis of the extreme value concepts. The understanding of the behaviour of extreme event cases is useful for understanding the whole behaviour of such cases both under the ordinary and extra-ordinary circumstances. Therefore,it is a mistake to separate the extreme events from the other events when it comes to modeling and analysis[5, 11].

Today, extreme point distributions have developed as one of the key statistical area for applied sciences. Analyzing extreme values therefore, requires parameter estimation and application of the probability of events that are more extreme than the previously experienced cases with the main goal of estimating the future expectations [2, 19]. Extreme value analysis provide a framework that assists for this type of research work that deals with extreme data sets. Gumbel distribution is not only widely used in various application in extreme value studies but also referred as the mother to the extreme value distributions (that is, Frechet and Weibull distribution types)[12, 19]. Not many research have been published on the extensive study of the Gumbel distribution even with its ability to fit data from many different areas of the extreme value observation like engineering, physics, climate among others.

In a statistical analysis Gumbel distribution, Exponential Gumbel distribution and Generalized Extreme Value distribution (three parameter Generalized Extreme Value distribution) were applied on two types of data sets. The results showed that Exponential Gumbel distribution could serve as an alternative to Generalized Extreme Value distribution because it gives narrower confidence intervals than the Generalized Extreme Value distribution. It was also noted that the Exponential Gumbel distribution produced the smallest Anderson-Darling statistical value compared to the Gumbel and Generalized Extreme Value distribution. This research recommended further studies on estimation methodology and analysis using Exponential Gumbel distribution [8, 12, 14]. This study also combined exponential and Gumbel distribution, it did not add a parameter to the univariate Gumbel distribution using Marshall Olkin proposed method.

Abdelaziz and Zoglat (2011), compared normal and Gumbel distribution after arguing that they are much alike in practical application in flood and engineering analysis. The researchers used the ratio of the Restricted Maximized Likelihood estimation as the test statistics under both normal and Gumbel (two parameter)distributions. Using Monte Carlo simulation, the probability of correct selection was compared with the asymptotic distribution results of the test statistics under the null hypothesis and it was found that ML can be used for differentiating between any two distributions of the scale and location parameter. This study showed Maximum Likelihood is the best estimation method for discriminating two or more distributions of the same family [1, 17]. This study compared the normal and Gumbel distribution, it did not add a parameter to the Gumbel distribution.

The research have found that adding a parameter to any existing distribution makes it more flexible and important for modeling and analyzing both simulated and real life data sets. This is because the newly introduced parameters in a distribution provides better estimates and makes it more robust and/or efficient than the baseline distributions. However, from the reviewed literature, it was realized that apart from combining Gumbel distribution with other distributions like exponential, gamma, geometric among others, no scholar have modeled a three parameters Gumbel distribution. For this study we wish to model a three parameters Gumbel distribution about which we consider three parameters (that is, shape, location and dispersion) using Marshall Olkin (1997) proposed method. Since the extreme value analysis address the extreme deviations from the centre of probability distribution and it focus on limiting distributions which are distinct from normal distribution. Extreme value distributions are always viewed to include families of Gumbel, Frechet and Weibull distributions. Of the three distributions, Gumbel distribution is frequently used in the extreme value theory analysis because majority of the authors refer to Gumbel distribution as the mother to the extreme value distributions from the fact that the Frechet and Weibull distributions can be transformed to Gumbel distribution by applying a simple transformation. Existing literature has shown that the addition of parameter to a distribution makes it robust and/or more flexible hence the study intends to improve the existing Gumbel distribution by making it more flexible through addition of shape parameter using Marshall and Olkin technique.

This research therefore intends to develop a new distribution called a three parameters Gumbel distribution to improve the flexibility of the already existing two parameter distribution. The new distribution will be developed by applying the Marshall Olkin method for adding a new parameter to an existing distribution. To estimate a parameters of the three parameters Gumbel distribution which we discuss in chapter three, this study intends to apply Maximum Likelihood Estimation method. This method of estimation was preferred over the other methods like Method of moments, Ordinary Least square, percentiles, Cramer-Von Mises etc because [3, 6, 18, 20] provide enough evidence supporting Maximum Likelihood Estimation as the best parameter estimation method since it provides better estimates for both small and large samples of data.

This research concentrates on three parameters namely; the location parameter, dispersion parameter and the shape parameter. The location parameter help in determining the shift of the distribution under study and as well tells us where the distribution is located/centered, the dispersion parameter helps in describing how the distribution is scattered around the center or simply how the distribution is spread and the shape parameter guide us on the shape of the distribution depending on the value of the shape parameter.

2. Methods

2.1. Marshall Olkin Method of Adding a Parameter

Marshall and Olkin (1997) proposed a procedure of introducing a new parameter to an existing distribution. This is because introducing a parameter to a well defined distribution is an honored procedure in that it helps in introducing a more flexible new family of the distribution for modeling various data types [4, 16]. A new parameter for the main objective of this study is to be added to the following baseline probability distribution which is a two parameter distribution with ω representing location of the variables and τ representing scale or dispersion parameter.

$$f(v) = \frac{1}{\tau} exp\big(-\frac{v-\omega}{\tau} - e^{-\frac{v-\omega}{\tau}} \big) \quad (1)$$

where $v \in \Re$ and $\omega > 0, \tau > 0$ are location and dispersion/scale parameters respectively.

