**American Journal of Applied Mathematics**
2026; 14(5): 348-358
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ijdsa.20261405.18

# Backward One-Step Block Hybrid Numerical Method for Solving Second-Order Initial Value Problems

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## To cite this article:

Olabode Bola Titilayo, Abidemi Afeez, Momoh Adelegan Lukuman, Fagbohun Oluwadamilola Moses. (2026). Backward One-Step Block Hybrid Numerical Method for Solving Second-Order Initial Value Problems. *American Journal of Applied Mathematics, 14*(5), 348-358. https://doi.org/10.11648/j.ijdsa.20261405.18

**Received:** 18 August 2026; **Accepted:** 18 August 2026 **Published:** 28 September 2026

**Abstract:** This study develops a backward hybrid block method for solving oscillatory second-order initial value problems (IVPs), which arise frequently in engineering and the physical sciences. Existing hybrid methods often struggle with stiff-oscillatory systems, lacking the robustness needed for accurate and stable solutions. To overcome this limitation, the proposed method incorporates backward-step information to significantly improve both accuracy and stability. The derivation employs multistep collocation and interpolation techniques, with the Chebyshev polynomial of the first kind serving as the basis function. This choice of basis function is particularly effective for approximating oscillatory behavior. Unknown parameters are efficiently determined using Gaussian elimination. The resulting continuous scheme is then evaluated at selected points to obtain the discrete block method. A detailed theoretical analysis establishes the method's order, error constant, consistency, and zero-stability, collectively confirming its convergence. To validate its performance, the method is applied to standard oscillatory test problems. Numerical results demonstrate that the proposed method achieves superior accuracy compared to existing methods in the literature. The findings confirm that this backward-step hybrid block method offers a robust, reliable, and computationally efficient solution for oscillatory IVPs. Its accuracy and stability make it a valuable tool for researchers and practitioners dealing with oscillatory problems in engineering and the physical sciences.

**Keywords:** Linear Multi-step Method, Hybrid Points, Chebyshev Polynomial, Oscillatory, Zero-Stability

## 1. Introduction

This study present a numerical solution to the general second-order ordinary differential equations expressed as follows:

$$y''(x) = f(x, y, y'), \quad y(0) = y_0, \quad y'(0) = y'_0 \tag{1}$$

where f is a real value function that is continuous Oscillatory differential equations are essential for modeling phenomena featuring periodic or wave-like solutions, Despite this, their slow-changing phases and amplitudes provide specific challenges in practical application of mathematics. When eigenvalues cluster close to the imaginary axis, which results in oscillatory solutions, traditional methods for tracking solutions over extended cycles have difficulty with accuracy. From oscillatory differential equations that describe various real-world systems like the cooling of a body, where factors like environmental fluctuations and medium conditions influence oscillatory temperature changes around an equilibrium to others. [1] Traditional approaches often reduce such equations to equivalent first-order systems; however, this strategy is computationally expensive, increases system dimensionality, and frequently introduces significant phase lag and amplitude errors over long integration intervals. [2, 3] To circumvent these limitations, recent research has focused on direct solvers, particularly block hybrid methods and collocation techniques.

For instance,[4, 5] developed continuous double-hybrid point and 3-step y-function hybrid approaches, establishing their consistency and zero-stability, while [6, 7] introduced a six-step block method demonstrating superior accuracy on standard test problems.

More recently, [9, 10] proposed a multi-derivative hybrid block technique with adaptive step-size control for higher-order IVPs. Despite these advancements, such methods are predominantly tailored for non-stiff or mildly oscillatory systems and do not explicitly address the simultaneous presence of stiffness and high-frequency oscillations. Conversely, the family of Backward Differentiation Formulas (BDF), pioneered by [11] and extensively analyzed by [12], has long been the gold standard for purely stiff problems due to their exceptional A-stability which is designed for stability in the complex left-half plane and fails to capture the oscillatory dynamics where eigenvalues lie near the imaginary axis; consequently, standard BDF schemes introduce unwanted numerical dissipation that artificially damps amplitudes and induces phase lag in oscillatory solutions. The literature thus reveals a conspicuous disconnect: hybrid and collocation methods prioritize oscillatory accuracy but lack the robust implicit stability of backward formulations, while the BDF community has largely overlooked the critical need for phase-preserving and amplitude-preserving properties in oscillatory regimes.This study addresses this critical gap by proposing a novel linear multistep hybrid formula with backward features that retains the favorable convergence and stability characteristics of classical BDF methods while explicitly minimizing phase-lag and amplification errors for oscillatory solutions. The proposed method is implemented in a block mode and eliminate the need for complex adaptive strategies, and we rigorously establish its zero-stability, consistency, and convergence, thereby offering a computationally efficient and highly accurate alternative for engineering and physical science applications where modeling dynamic, repetitive behaviors around equilibrium points is paramount.

## 2. Mathematical Formulation

Let the exact solution $y(x)$ of the second-order initial value problem of ordinary differential Equation (1) be approximated by a chebyshev polynomial of the first kind

$$y(x) = \sum_{j=0}^{(p+q)-1} a_j T_j(x), \tag{2}$$

where $a_j \in \mathbb{R}$ are unknown coefficients and $T_j(x)$ are continuously differentiable terms of Chebyshev polynomial and $p, q$ are the number of interpolating points and collocating points respectively, the first, second, and third derivative of (2) gives the following:

$$y'(x) = \sum_{j=1}^{(p+q)-1} a_j T_j'(x), \tag{3}$$

$$y''(x) = \sum_{j=2}^{(p+q)-1} a_j T_j''(x), \tag{4}$$

and

$$y'''(x) = \sum_{j=3}^{(p+q)-1} a_j T_j'''(x). \tag{5}$$

Equating (1) and (2) yields the differential system

$$f(x, y, y') = \sum_{j=4}^{(p+q)-1} a_j T_j''(x). \tag{6}$$

Evaluating (2) and (3) at $x_n$ and collocating (4) and (5) at $x_{n+j}$ where $j = \frac{-1}{2}\left(\frac{1}{2}\right)1$.

