

Dynamics of Generalized Unstable Nonlinear Schrödinger Equation: Instabilities, Solitons, and Rogue Waves

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Abstract: This study delves into the dynamics of the unstable Schrödinger equation, employing three distinct analytical methods: the complex envelope function ansatz, the generalized Tanh method, and the Bernoulli sub-ODE method. By leveraging the complex envelope function technique, we uncover solutions for various optical soliton types, including dark optical solitons, bright optical solitons, and bright-dark optical solitons. Notably, this method facilitates an in-depth examination of individual soliton intensity profiles, providing valuable insights into their behavior. Furthermore, we utilize the generalized Tanh method and the Bernoulli sub-ODE method to derive solutions involving hyperbolic and trigonometric functions. These solutions shed light on the intricate dynamics of nonlinear optical phenomena within the framework of the Schrödinger equation. The obtained solutions are graphically illustrated, showcasing dark, bright, dark-bright, and singular solitons. Our research contributes significantly to the understanding of unstable Schrödinger equation dynamics, offering a comprehensive analysis of optical soliton behavior. The conservation laws of the model equation are also constructed, providing a deeper understanding of the underlying physical principles. This study's findings have important implications for the development of advanced optical communication systems and the study of nonlinear optical phenomena.

Keywords: Optical Soliton, Unstable NLSE, Instability, Rogue Waves

1. Introduction

In the realm of wave propagation, the nonlinear Schrödinger equation (NLSE) serves as a fundamental model, offering valuable insights into a wide range of physical phenomena, from the dynamics of Bose-Einstein condensates to wave behavior in plasma and fluid systems. To better understand the complex dynamics in different physical settings, researchers have explored numerous variations of the NLSE over time [2].

One intriguing area of this research is the study of generalized unstable nonlinear Schrödinger equations (UNLSEs). By incorporating instabilities, these equations extend the traditional NLSE framework, introducing new layers of complexity and potential applications. The dynamic interaction between nonlinearity and instability in UNLSEs gives rise to phenomena such as rogue waves, the sudden emergence of localized structures, and intricate wave propagation patterns [3]. In today's era of interdisciplinary research and technological innovation, understanding and

harnessing the dynamics of generalized unstable nonlinear Schrödinger equations is vital. These equations have potential applications in fields such as fluid dynamics, optics, condensed matter physics, and plasma physics [4, 5, 7, 12].

The Generalized Unstable nonlinear Schrödinger Equation is given by [1].

$$iq_t + \alpha q_{xx} + \beta |q|^2 q + \gamma q = 0. \quad (1)$$

Where α , β , and γ are real constants, the changes in stability over time in various environments are mainly influenced by β and γ . The Unstable Nonlinear Schrödinger Equation (NLSE) is a fundamental model in nonlinear optics, especially for describing ultra-short signal propagation in optical fibers [1, 2, 8]. There are several methods for solving nonlinear partial differential equations (PDEs) and Schrödinger equations, including the Riccati-Bernoulli sub-ODE method [12], Bernoulli sub-ODE method [13], and extended simple equation method [6], among others. Various

techniques have been applied to study this model, such as the extended simple equation method, the $\tanh(\xi/2)$ -expansion method [11], New Jacobi elliptic function rational expansion [6] and Sine-Gordon expansion [9]. In this research, we explore the dynamics of generalized unstable nonlinear Schrödinger equations using innovative approaches like the complex envelope ansatz method, Bernoulli sub-ODE analysis, and the generalized tanh method. These techniques are effective in understanding the complex wave patterns, soliton dynamics, and instability behaviors associated with these equations.

2. Description of the Methods

2.1. Complex Envelope Function Ansatz Method

We begin by assuming a solution of the form [6]:

$$q(x, t) = A(x, t) \times e^{i\phi(x, t)}, \quad (2)$$

where

$$\phi(x, t) = -kx - \omega t + \theta. \quad (3)$$

In Equation (2), $A(x, t)$ represents the complex envelope function, while ϕ is the linear phase shift. The parameters k and ω relate to the phase components that define the wave frequency and wave number of the light pulse, respectively. Additionally, θ acts as the phase constant. The solution is adjusted using the complex amplitude approach suggested by Li et al. [6], which introduces an essential modification.

$$A(x, t) = i\Omega + \lambda \tanh[\eta(x \pm vt)] + i\rho \operatorname{sech}[\eta(x \pm vt)], \quad (4)$$