Marshall Olkin (1997) began with a parent survival function $\bar{F}(v)$ and considered a family of survival function given by:

$$\bar{S}(v) = \frac{\delta \bar{F}(v)}{1 - \bar{\delta}\bar{F}(v)} = \frac{\delta \bar{F}(v)}{F(v) + \delta \bar{F}(v)}, \infty < v < \infty \quad (2)$$

where $\delta > 0$, $\bar{\delta} = 1 - \delta$ and $\bar{F}(v) = 1 - F(v)$. For $\delta = 1, \bar{S}(v) = \bar{F}(v)$.

The two primary properties of the Marshall Olkin family of distributions is that it has a stability property, that is if the method is put in application twice it returns back to the original distribution and the introduced distribution satisfies the property of the geometric extreme stability [4].

After introducing the new distribution, the corresponding cumulative distribution function (cdf) and probability density function (pdf) are respectively obtained as given in the following equations:

$$T(v) = \frac{F(v)}{1 - \bar{\delta}\bar{F}(v)} \quad (3)$$

and

$$t(v) = \frac{\delta f(v)}{\Big(1 - \bar{\delta}\bar{F}(v)\Big)^2} \quad (4)$$

2.2. Maximum Likelihood Estimation Method

This subsection discusses the method of Maximum Likelihood Estimation which was used in this study for the purpose of estimating parameters of the new three parameters Gumbel distribution to be modeled. According to [13, 20], maximum likelihood method can be used in many problems since it has a strong instinctual appeal and it yield a better estimator(s). This method of maximum likelihood is widely put in application because it is more precise especially when dealing with large samples since it yields a more efficient estimator(s) when the sample is large. This is an evidence from the literature review where we realized that Maximum Likelihood Estimation method gives the best parameter estimates compared to the other estimation methods like Minimum Distance Estimation, Method of Moments etc.

According to [10], if supposing we have say $\hat{Z}$ of P is a solution to the maximization problem given as

$$\hat{Z} = argMaxLn(Z : v_1, v_2, ..., v_k), \quad (5)$$

where, $v_1, ..., v_k$ represents the data observations, then under suitable regularity conditions, where the first order condition is given as

$$\frac{\partial Ln}{\partial Z}(Z : v_1, v_2, ...v_k) = -k + \frac{1}{\hat{Z}}(\sum_{i=1}^{k} V_i) \quad (6)$$

These conditions are generally called the likelihood or log-likelihood equations. The first derivative or gradient of a condition (log-likelihood) which is solved at point $\hat{Z}$ need to satisfies the following equation

$$\frac{\partial Ln(Z : v_1, v_2, ..., v_k)}{\partial Z} = \frac{\partial Ln(\hat{Z} : v_1, v_2, ..., v_k)}{\partial Z} = 0 \quad (7)$$

The log-likelihood equation that coincide to linear or non-linear system of P equations with P unknown parameters $Z_1, Z_2, ...Z_P$ with K observations is given by

$$\frac{\partial Ln(Z : v_1, ..., v_k)}{\partial Z} = \left(\frac{\partial Ln(Z : v_1, ..., v_k)}{\partial Z_1}\right) = ... = \left(\frac{\partial Ln(Z : v_1, ..., v_k)}{\partial Z_P}\right) = 0 \quad (8)$$

Maximum Likelihood Estimation is a recommended technique for many distributions because it uses the values of the distribution parameters that makes the data more likely than any other parameters. This is achieved by maximizing the likelihood function of the parameters given the data. Some good features of maximum likelihood estimators is that they are asymptotically unbiased since the bias tends to zero as the sample size increases and also they are asymptotically efficient since they achieve the Cramer-Rao lower bound which states that for any unbiased estimator of population parameter P it gives a lower estimate for the variance of an unbiased estimator, as sample size approaches ∞ and lastly they are asymptotically normal [9, 10].

3. Model Formulation and Parameter Estimation

3.1. Three Parameters Gumbel Distribution Formulation

The formulation of a three parameters Gumbel distribution is done using Marshal Olkin method as given in equation (2). Suppose we have a random variable V, then the cdf is given as follows;

$\bar{F}(v) = 1 - F(v)$ whereby $F(v) = \int_{v=0}^{k} f(v)dv$, where $f(v)$ is pdf of the random variable V.

Therefore, having that,

$$F(k) = \int_{v=0}^{k} f(v)dv = \frac{1}{\tau} exp\big(-\frac{v-\omega}{\tau} - e^{-\frac{v-\omega}{\tau}} \big) dv \quad (9)$$

where,

ω is the location parameter

τ is the scale/dispersion parameter

Using the laws of indices on the expression for $f(v)$ gives,

$$F(k) = \int_{v=0}^{k} exp\big(e^{-\frac{v-\omega}{\tau}}\big)\frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}dv \tag{10}$$

Now, by letting,

$$p = -e^{-\frac{v-\omega}{\tau}}$$

it results to,

$$dp = \frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}dv$$

Using this substitution in equation (10), gives,

$$F(k) = \int_{v=0}^{k} e^{p}dp = e^{p}$$

$$= exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big|_{v=0}^{k}$$

$$F(k) = exp\big(-e^{-\frac{k-\omega}{\tau}}\big) - exp\big(-e^{\frac{\omega}{\tau}}\big). \tag{11}$$

Therefore given that;

$$\bar{F}(v) = 1 - F(v)$$

$$\bar{F}(v) = 1 + exp\big(-e^{\frac{\omega}{\tau}}\big) - exp\big(-e^{-\frac{v-\omega}{\tau}}\big) \tag{12}$$