$$\begin{aligned}
y_n &= a_0 + a_1x_n + a_2(-1+2x_n^2) + a_3(-3x_n + 4x_n^3) + a_4(1 - 8x_n^2 + 8x_n^4) \\
&\quad + a_5(5x_n - 20x_n^3 + 16x_n^5) + a_6(-1 + 18x_n^2 - 48x_n^4 + 32x_n^6) \\
&\quad + a_7(-7x_n + 56x_n^3 - 112x_n^5 + 64x_n^7) + a_8(1 - 32x_n^2 + 160x_n^4 - 256x_n^6 + 128x_n^8) \\
&\quad + a_9(9x_n - 120x_n^3 + 432x_n^5 - 576x_n^7 + 256x_n^9),
\end{aligned} \tag{7}$$

$$\begin{aligned}
y_n' &= a_1 + 4a_2x_n + a_3(-3 + 12x_n^2) + a_4(-16x_n + 32x_n^3) + a_5(5 - 60x_n^2 + 80x_n^4) \\
&\quad + a_6(36x_n - 192x_n^3 + 192x_n^5) + a_7(-7 + 168x_n^2 - 560x_n^4 + 448x_n^6) \\
&\quad + a_8(-64x_n + 640x_n^3 - 1536x_n^5 + 1024x_n^7) \\
&\quad + a_9(9 - 360x_n^2 + 2160x_n^4 - 4032x_n^6 + 2304x_n^8),
\end{aligned} \tag{8}$$

$$\begin{aligned}
f_{n-\frac{1}{2}} &= 4a_2 + 24a_3\left(-\frac{h}{2}+x_n\right) + a_4\left(-16+96\left(-\frac{h}{2}+x_n\right)^2\right) \\
&\quad + a_5\left(-120\left(-\frac{h}{2}+x_n\right)+320\left(-\frac{h}{2}+x_n\right)^3\right) + a_6\left(36-576\left(-\frac{h}{2}+x_n\right)^2+960\left(-\frac{h}{2}+x_n\right)^4\right) \\
&\quad + a_7\left(336\left(-\frac{h}{2}+x_n\right)-2240\left(-\frac{h}{2}+x_n\right)^3+2688\left(-\frac{h}{2}+x_n\right)^5\right) \\
&\quad + a_8\left(-64+1920\left(-\frac{h}{2}+x_n\right)^2-7680\left(-\frac{h}{2}+x_n\right)^4+7168\left(-\frac{h}{2}+x_n\right)^6\right) \\
&\quad + a_9\left(-720\left(-\frac{h}{2}+x_n\right)+8640\left(-\frac{h}{2}+x_n\right)^3-24192\left(-\frac{h}{2}+x_n\right)^5+18432\left(-\frac{h}{2}+x_n\right)^7\right), \qquad (9)
\end{aligned}$$

$$\begin{aligned}
f_n &= 4a_2 + 24a_3x_n + a_4(-16+96x_n^2) + a_5(-120x_n+320x_n^3) \\
&\quad + a_6(36-576x_n^2+960x_n^4) + a_7(336x_n-2240x_n^3+2688x_n^5) \\
&\quad + a_8(-64+1920x_n^2-7680x_n^4+7168x_n^6) \\
&\quad + a_9(-720x_n+8640x_n^3-24192x_n^5+18432x_n^7), \qquad (10)
\end{aligned}$$

$$\begin{aligned}
f_{n+\frac{1}{2}} = {} & 4a_2 + 24a_3\left(\tfrac{h}{2}+x_n\right) + a_4\left(-16+96\left(\tfrac{h}{2}+x_n\right)^2\right) + a_5\left(-120\left(\tfrac{h}{2}+x_n\right)+320\left(\tfrac{h}{2}+x_n\right)^3\right) \\
& +a_6\left(36-576\left(\tfrac{h}{2}+x_n\right)^2+960\left(\tfrac{h}{2}+x_n\right)^4\right) + a_7\left(336\left(\tfrac{h}{2}+x_n\right)-2240\left(\tfrac{h}{2}+x_n\right)^3+2688\left(\tfrac{h}{2}+x_n\right)^5\right) \\
& +a_8\left(-64+1920\left(\tfrac{h}{2}+x_n\right)^2-7680\left(\tfrac{h}{2}+x_n\right)^4+7168\left(\tfrac{h}{2}+x_n\right)^6\right) \\
& +a_9\left(-720\left(\tfrac{h}{2}+x_n\right)+8640\left(\tfrac{h}{2}+x_n\right)^3-24192\left(\tfrac{h}{2}+x_n\right)^5+18432\left(\tfrac{h}{2}+x_n\right)^7\right), \qquad (11)
\end{aligned}$$

$$\begin{aligned}
f_{n+1} &= 4a_2 + 24a_3(h+x_n) + a_4\left(-16+96(h+x_n)^2\right) + a_5\left(-120(h+x_n)+320(h+x_n)^3\right) \\
&\quad + a_6\left(36-576(h+x_n)^2+960(h+x_n)^4\right) + a_7\left(336(h+x_n)-2240(h+x_n)^3+2688(h+x_n)^5\right) \\
&\quad + a_8\left(-64+1920(h+x_n)^2-7680(h+x_n)^4+7168(h+x_n)^6\right) \\
&\quad + a_9\left(-720(h+x_n)+8640(h+x_n)^3-24192(h+x_n)^5+18432(h+x_n)^7\right), \qquad (12)
\end{aligned}$$