In this context, η and v represent the pulse width and velocity, respectively. When $\Omega = \lambda = 0$, the result is a bright optical soliton. On the other hand, setting $\rho = 0$ changes the solution in Equation (4) to a dark soliton. The parameters Ω , λ , and ρ in the ansatz of Equation (4) allow it to describe a dark-bright or combined optical soliton [6]. If $\Omega = 0$ and $\lambda, \rho \neq 0$, it indicates there is no distinct platform beneath the optical soliton. The corresponding envelope solution describes the behavior of a combined soliton on a zero background with constant amplitude and intensity. These solutions include two free parameters related to the peak intensity of the composite soliton and the amplitude of the background. Note that η , v , k , and ω are real values, while γ can be either real or complex [6]. Therefore, the amplitude of $A(x, t)$ is given by:

$$|A(x, t)| = \left(\lambda^2 + \Omega^2 + 2\Omega\rho \operatorname{sech}[\eta(x \pm vt)] + (\rho^2 - \lambda^2) \operatorname{sech}^2[\eta(x \pm vt)] \right)^{\frac{1}{2}}. \quad (5)$$

Furthermore, the nonlinear phase shift ψ_{NL} is given by

$$\psi_{NL} = \arctan \left[\frac{\beta + \rho \operatorname{sech}[\eta(x \pm vt)]}{\lambda \tanh[\eta(x \pm vt)]} \right]. \quad (6)$$

2.2. Bernoulli Sub ODE Methods

Contemplate a partial differential equation (PDE) formulated as [12].

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{tx}, \dots) = 0. \quad (7)$$

where $q(x, t) = q(\xi)$

Step 1:

Using the conversion

$$q(x, t) = q(\xi) \times e^{i\phi(x, t)}, \quad (8)$$

where $\xi = \lambda(x + vt)$ and $\phi(x, t) = -k_1x + \omega t + \theta$.

Equation (8) is susceptible to conversion into the subsequent ordinary differential equation (ODE) [12].

$$P(q, q', q'' \dots) = 0. \quad (9)$$

with $q'(\xi) = \frac{\partial q}{\partial \xi}$

Step 2:

Assuming that a solution of the form is applicable to Equation (9).

$$q(\xi) = \sum_{i=0}^n a_i G^i. \quad (10)$$

Here, $G = G(\xi)$ fulfills the equation[13]

$$G' + \lambda G = \mu G^2. \quad (11)$$

a_i are constants and $\mu \neq 0$, $\lambda \neq 0$.

The solution to Equation (11) takes the form of a Bernoulli equation.

$$G = -\frac{\lambda}{2\mu} (\operatorname{Tanh}[\frac{\lambda\xi}{2}] - 1). \quad (12)$$

$$G = -\frac{\lambda}{2\mu} (\operatorname{Coth}[\frac{\lambda\xi}{2}] - 1). \quad (13)$$

Step 3:

The positive integer n can be determined by equating the highest-order derivatives with the highest-order nonlinear term present in equation (10).

The balancing formula is given as[13]

$$D(\frac{d^a u}{d\xi^a}) = n + a, \quad D(u^b (\frac{d^a u}{d\xi^a})^c) = bn + c(n + a). \quad (14)$$

Therefore, the value of n in equation (10) can be derived by applying equation (14).

Step 4:

Replace equation (10) with equation (9), then apply equation (11) and collect all terms that have the same power of $G(\xi)$. Set each coefficient of G^i to zero, which will give a system of algebraic equations. Solve this system to find the values of a_i and other related parameters. Finally, substitute these values back into equation (10) to obtain the solution for equation (7).

2.3. Generalized Tanh Methods

Consider a partial differential equation (PDE) formulated as [13]

$$P(q, q_t, q_x, q_{tt}, q_{xx}, q_{tx}, \dots) = 0. \quad (15)$$

where $q(x, t) = q(\xi)$

Step 1:

Using the conversion

$$q(x, t) = q(\xi) \times e^{i\phi(x, t)}, \quad (16)$$

where $\xi = \lambda(x + vt)$ and $\phi(x, t) = -k_1x + \omega t + \theta$.

Equation (16) is susceptible to conversion into the subsequent ordinary differential equation (ODE).

$$P(q, q', q'' \dots) = 0. \quad (17)$$

where $q'(\xi) = \frac{\partial q}{\partial \xi}$

Step 2:

Assuming that a solution of the form below is applicable to Equation (17).

$$q(\xi) = \sum_{i=0}^n a_i Q^i. \quad (18)$$

Here, $Q = Q(\xi)$ fulfills the equation

$$\frac{\partial Q}{\partial \xi} = K + Q^2. \quad (19)$$

The solution to Equation (19) takes the form of a riccati equation. If $K < 0$ then

$$Q(\xi) = \sqrt{-K} \tanh[\sqrt{-K}\xi] \quad (20)$$

$$Q(\xi) = \sqrt{-K} \coth[\sqrt{-K}\xi] \quad (21)$$

If $K > 0$ then

$$Q(\xi) = \sqrt{K} \tan[\sqrt{K}\xi] \quad (22)$$

$$Q(\xi) = -\sqrt{K} \cot[\sqrt{K}\xi] \quad (23)$$

If $K = 0$ then

$$Q(\xi) = -\frac{1}{\xi} \quad (24)$$

Step 3:

The positive integer n can be determined by equating the highest-order derivatives with the highest-order nonlinear term present in equation (17).