With $\bar{F}(v)$, the survival function for the random variable V as illustrated in equation (2) is given by:

$$\bar{S}(v) = \frac{\delta\big[1 + exp\big(-e^{\frac{\omega}{\tau}}\big) - exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big]}{1 - \Big\{(1-\delta)\big[1 + exp\big(-e^{\frac{\omega}{\tau}}\big) - exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big]\Big\}} \tag{13}$$

with the corresponding cumulative distribution function obtained from equation (3) as follows,

$$F(v) = \frac{exp\big(-e^{-\frac{v-\omega}{\tau}}\big) - exp\big(-e^{\frac{\omega}{\tau}}\big)}{1 - \Big\{(1-\delta)\big[1 + exp\big(-e^{\frac{\omega}{\tau}}\big) - exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big]\Big\}} \tag{14}$$

and a probability distribution function given as follows (as from equation (4),

$$f(v) = \frac{\frac{\delta}{\tau}exp\big(-\frac{v-\omega}{\tau} - e^{-\frac{v-\omega}{\tau}}\big)}{\Big[1 - \Big\{(1-\delta)\big\{1 + exp\big(-e^{\frac{\omega}{\tau}}\big) - exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\}\Big\}\Big]^2} \tag{15}$$

where δ is the introduced shape parameter.

To show that the expression in equation (15) is a pdf (that is $\int_{v=0}^{\infty} f(v)dv = 1$), we simplify the function by letting

$$q = exp(-e^{\frac{\omega}{\tau}}).$$

which makes the denominator of the function in equation (15) to become,

$$\Big[1 - (1-\delta)\big\{1 + q - exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\}\Big]^2,$$

from which it gives,

$$1 - \Big[1 - \big\{exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\} + q - \delta + \delta\big\{exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\} - q\delta\Big]^2,$$

$$\Big[\big\{exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\} - q + \delta - \delta\big\{exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\} + q\delta\Big]^2,$$

$$\Big[\big\{exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\}(1-\delta) + \delta + q\delta - q\Big]^2$$

Therefore, expression in equation (15)can be written as:

$$f(v) = \frac{\delta}{\tau}\left[\frac{e^{-\frac{v-\omega}{\tau}}exp\big(-e^{-\frac{v-\omega}{\tau}}\big)}{\big[exp\big(-e^{-\frac{v-\omega}{\tau}}\big)(1-\delta) + \delta + q\delta - q\big]^2}\right], \tag{16}$$

Hence having,

$$\int f(v)dv = \delta\int\frac{exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}}{\big[exp\big(-e^{-\frac{v-\omega}{\tau}}\big)(1-\delta) + \delta + q\delta - q\big]^2} \tag{17}$$

From equation (17), by letting

$$p = -e^{-\frac{v-\omega}{\tau}},$$

and,

$$dp = \frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}dv,$$

for ease of integration, it gives

$$\int f(v)dv = \delta \int \frac{e^p dp}{\left[e^p(1-\delta)+\delta+q\delta-q\right]^2} \tag{18}$$

By rearranging the denominator in equation (18), it gives,

$$\int f(v)dv = \delta \int \frac{e^p dp}{\left[(1-\delta)(e^p-q)+\delta\right]^2} \tag{19}$$

from which again by letting

$$z = (1-\delta)(e^p-q)+\delta$$

then,

$$dz = (1-\delta)e^p dp,$$

implying that

$$e^p dp = \frac{dz}{1-\delta}.$$

Therefore, from equation (19), it gives,

$$\int f(v)dv = \frac{\delta}{1-\delta}\int \frac{dz}{z^2} = \frac{\delta}{(\delta-1)z}$$

whose solution becomes,

$$\frac{\delta}{(\delta-1)z} = \frac{\delta}{(\delta-1)\left[(1-\delta)(e^p-q)+\delta\right]},$$

and when by taking the integration from $v = 0$ to k, it gives

$$= \frac{\delta}{(\delta-1)\left[(1-\delta)(e^p-q)+\delta\right]}\Bigg|_{v=0}^{k} \tag{20}$$

The foregoing means that taking $v = \infty$ and also $v = 0$, gives for $v = \infty$, $e^p = exp\big(-e^{-\frac{v-\omega}{\tau}}\big) = 1$ and for $v = 0$, $e^p = exp\big(-e^{-\frac{v-\omega}{\tau}}\big) = exp\big(-e^{\frac{\omega}{\tau}}\big) = q$.

Hence, the value for the integral from equation (20) becomes:

$$\frac{\delta}{\delta-1}\Big[\frac{1}{(1-\delta)(1-q)+\delta} - \frac{1}{(1-\delta)(q-q)+\delta}\Big],$$

$$\frac{\delta}{\delta-1}\Big[\frac{1}{(1-\delta)(1-q)+\delta} - \frac{1}{\delta}\Big],$$

$$\frac{\delta}{\delta-1}\Big[\frac{\delta-1+q-q\delta}{\left[(1-\delta)(1-q)+\delta\right]\delta}\Big],$$

Since, $\delta-1+q-q\delta$ can be factored as $\delta-1+q(1-\delta) = 1(\delta-1)-q(\delta-1) = (1-q)(\delta-1)$. Also, $(1-\delta)(1-q)+\delta$ in the denominator can be simplified as $1-q-\delta+q\delta+\delta = 1-q+q\delta$, then it gives,

$$\frac{\delta}{\delta-1}\Big[\frac{(1-q)(\delta-1)}{\delta[1-q+q\delta]}\Big] = \frac{1-q}{1-q+q\delta}. \tag{21}$$

But, knowing that,

$$q = exp(-e^{\frac{\omega}{\tau}})$$

and since, $\omega > \tau$, then it follows that $q \to 0$. And therefore, equation (21) approaches 1, that is,

$$\frac{1-q}{1-q+q\delta} \to 1,$$

hence $f(v)$ is a pdf as required.