$$\begin{aligned}
g_{n-\frac{1}{2}} &= 24a_3 + 192a_4\left(-\frac{h}{2}+x_n\right) + a_5\left(-120+960\left(-\frac{h}{2}+x_n\right)^2\right) \\
&\quad + a_6\left(-1152\left(-\frac{h}{2}+x_n\right)+3840\left(-\frac{h}{2}+x_n\right)^3\right) + a_7\left(336-6720\left(-\frac{h}{2}+x_n\right)^2+13440\left(-\frac{h}{2}+x_n\right)^4\right) \\
&\quad + a_8\left(3840\left(-\frac{h}{2}+x_n\right)-30720\left(-\frac{h}{2}+x_n\right)^3+43008\left(-\frac{h}{2}+x_n\right)^5\right) \\
&\quad + a_9\left(-720+25920\left(-\frac{h}{2}+x_n\right)^2-120960\left(-\frac{h}{2}+x_n\right)^4+129024\left(-\frac{h}{2}+x_n\right)^6\right), \qquad (13)
\end{aligned}$$

$$\begin{aligned}
g_n &= 24a_3 + 192a_4x_n + a_5(-120+960x_n^2) + a_6(-1152x_n+3840x_n^3) \\
&\quad + a_7(336-6720x_n^2+13440x_n^4) + a_8(3840x_n-30720x_n^3+43008x_n^5) \\
&\quad + a_9(-720+25920x_n^2-120960x_n^4+129024x_n^6), \qquad (14)
\end{aligned}$$

$$
\begin{aligned}
g_{n+\frac{1}{2}} &= 24a_3 + 192a_4\left(\frac{h}{2}+x_n\right) + a_5\left(-120 + 960\left(\frac{h}{2}+x_n\right)^2\right) \\
&+ a_6\left(-1152\left(\frac{h}{2}+x_n\right) + 3840\left(\frac{h}{2}+x_n\right)^3\right) + a_7\left(336 - 6720\left(\frac{h}{2}+x_n\right)^2 + 13440\left(\frac{h}{2}+x_n\right)^4\right) \\
&+ a_8\left(3840\left(\frac{h}{2}+x_n\right) - 30720\left(\frac{h}{2}+x_n\right)^3 + 43008\left(\frac{h}{2}+x_n\right)^5\right) \\
&+ a_9\left(-720 + 25920\left(\frac{h}{2}+x_n\right)^2 - 120960\left(\frac{h}{2}+x_n\right)^4 + 129024\left(\frac{h}{2}+x_n\right)^6\right),
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
g_{n+1} &= 24a_3 + 192a_4(h+x_n) + a_5(-120 + 960(h+x_n)^2) \\
&+ a_6(-1152(h+x_n) + 3840(h+x_n)^3) + a_7(336 - 6720(h+x_n)^2 + 13440(h+x_n)^4) \\
&+ a_8(3840(h+x_n) - 30720(h+x_n)^3 + 43008(h+x_n)^5) \\
&+ a_9(-720 + 25920(h+x_n)^2 - 120960(h+x_n)^4 + 129024(h+x_n)^6).
\end{aligned}
\tag{16}
$$

Note:

$$g_{n+1} = f(x, y, y') \tag{17}$$

Solving this equations simultaneously, yields the $a_j$'s parameters and using the scaling function gives the continuous one-step scheme

$$
\begin{aligned}
y(t) = \alpha_0(t)y_n + \alpha_1(t)y'_n + h^2(\beta_{-\frac{1}{2}}(t)f_{n-\frac{1}{2}} + \beta_0(t)f_n + \\
\beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}} + \beta_1(t)f_1) + h^3(\gamma_{-\frac{1}{2}}(t)g_{n-\frac{1}{2}} \\
+\gamma_0(t)\gamma_n + \gamma_{\frac{1}{2}}(t)g_{n+\frac{1}{2}} + \gamma_1(t)g_1)
\end{aligned}
\tag{18}
$$

where $t = \frac{x-x_n}{h}, \alpha_0, \alpha_1, \beta_{-\frac{1}{2}}, \beta_0, \beta_{\frac{1}{2}}, \beta_1, \gamma_{-\frac{1}{2}}, g_{n-\frac{1}{2}}, \gamma_n, \gamma_{\frac{1}{2}}$ , and $\gamma_1$ are given below:

$$
\begin{aligned}
\alpha_1 &= 1 \\
\alpha_2 &= t
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\beta_{-\frac{1}{2}} &= \frac{14t^4}{81} - \frac{62t^5}{135} + \frac{20t^6}{81} + \frac{236t^7}{567} - \frac{104t^8}{189} + \frac{44t^9}{243} \\
\beta_0 &= \frac{t^2}{2} - \frac{11t^4}{12} + \frac{t^5}{10} + \frac{4t^6}{3} - \frac{8t^7}{21} - \frac{6t^8}{7} + \frac{4t^9}{9} \\
\beta_{\frac{1}{2}} &= \frac{2t^4}{3} + \frac{2t^5}{5} - \frac{4t^6}{3} - \frac{4t^7}{21} + \frac{8t^8}{7} - \frac{4t^9}{9} \\
\beta_1 &= \frac{25t^4}{324} - \frac{11t^5}{270} - \frac{20t^6}{81} + \frac{88t^7}{567} + \frac{50t^8}{189} - \frac{44t^9}{243} \\
\gamma_{-\frac{1}{2}} &= \frac{t^4}{54} - \frac{2t^5}{45} + \frac{t^6}{135} + \frac{2t^7}{27} - \frac{5t^8}{63} + \frac{2t^9}{81} \\
\gamma_0 &= \frac{t^3}{6} - \frac{t^4}{6} - \frac{7t^5}{20} + \frac{8t^6}{15} + \frac{4t^7}{21} - \frac{4t^8}{7} + \frac{2t^9}{9} \\
\gamma_{\frac{1}{2}} &= -\frac{1}{6}t^4 + \frac{7t^6}{15} - \frac{2t^7}{21} - \frac{3t^8}{7} + \frac{2t^9}{9} \\
\gamma_1 &= -\frac{1}{108}t^4 + \frac{t^5}{180} + \frac{4t^6}{135} - \frac{4t^7}{189} - \frac{2t^8}{63} + \frac{2t^9}{81}
\end{aligned}
\tag{20}
$$