The balancing formula is given as[13]

$$D\left(\frac{d^a u}{d\xi^a}\right) = n + a, \quad D(u^b (\frac{d^a u}{d\xi^a})^c) = bn + c(n + a). \quad (25)$$

Therefore, the value of n in equation (17) can be derived by applying equation (25).

Step 4:

Replace equation (10) with equation (9), then apply equation (11) and combine terms that have the same power of $G(\xi)$. Set each coefficient of G^i to zero, which will give a system of algebraic equations. Solve this system to find the values of a_i and other parameters. Finally, substitute these values back into equation (10) to obtain the solution for equation (7).

3. Applications of the Methods

3.1. Complex Envelope Function Ansatz Method to Generalized Unstable-NLSE

We begin by substituting eq. (2) into eq. (1) to obtain

$$i(A_t - 2\alpha k A_x) + \alpha A_{xx} + A(\beta |A|^2 + \gamma - \alpha k^2 - \omega) = 0 \quad (26)$$

Substituting eq. (4) into eq. (26), we get

$$\begin{aligned} & (\lambda \tanh(\tau) + i\rho \operatorname{sech}(\tau) + i\Omega) \left(\gamma + \alpha(-k^2) + \beta \left(\lambda^2 + (\rho^2 - \lambda^2) \operatorname{sech}^2(\tau) \right. \right. \\ & \left. \left. + 2\rho\Omega \operatorname{sech}(\tau) + \Omega^2 \right) - \omega \right) + i \left(-2\alpha k \left(\eta \lambda \operatorname{sech}^2(\tau) - i\eta \rho \tanh(\tau) \operatorname{sech}(\tau) \right) + \eta \lambda v \operatorname{sech}^2(\tau) - i\eta \rho v \tanh(\tau) \operatorname{sech}(\tau) \right) \\ & + \alpha \left(-2\eta^2 \lambda \tanh(\tau) \operatorname{sech}^2(\tau) + i\rho \left(\eta^2 \tanh^2(\tau) \operatorname{sech}(\tau) - \eta^2 \operatorname{sech}^3(\tau) \right) \right) = 0, \end{aligned} \quad (27)$$

where $\tau = \eta(x - vt)$. Collecting the coefficient of $\tanh(\tau)$ and $\operatorname{sech}(\tau)$ terms in Eq. (27) that are containing independent combinations to zero, we obtain the following independent parametric equations:

1. $\operatorname{sech}^0(\tau) \operatorname{sech}^0(\tau)$:

$$i\Omega(\beta(\lambda^2 + \Omega^2) + \gamma + \alpha(-k^2) - \omega) = 0, \quad (28)$$

2. $\operatorname{sech}(\tau) \tanh(\tau)$:

$$\rho(2\beta\lambda\Omega - 2\alpha\eta k + \eta v) = 0. \quad (29)$$

3. $\operatorname{sech}(\tau) \tanh^2(\tau)$:

$$i\alpha\eta^2\rho = 0, \quad (30)$$

4. $\tanh(\tau)\text{sech}^2(\tau)$:

$$\beta\lambda(\rho^2 - \lambda^2) - 2\alpha\eta^2\lambda = 0, \quad (31)$$

5. $\text{sech}^3(\tau)$:

$$i\rho(\alpha\eta^2 + \beta(\lambda - \rho)(\lambda + \rho)) = 0, \quad (32)$$

6. sech :

$$-i\rho(-\beta(\lambda^2 + 3\Omega^2) - \gamma + \alpha k^2 + \omega) = 0, \quad (33)$$

7. $\text{sech}^2(\tau)$:

$$i(\eta\lambda(v - 2\alpha k) - \beta\Omega(\lambda^2 - 3\rho^2)) = 0, \quad (34)$$

8. $\tanh(\tau)$:

$$\lambda(\beta(\lambda^2 + \Omega^2) + \gamma + \alpha(-k^2) - \omega) = 0, \quad (35)$$

Solving Eqs. (28)-(35), we obtain the following families and solutions

1. Family 1: When

$$\begin{aligned} \rho = 0; k &= \frac{\sqrt{\beta^2\Omega + \beta\lambda^2 + \gamma - \omega}}{\sqrt{\alpha}}; \eta = \frac{\beta\Omega\lambda}{v - 2\alpha k}; \\ v &= 2\alpha k + \sqrt{2}\sqrt{-3\alpha\beta\Omega^2 - 2\alpha\beta\lambda^2 - 2\alpha\gamma + 2\alpha\omega + 2\alpha^2k^2}. \end{aligned}$$

we get the following dark optical soliton.