3.1.1. Expected Value of a Three Parameters Gumbel Distribution

The derivation of the expected value of a three parameters Gumbel distribution because the research cannot assume that the location parameter is the mean since the measures of location are mean, mode and median.

Given $f(v)$ as the probability distribution function, the expected value ($E(V)$) is obtained as;

$$E(v) = \int v f(v)dv$$

$$E(v) = \int_{-\infty}^{\infty} \frac{v\frac{\delta}{\tau}exp\big(-\frac{v-\omega}{\tau} - e^{-\frac{v-\omega}{\tau}}\big)dv}{\Big[1-\Big\{(1-\delta)\big\{1+exp\big(-e^{\frac{\omega}{\tau}}\big) - exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\}\Big\}\Big]^2} \tag{22}$$

$$E(v) = \delta\int_{-\infty}^{\infty} \frac{v.exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}dv}{\Big[1-\Big\{(1-\delta)\big\{1+exp\big(-e^{\frac{\omega}{\tau}}\big) - exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\}\Big\}\Big]^2} \tag{23}$$

If we let

$$z = e^{-\frac{v-\omega}{\tau}} \Longrightarrow ln(z) = -\frac{v-\omega}{\tau} = \frac{-v}{\tau} + \frac{\omega}{\tau} \Longrightarrow v = \omega - \tau ln(z)$$

then having that,

$$dz = -\frac{1}{\tau} e^{-\frac{v-\omega}{\tau}} dv$$

by substituting v, z, and dz in equation (23), gives

$$E(v) = -\delta \int \frac{(\omega - \tau ln(z) e^{-z} dz}{\left[1 - \left((1-\delta)(1 - e^{-z} + q)\right)\right]^2} \tag{24}$$

where

$$q = exp\big(- e^{\frac{\omega}{\tau}} \big)$$

Therefore,

$$E(v) = -\delta \int \frac{\omega e^{-z} dz}{\left[1 - \left((1-\delta)(1 - e^{-z} + q)\right)\right]^2} + (\delta\tau) \int \frac{ln(z) e^{-z} dz}{\left[1 - \left((1-\delta)(1 - e^{-z} + q)\right)\right]^2} \tag{25}$$

The function $E(v)$ have two terms which are integrated as follows, for the first term from equation (25), that is;

$$-\delta \int \frac{\omega e^{-z} dz}{\left[1 - \left((1-\delta)(1 - e^{-z} + q)\right)\right]^2}$$

letting $p = (1-\delta)(e^{-z} - q) + \delta$ then, $dp = (1-\delta)(-e^{-z}) = (\delta - 1)e^{-z} dz$ implying that $e^{-z} dz = \frac{dp}{\delta - 1}$. This gives,

$$-\delta \int \frac{\omega e^{-z} dz}{\left[1 - \left((1-\delta)(1 - e^{-z} + k)\right)\right]^2} = \frac{-(\delta\omega)}{\delta - 1} \int \frac{dp}{p^2} = \frac{-(\delta\omega)}{\delta - 1} \big[\frac{1}{p}\big] = \frac{-(\delta\omega)}{\delta - 1} \big[\frac{1}{1-\delta)(e^{-z} - q) + \delta}\big]$$

which is given as follows after re-substituting z

$$= \frac{-(\delta\omega)}{\delta - 1} \big[\frac{1}{(1-\delta)(exp^{-e^{-\frac{v-\omega}{\tau}}} - q) + \delta}\big]\Big|_{-\infty}^{\infty}$$

this gives the following result after taking the integral with the limits of v from $-\infty$ to ∞

$$\begin{aligned} &= \frac{-(\delta\omega)}{\delta - 1} \big[\frac{1}{(1-\delta)(1-q) + \delta} - \frac{1}{-q(1-\delta) + \delta}\big] \\ &= \frac{-(\delta\omega)}{\delta - 1} \left[\frac{-q(1-\delta) + \delta - [(1-\delta)(1-q) + \delta]}{[(1-\delta)(1-q) + \delta][-q(1-\delta) + \delta]}\right] \end{aligned}$$

the integral for the first term finally becomes

$$\frac{\delta\omega}{(\delta q + \delta - q)(1 - q + \delta q)} \tag{26}$$

For the second terms in equation (25), which is given as,

$$(\delta\tau) \int \frac{ln(z) e^{-z} dz}{\left[1 - \left((1-\delta)(1 - e^{-z} + q)\right)\right]^2} = (\delta\tau) \int \frac{ln(z) e^{-z}}{[(1-\delta)(e^{-z} - q) + \delta]^2}\Big]$$