Evaluating (18) and its derivative at point $t = t_{n+j}$, j= $\frac{-1}{2}, \frac{1}{2}$ and 1 yields the following discrete schemes

$$\begin{aligned} y_{n-\frac{1}{2}} &= y_n - \frac{1}{2}hy'n + \frac{h^2}{2177280}\Big(50638f{-}\frac{1}{2} + 183276f_0 + 32994f_{\frac{1}{2}} + 5252f_1\Big) \\ &+ \frac{h^3}{2177280}\Big(3756g_{-\frac{1}{2}} - 35127g_0 - 9774g_{\frac{1}{2}} - 645g_1\Big), \end{aligned} \tag{21}$$

$$\begin{aligned} y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'n + \frac{h^2}{2177280}\Big(3842f{-}\frac{1}{2} + 187704f_0 + 77166f_{\frac{1}{2}} + 3448f_1\Big) \\ &+ \frac{h^3}{2177280}\Big(438g_{-\frac{1}{2}} + 16335g_0 - 11124g_{\frac{1}{2}} - 399g_1\Big), \end{aligned} \tag{22}$$

$$\begin{aligned} y_{n+1} &= y_n + hy'n + \frac{h^2}{34020}\Big(256f{-}\frac{1}{2} + 7587f_0 + 8208f_{\frac{1}{2}} + 959f_1\Big) \\ &+ \frac{h^3}{34020}\Big(30g_{-\frac{1}{2}} + 837g_0 - 54g_{\frac{1}{2}} - 78g_1\Big), \end{aligned} \tag{23}$$

and additional first derivative schemes

$$\begin{aligned} y'_{n-\frac{1}{2}} &= y'_n \quad + \frac{h}{362880}\Big( - 68930f_{-\frac{1}{2}} - 84510f_0 - 24030f_{\frac{1}{2}} - 3970f_1\Big) \\ &+ \frac{h^2}{362880}\Big( - 3849g_{-\frac{1}{2}} + 22977g_0 + 7263g_{\frac{1}{2}} + 489g_1\Big), \end{aligned} \tag{24}$$

$$\begin{aligned} y'_{n+\frac{1}{2}} &= y'_n \quad + \frac{h}{40320}\Big(270f_{-\frac{1}{2}} + 9810f_0 + 9810f_{\frac{1}{2}} + 270f_1\Big) \\ &+ \frac{h^2}{40320}\Big(31g_{-\frac{1}{2}} + 1017g_0 - 1017g_{\frac{1}{2}} - 31g_1\Big), \end{aligned} \tag{25}$$

*and*

$$\begin{aligned} y'_{n+1} &= y'_n \quad + \frac{h}{11340}\Big(200f_{-\frac{1}{2}} + 3510f_0 + 5400f_{\frac{1}{2}} + 2230f_1\Big) \\ &+ \frac{h^2}{11340}\Big(24g_{-\frac{1}{2}} + 513g_0 + 432g_{\frac{1}{2}} - 129g_1\Big). \end{aligned} \tag{26}$$

Equation (24) to (26) constituted the block method, which can be written in block matrix as given in the expression below:

$$A_1 Y_{\mu+1} = A_0 Y_\mu + h(B_0 W_\mu) + h^2(B_1 L_\mu) + h^3(B_2 M_\mu) \tag{27}$$

where

$$Y_\mu = \Big(y_{n-\frac{1}{2}}, y_{n-1}, y_n, y'_{n-\frac{1}{2}}, y'_{n-1}, y'_n\Big)^{\mathrm{T}},$$

$$W_\mu = \Big(y_{n-\frac{1}{2}}, y_{n-1}, y_n, y'_{n-\frac{1}{2}}, y'_{n-1}, y'_n\Big)^{\mathrm{T}},$$

$$L_\mu = \Big(f_{n-2}, f_{n-1} f_{n-\frac{1}{2}}, f_n, f_{n+\frac{1}{2}}, f_{n+1}\Big)^{\mathrm{T}},$$

$$M_\mu = \Big(g_{n-2}, g_{n-1} g_{n-\frac{1}{2}}, g_n, g_{n+\frac{1}{2}}, g_{n+1}\Big)^{\mathrm{T}},$$

$$Y_{\mu+1} = \Big(y_{n-\frac{1}{2}}, y_{n+\frac{1}{2}}, y_{n+1}, y'_{n-\frac{1}{2}}, y'_{n+\frac{1}{2}}, y'_{n+1}\Big)^{\mathrm{T}},$$

with

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & \frac{64}{8505} & \frac{281}{1260} & \frac{76}{315} & \frac{137}{4860} \\ 0 & 0 & \frac{3617}{155520} & \frac{1697}{20160} & \frac{611}{40320} & \frac{1313}{544320} \\ 0 & 0 & \frac{1921}{1088640} & \frac{869}{10080} & \frac{1429}{40320} & \frac{431}{272160} \\ 0 & 0 & -\frac{6893}{36288} & -\frac{313}{1344} & -\frac{89}{1344} & -\frac{397}{36288} \\ 0 & 0 & \frac{3}{448} & \frac{109}{448} & \frac{109}{448} & \frac{3}{448} \\ 0 & 0 & \frac{10}{567} & \frac{13}{42} & \frac{10}{21} & \frac{223}{1134} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & \frac{1}{1134} & \frac{31}{1260} & -\frac{1}{630} & -\frac{13}{5670} \\ 0 & 0 & \frac{313}{181440} & -\frac{1301}{80640} & -\frac{181}{40320} & -\frac{43}{145152} \\ 0 & 0 & \frac{73}{362880} & \frac{121}{16128} & -\frac{103}{20160} & -\frac{19}{103680} \\ 0 & 0 & -\frac{1283}{120960} & \frac{851}{13440} & \frac{269}{13440} & \frac{163}{120960} \\ 0 & 0 & \frac{31}{40320} & \frac{113}{4480} & -\frac{113}{4480} & -\frac{31}{40320} \\ 0 & 0 & \frac{2}{945} & \frac{19}{420} & \frac{4}{105} & -\frac{43}{3780} \end{pmatrix}.$$