$$\begin{aligned} q(x, t) &= \frac{e^{i\left(\theta + t\omega - \frac{x\sqrt{\beta\lambda^2 + \beta\Omega^2 + \gamma - \omega}}{\sqrt{\alpha}}\right)}}{\sqrt{2}\sqrt{2\alpha(\beta\lambda^2 + \beta\Omega^2 + \gamma - \omega) - 2\alpha\beta\lambda^2 - 3\alpha\beta\Omega^2 - 2\alpha\gamma + 2\alpha\omega}} \\ &\quad \left(i\Omega + \lambda \tanh(\beta\lambda\Omega(x + t(2\sqrt{\alpha}\sqrt{\beta\lambda^2 + \beta\Omega^2 + \gamma - \omega} \right. \\ &\quad \left. + \sqrt{2}\sqrt{2\alpha(\beta\lambda^2 + \beta\Omega^2 + \gamma - \omega) - 2\alpha\beta\lambda^2 - 3\alpha\beta\Omega^2 - 2\alpha\gamma + 2\alpha\omega})) \right) \end{aligned} \quad (36)$$

The corresponding intensity $|A|^2$ appear in the form of the following bright optical soliton

$$\begin{aligned} |A(x, t)|^2 &= \\ &\left(\lambda^2 + \lambda^2(-\text{sech}^2 \frac{\left(\beta\lambda\Omega(t(2\sqrt{\alpha}\sqrt{\beta\lambda^2 + \beta\Omega^2 + \gamma - \omega} + \sqrt{2} \right. \right. \\ &\quad \left. \left. \sqrt{2}\sqrt{2\alpha(\beta\lambda^2 + \beta\Omega^2 + \gamma - \omega) - 2\alpha\beta\lambda^2 - 3\alpha\beta\Omega^2 - 2\alpha\gamma + 2\alpha\omega} \right. \right. \\ &\quad \left. \left. \times \sqrt{2\alpha(\beta\lambda^2 + \beta\Omega^2 + \gamma - \omega) - 2\alpha\beta\lambda^2 - 3\alpha\beta\Omega^2 - 2\alpha\gamma + 2\alpha\omega} + x \right) + \Omega^2 \right)^{\frac{1}{2}} \end{aligned} \quad (37)$$

The solitons will exist provided

$$\left(\beta^2\Omega + \beta\lambda^2 + \gamma - \omega \right) \left(-3\alpha\beta\Omega^2 - 2\alpha\beta\lambda^2 - 2\alpha\gamma + 2\alpha\omega + 2\alpha^2k^2 \right) > 0.$$

The nonlinear phase shift for this case is

$$\psi_{NL} = \arctan \left[\frac{\Omega}{\lambda \tanh[\eta(x - vt)]} \right]. \quad (38)$$

2. Family 2: When

$$\alpha = 0; \gamma = \omega; \lambda = \rho; \beta = \frac{\omega - \gamma}{\rho^2}; \eta = \frac{2\beta\Omega\lambda}{v}. \quad (39)$$

we get the following bright- dark optical soliton

$$q(x, t) = (i\rho + i\Omega)e^{i(\theta - kx + t\omega)}. \quad (40)$$

The intensity $|A|^2$ appear in the form of the following bright optical soliton

$$|A(x, t)|^2 = \sqrt{\rho^2 + 2\rho\Omega + \Omega^2} \quad (41)$$

The nonlinear phase shift for this case is

$$\psi_{NL} = \arctan \left[\frac{\Omega + \rho \operatorname{sech}[\eta(x - vt)]}{\lambda \tanh[\eta(x - vt)]} \right] \quad (42)$$

3. Family 3: When

$$\rho = 0; \Omega = 0; v = 2\alpha k; \eta = \frac{i\sqrt{\beta}\lambda}{\sqrt{2}\sqrt{\alpha}}; k = \frac{\sqrt{\beta\lambda^2 + \gamma - \omega}}{\sqrt{\alpha}}.$$

we get the following bright- dark optical soliton

$$q(x, t) = i\lambda e^{i\left(\theta + t\omega - \frac{x\sqrt{\beta\lambda^2 + \gamma - \omega}}{\sqrt{\alpha}}\right)} \tan \left(\frac{\sqrt{\beta}\lambda \left(2\sqrt{\alpha}t\sqrt{\beta\lambda^2 + \gamma - \omega} + x\right)}{\sqrt{2}\sqrt{\alpha}} \right). \quad (43)$$

The intensity $|A|^2$ appear in the form of the following bright optical soliton

$$|A(x, t)|^2 = \sqrt{\lambda^2 - \lambda^2 \sec^2 \left(\frac{\sqrt{\beta}\lambda \left(2\sqrt{\alpha}t\sqrt{\beta\lambda^2 + \gamma - \omega} + x\right)}{\sqrt{2}\sqrt{\alpha}} \right)}. \quad (44)$$

The nonlinear phase shift does not exist for this family.

3.2. Application of BSMODE Method to Generalized Unstable-NLSE

3.2.1. Kerr Law Non-linearity

For Kerr law non-linearity, $F(q) = q$.

$$iq_t + \alpha q_{xx} + \beta q|q|^2 + \gamma q = 0 \quad (45)$$

Using eq. (8) in eq. (45) and separate the real and imaginary parts of the equation, we obtained.