Using integration by parts to integrate the function. And knowing that by applying the integration by parts, then $wp - \int pdw$ is applied to solve the function. Letting $w = ln(z)$ giving $dw = \frac{dz}{z}$. Also if we let

$$dp = \frac{e^{-z} dz}{\left[(1-\delta)(e^{-z} - q) + \delta\right]^2}$$

implying that

$$p = \int \frac{e^{-z} dz}{\left[(1-\delta)(e^{-z} - q) + \delta\right]^2} = \frac{-1}{(\delta - 1)[(1-\delta)(e^{-z} - q) + \delta]}$$

this gives the integral result as

$$\int \frac{ln(z)e^{-z}}{[(1-\delta)(e^{-z}-q)+\delta]^2}\Big] = wp - \int pdw$$

$$= \frac{-ln(z)}{(\delta-1)[(1-\delta)(e^{-z}-q)+\delta]} + \frac{1}{\delta-1}\int \frac{dz}{z[(1-\delta)(e^{-z}-q)+\delta]} \tag{27}$$

Equation (27) shows that the function still requires integration by parts for the second term, that is

$$\frac{1}{\delta-1}\int \frac{dz}{z[(1-\delta)(e^{-z}-q)+\delta]}.$$

By letting

$$w = \frac{1}{[(1-\delta)(e^{-z}-q)+\delta]}$$

implying that

$$dw = -[(1-\delta)(e^{-z}-q)+\delta]^{-2}(\delta-1)e^{-z}.$$

Also, by taking $dp = \frac{dz}{z}$ we have $p = ln(z)$. Therefore, it gives

$$\frac{1}{\delta-1}\Big[\frac{ln(z)}{[(1-\delta)(e^{-z}-q)+\delta]} + (\delta-1)\int \frac{ln(z)e^{-z}}{[(1-\delta)(e^{-z}-q)+\delta]^2}\Big] \tag{28}$$

Replacing equation (28) in equation (27), it gives 0 implying that the function is indeterminate and therefore it is undefined. The expected value is therefore given as;

$$\begin{aligned} E(v) &= \frac{\delta\omega}{(\delta q+\delta-q)(1-q+\delta q)} \\ &= \frac{\delta\omega}{[\delta exp(-e^{\frac{\omega}{\tau}})+\delta-exp(-e^{\frac{\omega}{\tau}})][1-exp(-e^{\frac{\omega}{\tau}})+\delta exp(-e^{\frac{\omega}{\tau}})]} \end{aligned} \tag{29}$$

3.1.2. The Variance of a Three Parameters Gumbel Distribution

For the variance of the three parameters Gumbel distribution, recalling that, $Var(v) = E(v^2) - [E(v)]^2$. The formular for $E(v)$ is already obtained in subsection 3.1.1. The $E(v^2)$ is then obtained as follows.

$$E(v^2) = \delta\int_{-\infty}^{\infty} \frac{v^2.exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}dv}{\Big[1-\Big\{(1-\delta)\big\{1+exp\big(-e^{\frac{\omega}{\tau}}\big)-exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\big\}\Big\}\Big]^2}$$

$$E(v^2) = \delta\int_{-\infty}^{\infty} \frac{v^2.exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}dv}{\Big[(1-\delta)\big(exp^{(-e^{\frac{-v-\omega}{\tau}})}-q\big)+\delta\Big]^2}$$

In subsection 3.1.1, having let $q = exp\big(-e^{\frac{\omega}{\tau}}\big)$ and $z = e^{-\frac{v-\omega}{\tau}} \Longrightarrow v = \omega - \tau ln(z)$ hence, $v^2 = (\omega - \tau ln(z))^2 \Longrightarrow v^2 = \omega^2 - 2\omega\tau ln(z) + \tau^2[ln(z)]^2$. From z function, it gives

$$dz = -\frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}dv$$

Therefore, obtaining,

$$\begin{aligned} E(v^2) &= \delta\int_{-\infty}^{\infty} \frac{v^2.exp\big(-e^{-\frac{v-\omega}{\tau}}\big)\frac{1}{\tau}e^{-\frac{v-\omega}{\tau}}}{\Big[(1-\delta)\big(exp^{(-e^{\frac{-v-\omega}{\tau}})}-q\big)+\delta\Big]^2} \\ &= -\delta\int \frac{\big(\omega^2-2\omega\tau ln(z)+\tau^2[ln(z)]^2\big)e^{-z}dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} \end{aligned}$$

$$E(v^2) = -\delta\omega^2 \int \frac{e^{-z}dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} + 2\delta\omega\tau \int \frac{e^{-z}ln(z)dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} - \delta\tau^2 \int \frac{e^{-z}[ln(z)]^2dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} \tag{30}$$

As evidence in subsection 3.1.1, the first term of $E(v^2)$ from equation (30) which is

$$\int \frac{e^{-z}dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} = \frac{1}{(\delta q+\delta-q)(1-q+q\delta)}.$$

Also, the integral for the second term, that is

$$\delta\omega\tau \int \frac{e^{-z}ln(z)dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} = 0$$

as evidence from equations (27) and (28).

This means that we are only remaining with the integral solution of the third term of $E(v^2)$ in equation (30) which is given as

$$-\delta\tau^2 \int \frac{e^{-z}[ln(z)]^2dz}{[(1-\delta)(e^{-z}-q)+\delta]^2}$$

to integrate this function, we apply integration by part ($wp - \int pdw$).