# 3. Analysis of the Properties of the Method

This section presents the analysis of the basic properties of the proposed method

### 3.1. Order and Error Constant of One-step Method

Expressing equations (23) in Taylor's series gives

$$L\Bigg[y[x_n+h] - y[x_n] - hy'[x_n] - \frac{1}{34020}h^2\left(256\,y''\left[x_n - \frac{1}{2}h\right] + 7587\,y''[x_n] + 8208\,y''\left[x_n + \frac{1}{2}h\right] + 959\,y''[x_n+h]\right) - \frac{1}{34020}h^3\left(30\,y'''\left[x_n - \frac{1}{2}h\right] + 837\,y'''[x_n] - 54\,y'''\left[x_n + \frac{1}{2}h\right] - 78\,y'''[x_n+h]\right)\Bigg]$$

where $y(x)$ represent an arbitrary test function that is continuously differentiable in the interval [a, b]. Taylor series expansion of $y(x+rh)$ and $y''(x+rh), r = 1, 2, ..., m$ about $x_n$ and collecting the like terms in $h$ and $y$ yields

$$L[y(x);h] = \bar{c}_0 y(x) + \bar{c}_1 hy'(x) + \bar{c}_2 h^2 y''(x) + ... + \bar{c}_p h^p y^p(x) \quad (28)$$

From the above $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = 0$ and $C_{10} = \frac{1}{90316800}$. Since $c_{p+2} = 10$, $p = 8$, that the order is 8 and the error constant is $\frac{1}{90316800}$. The same approach was used for (21) -(26) with the uniform order of 8 and the error constant presented below:

***Table 1.*** *Error Constant of One-Step Method.*

| methods | Error Constant |
|---|---|
| (21) | $\frac{89y^{(10)}(x_n)}{17340825600} + O(h^{11})$ |
| (22) | $\frac{103y^{(10)}(x_n)}{52022476800} + O(h^{11})$ |
| (23) | $\frac{y^{(10)}(x_n)}{90316800} + O(h^{11})$ |
| (24) | $-\frac{313y^{(10)}(x_n)}{13005619200} + O(h^{10})$ |
| (25) | $\frac{103y^{(10)}(x_n)}{13005619200} + O(h^{10})$ |
| (26) | $\frac{13y^{(10)}(x_n)}{406425600} + O(h^{10})$ |

### 3.2. Consistency of the Method

*Defination*: The linear multistep method is said to be consistent if it has order p $\geq 1$. It is obvious that the present method is consistent, because it is of the eight-order..

### 3.3. Zero Stability of the Block Method

Using the block matrix (27), the zero stability of the suggested method can be obtain using the characteristics equation. The zero-stability refers to stability of the difference system in (27) as $h \to 0$. As such, if $h \to 0$ in (27), we obtain

$$A_1 Y_{\mu+1} = A_0 Y_\mu \quad (29)$$

$$\left|\lambda A_1 - A_0\right| = \left|\lambda \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right| \quad (30)$$

$$det(\lambda A_1 - A_0) = 0 \quad (31)$$

The characteristic equation of (29) is given as $\rho(\lambda) = \lambda^4(\lambda - 1)^2$. Solving for the root of $\rho(\lambda) = 0$, gives $\lambda = 1, 1, 0, 0, 0, 0$.

Linear multistep methods (22-23) which by extension is in (29) is said to be zero-stable if the root of the polynomial $\rho(\lambda) = 0$ satisfies the strict root condition, i.e., if all its roots lie inside the unit circle, $|\lambda| < 1$ and with those on the boundary being simple.

### 3.4. Convergence of the Block Method

The necessary and sufficient condition for any method to be convergent is for it to be consistent and zero stable [15].Therefore,it has been proved above successfully that the new three off-step hybrid method with three off-step points is consistent and zero stable. Hence, the method is convergent [15].

### 3.5. Region of Absolute Stability of One-step Method

To analyse the stability region of the method is applied to the test equation

$$y'' + 2v\,y' + v^2 y = 0 \quad (32)$$

that is, the function and it's derivative are replaced as indicated below:

$$f_{n-\frac{1}{2}} \to -v^2 y_{\frac{-1}{2}} - 2v y'_{n-\frac{1}{2}}, \quad f_n \to -v^2 y_0 - 2v y'_0$$
$$f_{n+\frac{1}{2}} \to -v^2 y_{\frac{1}{2}} - 2v y'_{n+\frac{1}{2}}, \quad f_n \to -v^2 y_1 - 2v y'_1$$
$$g_{n-\frac{1}{2}} \to v^2(3y_{\frac{-1}{2}} + 2v y'_{n-\frac{1}{2}}), \quad g_0 \to v^2(3y'_0 + 2v y_0)$$
$$g_{n+\frac{1}{2}} \to v^2 3(y_{\frac{1}{2}} + 2v y'_{n+\frac{1}{2}}), \quad g_1 \to v^2(3y'_1 + 2v y_1).$$

This leads to a system of equations that can be written as

$$Ay_0 + By_n = 0 \qquad (33)$$

where

$$y_{n+1} = \left[y_{-\frac{1}{2}}, y_{\frac{1}{2}}, y_1, y'_{-\frac{1}{2}}, y'_{\frac{1}{2}}, y'_1\right]^{\mathrm{T}} \text{ and}$$
$$y_n = \left[y_{-\frac{1}{2}}, y_{-\frac{1}{2}}, y_0, y'_{-\frac{1}{2}}, y'_{-\frac{1}{2}}, y'_0\right]^{\mathrm{T}}$$

A, B, are 6 x 6 matrices whose entries are large to be reported in the paper.