The imaginary part as:

$$v = 2k\alpha. \quad (46)$$

The real part as:

$$-\alpha k^2 q + \alpha q'' + \beta q^3 + \gamma q - \omega q = 0 \quad (47)$$

Balancing q^3 and q'' eq. (47), we get $n = 1$. Substituting $n = 1$ in eq. (10), we obtain.

$$q(\xi) = a_0 + a_1 G(\xi). \quad (48)$$

Where a_0 and a_1 are constants to be determined. Substituting equation (48) and its derivatives into equation (47) gives us an over-determined set of expressions. By gathering the terms of G^i and performing the necessary calculations, we obtain.

G^0 :

$$a_0 (a_0^2 \beta + \gamma - \alpha k^2 - \omega) = 0. \quad (49)$$

G^1 :

$$a_1 (3a_0^2 \beta + \alpha \lambda^2 + \gamma - \alpha k^2 - \omega) = 0. \quad (50)$$

G^2 :

$$3a_1 (a_0 a_1 \beta - \alpha \lambda \mu) = 0. \quad (51)$$

G^3 :

$$2\alpha a_1 \mu^2 + a_1^3 \beta = 0. \quad (52)$$

From Solving Eq. (49) - (52) we obtained the following Set values.

$$a_1 = -\frac{2a_0\mu}{\lambda}; a_0 = \frac{i\sqrt{\alpha}\lambda}{\sqrt{2}\sqrt{\beta}}; k = \frac{\sqrt{-\alpha\lambda^2+2\gamma-2\omega}}{\sqrt{2}\sqrt{\alpha}}.$$

The solution of the obtained values are given as follows.

$$q_{2,1}(x, t) = \frac{i\sqrt{\alpha}\lambda \coth\left(\frac{1}{2}\lambda\left(\sqrt{2}\sqrt{\alpha}t\sqrt{-\alpha\lambda^2+2\gamma-2\omega}+x\right)\right)}{\sqrt{2}\sqrt{\beta}} \times \exp\left(i\left(\theta+t\omega-\frac{x\sqrt{-\alpha\lambda^2+2\gamma-2\omega}}{\sqrt{2}\sqrt{\alpha}}\right)\right). \quad (53)$$

and

$$q_{2,2}(x, t) = \frac{i\sqrt{\alpha}\lambda \tanh\left(\frac{1}{2}\lambda\left(\sqrt{2}\sqrt{\alpha}t\sqrt{-\alpha\lambda^2+2\gamma-2\omega}+x\right)\right)}{\sqrt{2}\sqrt{\beta}} \times \exp\left(i\left(\theta+t\omega-\frac{x\sqrt{-\alpha\lambda^2+2\gamma-2\omega}}{\sqrt{2}\sqrt{\alpha}}\right)\right) \quad (54)$$

3.2.2. Quadratic Cubic Law Non-linearity

For Quadratic Cubic law Non-linearity, $F(q) = \sqrt{q} + q$.

$$iq_t + \alpha q_{xx} + (\beta_1 |q| + \beta_2 |q|^2)q + \gamma q = 0. \quad (55)$$

Using eq. (8) in eq. (55) and separate the real and imaginary parts of the equation, we obtained.

The imaginary part as:

$$v = 2k\alpha. \quad (56)$$

The real part as:

$$-\alpha k^2 q + \alpha q'' + \beta_1 q^2 + \beta_2 q^3 + \gamma q - \omega q = 0 \quad (57)$$

Balancing q^3 and q'' eq. (57), we get $n = 1$. Substituting $n = 1$ in eq. (10), we obtain.

$$q(\xi) = a_0 + a_1 G(\xi). \quad (58)$$

Where a_0, a_1 are constant to be determine.

Substituting eq. (58) and its derivatives into eq. (57), we obtain an over-determining expressions. Collecting the terms of G^i and conducting all the necessary computations, we obtain.

G^0 :

$$a_0 (a_0 \beta_1 + a_0^2 \beta_2 + \gamma + \alpha (-k^2) - \omega) = 0. \quad (59)$$

G^1 :

$$a_1 (2a_0 \beta_1 + 3a_0^2 \beta_2 + \alpha \lambda^2 + \gamma - \alpha k^2 - \omega) = 0. \quad (60)$$

G^2 :

$$a_1 (a_1 (3a_0 \beta_2 + \beta_1) - 3\alpha \lambda \mu) = 0. \quad (61)$$

G^3 :

$$2\alpha a_1 \mu^2 + a_1^3 \beta_2 = 0. \quad (62)$$

From Solving Eq. (59) - (62) we obtained the following Set values.