By letting $w = (ln(z))^2 \Longrightarrow dw = \frac{2ln(z)dz}{z}$. Again, if we let,

$$dp = \frac{e^{-z}dz}{[(1-\delta)(e^{-z}-q)+\delta]^2}$$

implying that

$$p = \int \frac{e^{-z}dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} = \frac{-1}{(\delta-1)[(1-\delta)(e^{-z}-q)+\delta]}$$

Therefore, having the $wp - \int pdw$ for the function given as;

$$\int \frac{e^{-z}[ln(z)]^2dz}{[(1-\delta)(e^{-z}-q)+\delta]^2} = \frac{-[ln(z)]^2}{(\delta-1)[(1-\delta)(e^{-z}-q)+\delta]} + \frac{2}{\delta-1}\int \frac{ln(z)dz}{z[(1-\delta)(e^{-z}-q)+\delta]} \tag{31}$$

Using integration by parts again in the second term of equation (31), by letting

$$w = \frac{1}{[(1-\delta)(e^{-z}-q)+\delta]} \Longrightarrow dw = -[(1-\delta)(e^{-z}-q)+\delta]^2(\delta-1)e^{-z}.$$

And, also letting

$$dp = \frac{ln(z)dz}{z} \Longrightarrow p = \frac{[ln(z)]^2}{2}.$$

This gives,

$$\frac{2}{\delta-1}\int \frac{ln(z)dz}{z[(1-\delta)(e^{-z}-q)+\delta]} = \frac{2}{\delta-1}\Big[\frac{[ln(z)]^2}{2[(1-\delta)(e^{-z}-q)+\delta]} + \frac{\delta-1}{2}\int \frac{[ln(z)]^2e^{-z}}{[(1-\delta)(e^{-z}-q)+\delta]^2}\Big] \tag{32}$$

Replacing equation (32) in equation (31), gives 0 implying that the function is indeterminate and therefore it is undefined. The $E(v^2)$ is therefore given as;

$$E(v^2) = \frac{\delta\omega^2}{(\delta q+\delta-q)(1-q+q\delta)} \tag{33}$$

Therefore, $Var(v) = E(v^2) - [E(v)]^2$ is obtained from equations (29) and (33) as folows;

$$\begin{aligned}
Var(v) &= \frac{\delta\omega^2}{(\delta q+\delta-q)(1-q+q\delta)} - \Big[\frac{\delta\omega}{(\delta q+\delta-q)(1-q+\delta q)}\Big]^2 \\
&= \frac{\delta\omega^2}{(\delta q+\delta-q)(1-q+q\delta)} - \frac{\delta^2\omega^2}{[(\delta q+\delta-q)(1-q+\delta q)]^2} \\
&= \frac{\delta\omega^2[(\delta q+\delta-q)(1-q+\delta q)]-\delta^2\omega^2}{[(\delta q+\delta-q)(1-q+\delta q)]^2} \\
&= \frac{\delta\omega^2\big([(\delta q+\delta-q)(1-q+\delta q)]-\delta\big)}{[(\delta q+\delta-q)(1-q+\delta q)]^2}
\end{aligned}$$

Replacing q gives us,

$$Var(v) = \frac{\delta\omega^2\big([(\delta exp(-e^{\frac{\omega}{\tau}})+\delta-exp(-e^{\frac{\omega}{\tau}}))(1-exp(-e^{\frac{\omega}{\tau}})+\delta exp(-e^{\frac{\omega}{\tau}}))]-\delta\big)}{[(\delta exp(-e^{\frac{\omega}{\tau}})+\delta-exp(-e^{\frac{\omega}{\tau}}))(1-exp(-e^{\frac{\omega}{\tau}})+\delta exp(-e^{\frac{\omega}{\tau}}))]^2} \tag{34}$$

3.2. Maximum Likelihood Estimation for the Parameters

This subsection shows the process of estimating each of the three parameters in a three parameters distribution namely; the location parameters (ω), scale/dispersion parameter (τ) and the shape parameter (δ) using maximum likelihood estimation method. The maximum likelihood estimation method involves three steps, (that is getting the likelihood function, the log of the likelihood function and the derivative with respect to the required parameter).

Considering the probability distribution function given in equation (15), its likelihood function is given follows,

$$\begin{aligned}
R &= \prod_{i=1}^{k} f(v_i) \\
&= \prod_{i=1}^{k} \frac{\delta}{\tau}\Bigg[\frac{exp\big(-\frac{v_i-\omega}{\tau}-e^{\frac{-v_i-\omega}{\tau}}\big)}{\big[1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)\big]^2}\Bigg]
\end{aligned}$$

where R is the symbol used in this study to represent likelihood function

$$R(v_i;\delta,\omega,\tau) = \big(\frac{\delta}{\tau}\big)^k \frac{exp\sum_{i=1}^{k}\big(-\frac{v_i-\omega}{\tau}-e^{\frac{-v_i-\omega}{\tau}}\big)}{\Big[exp\sum_{i=1}^{k} ln\big[1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)\big]\Big]^2} \tag{35}$$

The likelihood function R, can be expressed with a variable together with parameters to be estimated as shown in equation (35), and its log-likelihood function which maximizes the parameters becomes

$$\begin{aligned}
ln(R) = k\big\{ln\big(\frac{\delta}{\tau}\big)\big\} + \sum_{i=1}^{k}\Big[-\big(\frac{v_i-\omega}{\tau}\big)-e^{-\big(\frac{v_i-\omega}{\tau}\big)}\Big] \\
-2\sum_{i=1}^{k} ln\big[1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)\big]
\end{aligned} \tag{36}$$