The graph of the region of the absolute stability is shown below:

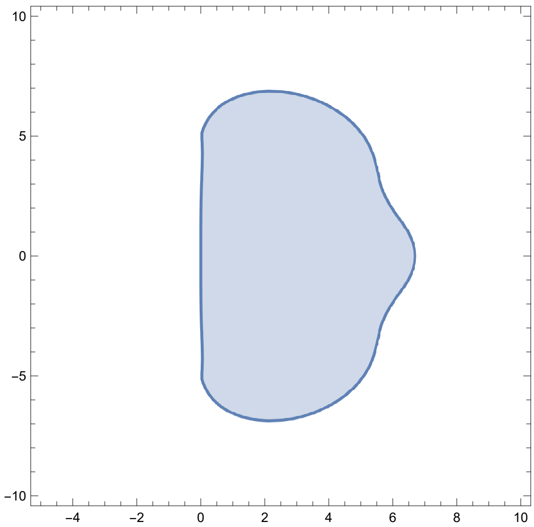


***Figure 1.*** *Region of absolute stability of One-step Method.*

The absolute stability region of one-step method is shown in the Figure 1, the shaded portion indicates the (z = h) values for which the method produces stable results, while the unshaded portion indicates (z = h) values for which it produces unstable results.

# 4. Numerical Experiments

To test the performance of the proposed method, three sample problems are considered as numerical examples for the numerical experiments. The accuracy are measured by calculating the absolute errors generated when method was applied to the sample problems.

*Example 1*

Considering a moderately stiff problem

$$y''(x) = x(y')^2, \qquad y(0) = 1, \qquad y(0) = \frac{1}{2}, \qquad h = 0.1$$

which appear in [16], whose theoretical solution is

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

The numerical results of problem 1 is presented in Table (2). The table covers the results from $x$ from 0 to 1.0 with step size $h = 0.1$. At $x = 0$, the error is exactly 0, as expected (initial condition satisfied perfectly). The errors grow gradually from about $6.25 \times 10^{-17}$ at $x = 0.1$ to approximately $6.76 \times 10^{-13}$ at $x = 1.0$. The $y$ values increase monotonically from 1 to about 1.54931, which matches the logarithmic growth of the exact solution. The very small errors (all in the range $10^{-17}$ to $10^{-13}$) demonstrate that the One-step method approximates this moderately stiff problem with exceptional accuracy.

***Table 2.*** *Solution of Problem 1 - One-step Method (Comparison with Literature).*

| $x$ | $y_{\text{approx}}$ | $y_{\text{Exact}}$ | Error (One-step) | Error in [16] | Error in [7] |
|---|---|---|---|---|---|
| 0.0 | 1. | 1. | 0. | – | – |
| 0.1 | 1.05004 | 1.05004 | $6.24501 \times 10^{-17}$ | $1.2270 \times 10^{-16}$ | $1.6898 \times 10^{-11}$ |
| 0.2 | 1.10034 | 1.10034 | $6.9389 \times 10^{-17}$ | $3.0480 \times 10^{-16}$ | $8.0559 \times 10^{-10}$ |
| 0.3 | 1.15114 | 1.15114 | $1.4155 \times 10^{-15}$ | $6.2070 \times 10^{-16}$ | $8.6157 \times 10^{-9}$ |
| 0.4 | 1.20273 | 1.20273 | $5.2181 \times 10^{-15}$ | $9.3350 \times 10^{-16}$ | $4.0241 \times 10^{-8}$ |
| 0.5 | 1.25541 | 1.25541 | $1.3878 \times 10^{-14}$ | $2.1353 \times 10^{-15}$ | $1.4066 \times 10^{-7}$ |
| 0.6 | 1.30952 | 1.30952 | $3.1641 \times 10^{-14}$ | $3.3238 \times 10^{-15}$ | $3.8161 \times 10^{-7}$ |
| 0.7 | 1.36544 | 1.36544 | $6.7280 \times 10^{-14}$ | $9.0660 \times 10^{-15}$ | $9.2647 \times 10^{-7}$ |
| 0.8 | 1.42365 | 1.42365 | $1.4199 \times 10^{-13}$ | $1.4733 \times 10^{-14}$ | $2.0024 \times 10^{-6}$ |
| 0.9 | 1.4847 | 1.4847 | $3.0420 \times 10^{-13}$ | $5.0508 \times 10^{-14}$ | $4.1184 \times 10^{-6}$ |
| 1.0 | 1.54931 | 1.54931 | $6.7624 \times 10^{-13}$ | – | – |

*Example 2:*
Consider a linear second order problem

$$y'' = y', \quad y(0) = 0,\ y'(0) = -1,\ h = 0.01$$

which appear in [8], with Exact solution:

$$y(x) = 1 - e^{x}$$

Table (3) shows the comparison of maximum absolute errors in One-step method, [8] and [13]. The first, second, third, fourth, fifth and sixth columns indicate number of steps, exact solution, computed solution, error in one-step, error in [8] and error in [13] respectively. The solution from one-step agrees with the exact solution up to at least $10^{-16}$ while that of [8] agrees up to $10^{-12}$ and that of [13] agrees up to $10^{-9}$ indicating that the current method is more accurate. Figures 2 shows the solution plot and error plots respectively for example 2.