Set 1: $a_0 = 0; \alpha = -\frac{2\beta_1^2}{9\beta_2\lambda^2}; a_1 = -\frac{2\beta_1\mu}{3\beta_2\lambda}; k = \frac{\sqrt{\alpha\lambda^2+\gamma-\omega}}{\sqrt{\alpha}}.$

The solution of the obtained values are given as follows.

$$q_{3,1}(x, t) = \frac{\beta_1 \left(\coth\left(\frac{1}{18}\lambda\left(2\sqrt{2}t\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}\sqrt{-\frac{2\beta_1^2}{\beta_2}+9\gamma-9\omega+9x}\right)\right) - 1 \right)}{3\beta_2} \times \exp\left(i\left(\theta+t\omega-\frac{3x\sqrt{-\frac{2\beta_1^2}{9\beta_2}+\gamma-\omega}}{\sqrt{2}\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}}\right)\right). \quad (63)$$

and

$$q_{3,2}(x, t) = -\frac{\beta_1 \left(\tanh \left(\frac{1}{18} \lambda \left(2\sqrt{2}t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} \sqrt{-\frac{2\beta_1^2}{\beta_2} + 9\gamma - 9\omega + 9x} \right) \right) - 1 \right)}{3\beta_2} \times \exp \left(i \left(\theta + t\omega - \frac{3x \sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega}}{\sqrt{2} \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}}} \right) \right) \quad (64)$$

Set 2:

$$a_0 = -\frac{2\beta_1}{3\beta_2}; \gamma = \omega; k = \lambda; \alpha = -\frac{2\beta_1^2}{9\beta_2 \lambda^2}; a_1 = -\frac{a_0 \mu}{\lambda}.$$

The solution of the obtained values are given as follows.

$$q_{3,3}(x, t) = -\frac{\beta_1 \left(\coth \left(\frac{\lambda x}{2} - \frac{2\beta_1^2 t}{9\beta_2} \right) + 1 \right) e^{i(\theta + t\omega - \lambda x)}}{3\beta_2} \quad (65)$$

and

$$q_{3,4}(x, t) = -\frac{\beta_1 \left(\tanh \left(\frac{\lambda x}{2} - \frac{2\beta_1^2 t}{9\beta_2} \right) + 1 \right) e^{i(\theta + t\omega - \lambda x)}}{3\beta_2} \quad (66)$$

Set 3:

$$a_0 = 0; \gamma = \omega; \alpha = -\frac{2\beta_1^2}{9\beta_2 \lambda^2}; a_1 = -\frac{2\beta_1 \mu}{3\beta_2 \lambda}; k = \frac{\sqrt{\alpha \lambda^2 + \gamma - \omega}}{\sqrt{\alpha}}.$$

The solution of the obtained values are given as follows.

$$q_{3,5}(x, t) = \frac{\beta_1 \left(\coth \left(\frac{1}{18} \lambda \left(4\sqrt{-\frac{\beta_1^2}{\beta_2}} t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} + 9x \right) \right) - 1 \right)}{3\beta_2} \times \exp \left(i \left(\theta + t\omega - \frac{\sqrt{-\frac{\beta_1^2}{\beta_2}} x}{\sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}}} \right) \right) \quad (67)$$

and

$$q_{3,6}(x, t) = \frac{\beta_1 \left(\tanh \left(\frac{1}{18} \lambda \left(4\sqrt{-\frac{\beta_1^2}{\beta_2}} t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} + 9x \right) \right) - 1 \right)}{3\beta_2} \times \exp \left(i \left(\theta + t\omega - \frac{\sqrt{-\frac{\beta_1^2}{\beta_2}} x}{\sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}}} \right) \right) \quad (68)$$

$$\text{Set 4: } a_0 = -\frac{2\beta_1}{3\beta_2}; k = \frac{i\lambda}{2\sqrt{2}}; a_1 = -\frac{a_0 \mu}{\lambda}; \gamma = \frac{4\beta_2 \omega + \beta_1^2}{4\beta_2}; \alpha = -\frac{8(\gamma - \omega)}{9\lambda^2}.$$

The solution of the obtained values are given as follows.

$$q_{3,7}(x, t) = -\frac{\beta_1 \left(1 + \coth \left(\frac{\lambda x}{2} - \frac{i\beta_1^2 t}{9\sqrt{2}\beta_2} \right) \right) e^{i\left(\theta + t\omega - \frac{i\lambda x}{2\sqrt{2}}\right)}}{3\beta_2} \quad (69)$$

and

$$q_{3,8}(x, t) = -\frac{\beta_1 \left(1 + \tanh \left(\frac{\lambda x}{2} - \frac{i\beta_1^2 t}{9\sqrt{2}\beta_2} \right) \right) e^{i\left(\theta + t\omega - \frac{i\lambda x}{2\sqrt{2}}\right)}}{3\beta_2} \quad (70)$$