3.2.1. Parameters Estimation for δ

With the log likelihood function, the estimate for δ is obtained by performing the partial derivative for the log likelihood function with respect to δ and equating the result to zero (that is, by computing $\frac{\partial LnR}{\partial\delta}$). This is done by considering on the terms in equation (36) containing δ because of the fact that terms independent of δ definitely give a derivative result of zero. In equation (36), the first term (that is, $k\big\{ln\big(\frac{\delta}{\tau}\big)\big\}$. Differentiating with respect to δ as follows:

$$d = k\big\{ln\big(\frac{\delta}{\tau}\big)\big\}$$

then,

$$\frac{\partial d}{\partial\delta} = \frac{\tau}{\delta}.\frac{k}{\tau} = \frac{k}{\delta} \tag{37}$$

And letting c to represent the second term with δ then,

$$c=-2\sum_{i=1}^{k} ln\big[1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)\big],$$

which is differentiated with respect to δ as follows:
by letting,

$$\begin{aligned} p &= 1-(1-\delta)\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big] \\ &= 1-1\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big]+\delta\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big] \end{aligned}$$

Therefore, it gives,

$$\frac{dp}{\partial\delta}=\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big]$$

Now differentiating c with respect to p, gives,

$$\frac{dc}{dp}=\sum_{i=1}^{k}(lnp)=2\sum_{i=1}^{k}\frac{1}{p} \tag{38}$$

This leads to,

$$\frac{\partial c}{\partial\delta}=2\sum_{i=1}^{k}\frac{\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big]}{p} \tag{39}$$

Thus having the estimate of δ obtained from equations(37) and (39) as,

$$\hat{\delta}=\frac{\partial ln(L)}{\partial\delta}=\frac{k}{\delta}-2\sum_{i=1}^{k}\frac{\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big]}{p}=0, \tag{40}$$

where,

$$p=1-(1-\delta)\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big]$$

3.2.2. Parameters Estimation for ω

The process of estimating the location parameter ω is shown by the use of maximum likelihood estimate. By using the log likelihood function in equation (36) and working out the partial derivative while equating to zero, and proceed as follows: Assuming that the term in the equation without ω have their partial derivatives equals to zero hence leaving the study to work out partial derivatives of only terms having ω as follows;

For the second term whereby there is

$$\sum_{i=1}^{k}\Big[-\big(\frac{v_i-\omega}{\tau}\big)-e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big]$$

letting

$$r=-\big(\frac{v_i-\omega}{\tau}\big)=-\frac{v_i}{\tau}+\frac{\omega}{\tau}$$

This results to,

$$\frac{\partial r}{\partial\omega}=\frac{1}{\tau} \tag{41}$$

Also, by letting

$$s=e^{-\left(\frac{v_i-\omega}{\tau}\right)}$$

and,

$$r=-\big(\frac{v_i-\omega}{\tau}\big)$$

then

$$\frac{\partial s}{\partial\omega}=\frac{1}{\tau}e^{r}=\frac{1}{\tau}e^{-\left(\frac{v_i-\omega}{\tau}\right)} \tag{42}$$

Combining $\frac{\partial r}{\partial\omega}$, $\frac{\partial s}{\partial\omega}$, having the derivative of the second term with respect to ω given as;

$$\sum_{i=1}^{k}\Big[\frac{1}{\tau}-\frac{1}{\tau}e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big]$$
$$\frac{1}{\tau}\sum_{i=1}^{k}\Big[1-e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big] \tag{43}$$

The third term, letting,

$$t=-2\sum_{i=1}^{k} ln\big[1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)\big]$$

and by letting,

$$p = 1-(1-\delta)\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big],$$

gives,

$$\frac{\partial t}{\partial \omega} = 2(1-\delta)\sum_{i=1}^{k}\left[\frac{exp\Big(-e^{\left(-\frac{v_i-\omega}{\tau}\right)}\Big).e^{\left(-\frac{v_i-\omega}{\tau}\right)}-e^{\frac{\omega}{\tau}}.exp(-e^{\frac{\omega}{\tau}})}{p\tau}\right] \tag{44}$$

Combining $\frac{\partial r}{\partial \omega}$, $\frac{\partial s}{\partial \omega}$ and $\frac{\partial t}{\partial \omega}$ as given in equations (41), (42) and (44) respectively gives the maximum likelihood estimate for ω as;

$$\begin{aligned}\hat{\omega} &= \frac{\partial ln(L)}{\partial \omega}\\ &= \frac{1}{\tau}\sum_{i=1}^{k}\Big[1-e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big]+2(1-\delta)\sum_{i=1}^{k}\left[\frac{exp\Big(-e^{\left(-\frac{v_i-\omega}{\tau}\right)}\Big).e^{\left(-\frac{v_i-\omega}{\tau}\right)}-e^{\frac{\omega}{\tau}}.exp(-e^{\frac{\omega}{\tau}})}{p\tau}\right]=0\end{aligned} \tag{45}$$

where,

$$p = 1-(1-\delta)\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big]$$

3.2.3. Parameters Estimation for τ

The maximum estimator for τ is obtained by partially differentiating the log likelihood function given below with respect to τ and equating to zero as follows.

$$ln(R) = k\{ln(\frac{\delta}{\tau})\}+\sum_{i=1}^{k}\Big[-(\frac{v_i-\omega}{\tau})-e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big]-2\sum_{i=1}^{k}ln\big[1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)\big]$$

The first term of the likelihood function is differentiated using chain rule as follows, let,

$$p = k\{ln(\frac{\delta}{\tau})\}$$

therefore,

$$\frac{\partial p}{\partial \tau} = \frac{-k}{\tau} \tag{46}$$

For the second term of the log likelihood function, letting

$$m = -(\frac{v_i-\omega}{\tau}),$$

$$\frac{\partial m}{\partial \tau} = (v_i-\omega)\tau^{-1} = (v_i-\omega)\tau^{-2} = \frac{v_i-\omega}{\tau^2} \tag{47}$$