***Table 3.*** *Solution of Problem 2 - One-step Method (Comparison with Literature).*

| $x$ | $y_{\text{approx}}$ | $y_{\text{Exact}}$ | Error (One-step) | Error in [8] | Error in [13] |
|---|---|---|---|---|---|
| 0.1 | -0.105171 | -0.105171 | 0. | $5.8319 \times 10^{-12}$ | $2.0958 \times 10^{-10}$ |
| 0.2 | -0.221403 | -0.221403 | 0. | $1.0878 \times 10^{-11}$ | $2.0927 \times 10^{-9}$ |
| 0.3 | -0.349859 | -0.349859 | 0. | $1.5118 \times 10^{-11}$ | $7.8426 \times 10^{-9}$ |
| 0.4 | -0.491825 | -0.491825 | $2.2205 \times 10^{-16}$ | $1.8538 \times 10^{-11}$ | $2.0095 \times 10^{-8}$ |
| 0.5 | -0.648721 | -0.648721 | $2.2205 \times 10^{-16}$ | $2.1121 \times 10^{-11}$ | $4.1998 \times 10^{-8}$ |
| 0.6 | -0.822119 | -0.822119 | $2.2205 \times 10^{-16}$ | $2.2850 \times 10^{-11}$ | $7.7288 \times 10^{-8}$ |
| 0.7 | -1.01375 | -1.01375 | $4.4409 \times 10^{-16}$ | $2.3707 \times 10^{-11}$ | $1.3038 \times 10^{-7}$ |
| 0.8 | -1.22554 | -1.22554 | $4.4409 \times 10^{-16}$ | $2.3675 \times 10^{-11}$ | $2.0648 \times 10^{-7}$ |
| 0.9 | -1.4596 | -1.4596 | $8.8818 \times 10^{-16}$ | $2.2734 \times 10^{-11}$ | $3.1168 \times 10^{-7}$ |
| 1.0 | -1.71828 | -1.71828 | $4.4409 \times 10^{-16}$ | $2.0868 \times 10^{-11}$ | $4.5310 \times 10^{-7}$ |

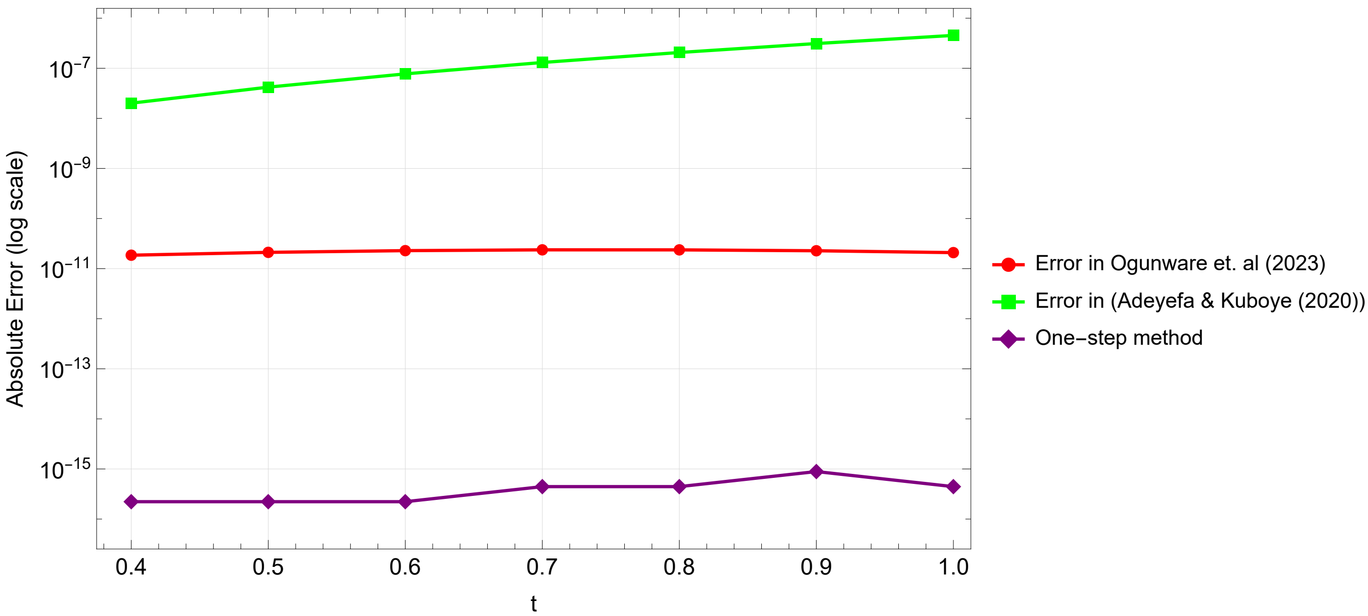


***Figure 2.*** *Comparison of results of example 2*

*Example 3:*
Consider a highly stiff initial value problem

$$y'' = 100y, \quad y(0) = 1,\ y'(0) = -10,\ h = 0.01$$

see in [8], whose theoretical solution is

$$y(x) = e^{-10x}$$

The solutions to (4) were obtained within [0; 1] over 10 iterations and are compared with the exact solution, as presented in Figure 3. It is clear from (4) and Figure 3 that methods shows good performance over the methods in [8] and [14]