Set 5:

$$\alpha = -\frac{2\beta_1^2}{9\beta_2 \lambda^2}; k = \frac{\sqrt{\alpha \lambda^2 + \gamma - \omega}}{\sqrt{\alpha}}; a_0 = -\frac{\beta_1 (\alpha \lambda^2 - \gamma + \alpha k^2 + \omega)}{\beta_2 \alpha \lambda^2 - 2\gamma + 2\alpha k^2 + 2\omega}; a_1 = -\frac{3\alpha a_0 \lambda \mu}{2\alpha \lambda^2 - \gamma + \alpha k^2 + \omega}.$$

The solution of the obtained values are given as follows.

$$q_{3,9}(x, t) = -\frac{\beta_1 \left(\coth \left(\frac{1}{18} \lambda \left(2\sqrt{2}t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} \sqrt{-\frac{2\beta_1^2}{\beta_2} + 9\gamma - 9\omega + 9x} \right) \right) + 1 \right)}{3\beta_2} \times \exp \left(i \left(\theta + t\omega - \frac{3x \sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega}}{\sqrt{2} \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}}} \right) \right) \quad (71)$$

and

$$q_{3,10}(x, t) = -\frac{\beta_1 \left(\tanh \left(\frac{1}{18} \lambda \left(2\sqrt{2}t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} \sqrt{-\frac{2\beta_1^2}{\beta_2} + 9\gamma - 9\omega + 9x} \right) \right) + 1 \right)}{3\beta_2} \times \exp \left(i \left(\theta + t\omega - \frac{3x \sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega}}{\sqrt{2} \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}}} \right) \right) \quad (72)$$

3.2.3. Parabolic Law Non-linearity

For Quadratic Cubic law Non-linearity, $F(q) = q + q^2$.

$$iq_t + \alpha q_{xx} + (\beta_1 |q|^2 + \beta_2 |q|^4)q + \gamma q = 0. \quad (73)$$

Using eq. (8) in eq. (73) and separate the real and imaginary parts of the equation, we obtained.

The imaginary part as:

$$v = 2k\alpha. \quad (74)$$

The real part as:

$$-\alpha k^2 q + \alpha q'' + \beta_2 q^5 + \beta_1 q^3 + \gamma q - \omega q = 0 \quad (75)$$

Setting

$$q = V^{\frac{1}{2}} \quad (76)$$

eq. (75) is transform to

$$-4V^2 (-\gamma + \alpha k^2 + \omega) + 2\alpha V V'' - \alpha V'^2 + 4\beta_2 V^4 + 4\beta_1 V^3 = 0 \quad (77)$$

Balancing between highest order non linearity and highest order derivatives in eq. (77), we get $n = 1$. Substituting $n = 1$ in eq. (8), we obtain.

$$q(\xi) = a_0 + a_1 G(\xi). \quad (78)$$

Where a_0, a_1 are constant to be determine.

Substituting eq. (78) and its derivatives into eq. (77), we obtain an over-determining expressions. Collecting the terms of G^i and conducting all the necessary computations, we obtain.

G^0 :

$$4a_0^2 (a_0 \beta_1 + a_0^2 \beta_2 + \gamma + \alpha (-k^2) - \omega) = 0. \quad (79)$$

G^1 :

$$2a_0 a_1 (6a_0 \beta_1 + 8a_0^2 \beta_2 + \alpha \lambda^2 + 4\gamma - 4\alpha k^2 - 4\omega) = 0. \quad (80)$$

G^2 :

$$a_1 (-6a_0 (\alpha \lambda \mu - 2a_1 \beta_1) + 24a_1 a_0^2 \beta_2 + a_1 (\alpha \lambda^2 + 4\gamma - 4\alpha k^2 - 4\omega)) = 0. \quad (81)$$

G^3 :

$$4a_1 (a_1 (a_1 \beta_1 - \alpha \lambda \mu) + a_0 (4a_1^2 \beta_2 + \alpha \mu^2)) = 0. \quad (82)$$

G^4 :

$$3\alpha a_1^2 \mu^2 + 4a_1^4 \beta_2 = 0 \quad (83)$$

From Solving Eq. (79) - (83) we obtained the following Set values.

Set 1:

$$a_0 = 0; \alpha = -\frac{3\beta_1^2}{4\beta_2\lambda^2}; a_1 = -\frac{3\beta_1\mu}{4\beta_2\lambda}; k = \frac{\sqrt{\alpha\lambda^2 + 4\gamma - 4\omega}}{2\sqrt{\alpha}}.$$

The solution of the obtained values are given as follows.

$$q_{4,1}(x, t) = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\frac{\beta_1 \left(\coth \left(\frac{1}{8}\lambda \left(t\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}\sqrt{-\frac{9\beta_1^2}{\beta_2}} + 48\gamma - 48\omega + 4x \right) \right) - 1 \right)}{\beta_2}} \times \exp \left(i \left(\theta + t\omega - \frac{x\sqrt{-\frac{3\beta_1^2}{4\beta_2} + 4\gamma - 4\omega}}{\sqrt{3}\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}} \right) \right) \quad (84)$$

and

$$q_{4,2}(x, t) = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\frac{\beta_1 \left(\tanh \left(\frac{1}{8}\lambda \left(t\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}\sqrt{-\frac{9\beta_1^2}{\beta_2}} + 48\gamma - 48\omega + 4x \right) \right) - 1 \right)}{\beta_2}} \times \exp \left(i \left(\theta + t\omega - \frac{x\sqrt{-\frac{3\beta_1^2}{4\beta_2} + 4\gamma - 4\omega}}{\sqrt{3}\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}} \right) \right) \quad (85)$$