Again by letting $r = e^{-\left(\frac{v_i-\omega}{\tau}\right)}$, and first dealing with the term in the parenthesis, letting this term be $m = -\left(\frac{v_i-\omega}{\tau}\right)$, it gives,

$$\frac{\partial r}{\partial \tau} = (e^r).\frac{v_i-\omega}{\tau^2} = \frac{v_i-\omega}{\tau^2}.e^{-\left(\frac{v_i-\omega}{\tau}\right)} \tag{48}$$

Therefore, the derivative of the second term with respect to τ becomes,

$$\begin{aligned}&\sum_{i=1}^{k}\Big[\frac{v_i-\omega}{\tau^2}-\frac{v_i-\omega}{\tau^2}.e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big]\\ &\frac{1}{\tau^2}\sum_{i=1}^{k}\Big[v_i-\omega\Big]\Big[1-e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big]\end{aligned} \tag{49}$$

The third term of the function is,

$$t = -2\sum_{i=1}^{k} ln\big[1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)\big],$$

about which letting,

$$p = 1-(1-\delta)\big(1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big)$$

The function for t has two terms with parameter τ which are differentiated with respect to τ as follows.
For,

$$-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big),$$

letting

$$h = -e^{\frac{-v_i-\omega}{\tau}},$$

and,

$$j = \frac{-v_i-\omega}{\tau},$$

then,

$$\frac{dh}{d\tau} = \frac{d}{dj}.\frac{dj}{d\tau} = -e^t.\frac{v_i-\omega}{\tau^2} = -e^{\frac{-v_i-\omega}{\tau}}.\frac{v_i-\omega}{\tau^2}$$

Hence,

$$\begin{aligned}\frac{d}{dh}.\frac{dh}{d\tau} &= -e^h.-e^{\frac{-v_i-\omega}{\tau}}.\frac{v_i-\omega}{\tau^2}\\ &= exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big).e^{\frac{-v_i-\omega}{\tau}}.\frac{v_i-\omega}{\tau^2}\end{aligned} \tag{50}$$

The last term in the t function is $exp(-e^{\frac{\omega}{\tau}})$ for which is written by letting $h = -e^{\frac{\omega}{\tau}}$ and the power is written as $j = \frac{\omega}{\tau}$.
From which,

$$\frac{dh}{d\tau} = \frac{d}{dj}.\frac{dj}{d\tau} = -e^t.-\frac{\omega}{\tau^2} = \frac{\omega}{\tau^2}e^{\frac{\omega}{\tau}},$$

Hence,

$$\frac{d}{dh}.\frac{dh}{d\tau} = e^h.\frac{\omega}{\tau^2}e^{\frac{\omega}{\tau}} = \frac{\omega}{\tau^2}exp(-e^{\frac{\omega}{\tau}})e^{\frac{\omega}{\tau}} \tag{51}$$

Using equations (50) and (51), the derivative for the t function with respect to τ becomes,

$$\frac{\partial p}{\partial \tau} = -(1-\delta)\Big[\frac{v_i-\omega}{\tau^2}exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big).e^{\frac{-v_i-\omega}{\tau}} + \frac{\omega}{\tau^2}exp(-e^{\frac{\omega}{\tau}})e^{\frac{\omega}{\tau}}\Big]$$

Because of equation (38), the final derivative for the third term in the log likelihood function with respect to τ is

$$\frac{\partial t}{\partial \tau} = -2(1-\delta)\sum_{i=1}^{k}\frac{(v_i-\omega)exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big).e^{\frac{-v_i-\omega}{\tau}} + \omega.exp(-e^{\frac{\omega}{\tau}})e^{\frac{\omega}{\tau}}}{\tau^2 p} = Q \tag{52}$$

where,

$$p = 1-(1-\delta)\big[1-exp\big(-e^{\frac{-v_i-\omega}{\tau}}\big)+exp(-e^{\frac{\omega}{\tau}})\big].$$

Therefore, the estimator for τ is given as follows from equations (46), (49) and (52) above

$$\hat{\tau} = \frac{\partial ln(L)}{\partial \tau} = \frac{-k}{\tau} + \frac{1}{\tau^2}\sum_{i=1}^{k}\Big[v_i-\omega\Big]\Big[1-e^{-\left(\frac{v_i-\omega}{\tau}\right)}\Big] + Q = 0 \tag{53}$$

where Q is defined in equation (52).

4. Conclusion

Introducing a new parameter to an existing distribution makes it more flexible and robust for application. In this research, a new parameter called the shape parameter was introduced to the existing two parameter Gumbel distribution. The introduced three parameters Gumbel distribution is a probability distribution function which can be used in modelling statistical data since it is more flexible. The maximum likelihood estimates for the three parameters namely shape (δ), location (ω) and dispersion (τ) are efficient, sufficient and consistent and this makes the function more flexible and better for application.

5. Recommendation

The three parameters Gumbel distribution can be used in modeling and analysis of normal data, skewed data and extreme data since it will provide efficient, sufficient and consistent estimates. For the purpose of future improvement on a three parameters Gumbel distribution, future researches can investigate the behaviour of its location parameter.

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Conflicts of Interest

The authors declare no conflict of interest.

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