***Table 4.*** *Solution of Problem 3 - One-step Method (Comparison with Literature).*

| $x$ | $y_{\text{approx}}$ | $y_{\text{Exact}}$ | Error (One-step) | Error in [8] | Error in [14] |
|---|---|---|---|---|---|
| 0.02 | 0.818731 | 0.818731 | 0. | $6.5075 \times 10^{-11}$ | $1.6280 \times 10^{-10}$ |
| 0.03 | 0.740818 | 0.740818 | $2.2205 \times 10^{-16}$ | $8.8324 \times 10^{-11}$ | $2.5675 \times 10^{-10}$ |
| 0.04 | 0.67032 | 0.67032 | $1.1102 \times 10^{-16}$ | $1.0656 \times 10^{-10}$ | $3.4946 \times 10^{-10}$ |
| 0.05 | 0.606531 | 0.606531 | $1.1102 \times 10^{-16}$ | $1.2052 \times 10^{-10}$ | $4.7320 \times 10^{-10}$ |
| 0.06 | 0.548812 | 0.548812 | $1.1102 \times 10^{-16}$ | $1.3086 \times 10^{-10}$ | $7.0419 \times 10^{-10}$ |
| 0.07 | 0.496585 | 0.496585 | $2.2205 \times 10^{-16}$ | $1.3815 \times 10^{-10}$ | $9.6018 \times 10^{-10}$ |
| 0.08 | 0.449329 | 0.449329 | $2.7756 \times 10^{-16}$ | $1.4286 \times 10^{-10}$ | $1.2232 \times 10^{-9}$ |
| 0.09 | 0.40657 | 0.40657 | $3.3307 \times 10^{-16}$ | $1.4542 \times 10^{-10}$ | $1.4962 \times 10^{-9}$ |
| 0.10 | 0.367879 | 0.367879 | $3.8858 \times 10^{-16}$ | $1.4620 \times 10^{-10}$ | $1.8005 \times 10^{-9}$ |

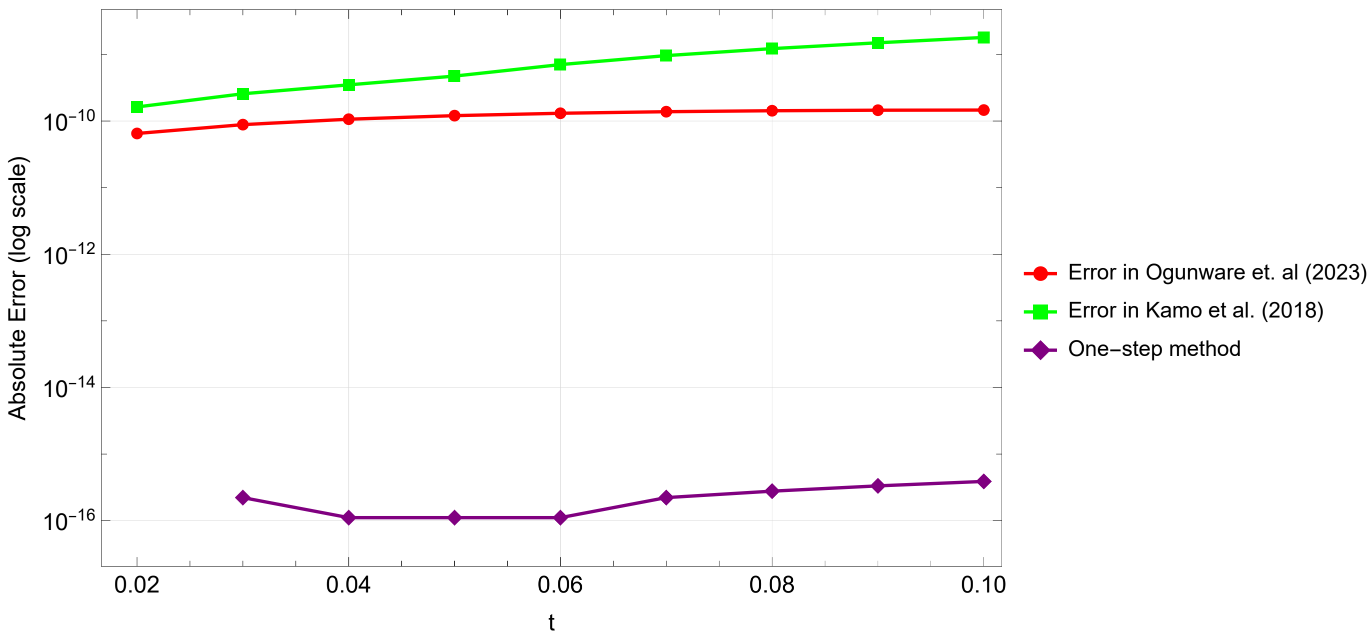


***Figure 3.*** *Comparison of results of example 3*

## 5. Discussion of Results

The developed block method for solving higher order ordinary differential equation on implicit one-step second derivative was studied in this research. The method was derived using interpolation and collocate as a basic function. The properties of the one-step block method was analyzed. The method was tested with some numerical examples solved by [4, 7, 8, 13, 16] it is obvious that our method found to give better accuracy when compared. The solution graph shown the convergence of the method in contrast with the exact solutions.

## 6. Conclusion

A new strategy for developing block methods for solving singular initial value problems has been introduced in this paper. The suggested strategy eliminates the use of ad-hoc formulas to obtain the approximation of the problem at the initial points. Some examples are considered for numerical experiments, and the results established the good performance of the technique.

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## Abbreviations

| | |
|---|---|
| $x$ | Value of the Independent Variable Where a Numerical Value Is Taken |
| $yExact$ | Exact Result at $x$ |
| $yapprox$ | Numerical Result of $x$ |

## Author Contributions

**Olabode Bola Titilayo:** Conceptualization, Methodology, Resources, Supervision

**Abidemi Afeez:** Conceptualization, Methodology, Resources, Supervision

**Momoh Adelegan Lukuman:** Project administration, Validation, Writing – review & editing

**Fagbohun Oluwadamilola Moses:** Formal Analysis, Investigation, Methodology, Software, Writing – original draft

# Conflicts of Interest

The authors declare no conflicts of interest.

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