Set 2:

$$a_0 = -\frac{3\beta_1}{4\beta_2}; \gamma = \omega; k = \frac{\lambda}{2}; \alpha = -\frac{3\beta_1^2}{4\beta_2\lambda^2}; a_1 = -\frac{a_0\mu}{\lambda}.$$

The solution of the obtained values are given as follows.

$$q_{4,3}(x, t) = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{-\frac{\beta_1 \left(\tanh \left(\frac{\lambda x}{2} - \frac{3\beta_1^2 t}{8\beta_2} \right) + 1 \right)}{\beta_2}} e^{i(\theta + t\omega - \frac{\lambda x}{2})} \quad (86)$$

and

$$q_{4,4}(x, t) = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{-\frac{\beta_1 \left(\tanh \left(\frac{\lambda x}{2} - \frac{3\beta_1^2 t}{8\beta_2} \right) + 1 \right)}{\beta_2}} e^{i(\theta + t\omega - \frac{\lambda x}{2})} \quad (87)$$

Set 3:

$$\alpha = -\frac{3\beta_1^2}{4\beta_2\lambda^2}; k = \frac{\sqrt{-4\alpha\beta_2\lambda^2 + 32\beta_2\gamma - 32\beta_2\omega - 9\beta_1^2}}{4\sqrt{2}\sqrt{\alpha}\sqrt{\beta_2}}; a_0 = -\frac{\beta_1\alpha\lambda^2 - 2\gamma + 2\alpha k^2 + 2\omega}{\beta_2\alpha\lambda^2 - 4\gamma + 4\alpha k^2 + 4\omega};$$

$$a_1 = -\frac{6\alpha a_0\lambda\mu}{5\alpha\lambda^2 - 4\gamma + 4\alpha k^2 + 4\omega}.$$

The solution of the obtained values are given as follows.

$$q_{4,5}(x, t) = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{-\frac{\beta_1 \left(\tanh \left(\frac{1}{8}\lambda \left(\frac{\sqrt{3}t\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}\sqrt{16\beta_2(\gamma-\omega)-3\beta_1^2}}{\sqrt{\beta_2}} + 4x \right) \right) + 1 \right)}{\beta_2}} \times \exp \left(i \left(\theta + t\omega - \frac{x\sqrt{32\beta_2\gamma - 32\beta_2\omega - 6\beta_1^2}}{2\sqrt{6}\sqrt{\beta_2}\sqrt{-\frac{\beta_1^2}{\beta_2\lambda^2}}} \right) \right) \quad (88)$$

and

$$q_{4,6}(x, t) = \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{-\frac{\beta_1 \left(\tanh \left(\frac{1}{8} \lambda \left(\frac{\sqrt{3} t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} \sqrt{16 \beta_2 (\gamma - \omega) - 3 \beta_1^2}}}{\sqrt{\beta_2}} + 4x \right) \right) + 1}{\beta_2}} \right)} \times \exp \left(i \left(\theta + t\omega - \frac{x \sqrt{32 \beta_2 \gamma - 32 \beta_2 \omega - 6 \beta_1^2}}{2 \sqrt{6} \sqrt{\beta_2} \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}}} \right) \right) \quad (89)$$

Set 4:

$$\gamma = \frac{4\beta_2\omega + \beta_1^2}{4\beta_2}; \alpha = -\frac{3(\gamma - \omega)}{\lambda^2}; k = \frac{\sqrt{-\alpha\lambda^2 - \gamma + \omega}}{2\sqrt{2}\sqrt{\alpha}};$$

$$a_0 = -\frac{\beta_1\alpha\lambda^2 - 2\gamma + 2\alpha k^2 + 2\omega}{\beta_2\alpha\lambda^2 - 4\gamma + 4\alpha k^2 + 4\omega}; a_1 = -\frac{6\alpha a_0 \lambda \mu}{5\alpha\lambda^2 - 4\gamma + 4\alpha k^2 + 4\omega};$$

The solution of the obtained values are given as follows.

$$q_{4,7}(x, t) = \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{-\frac{\beta_1 \left(\tanh \left(\frac{1}{8} \lambda \left(\sqrt{3} \sqrt{\frac{\beta_1^2}{\beta_2}} t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} + 4x \right) \right) + 1}{\beta_2}} \right)} \times \exp \left(i \left(\theta + t\omega - \frac{x \sqrt{3 \left(\frac{4\beta_2\omega + \beta_1^2}{4\beta_2} - \omega \right) - \frac{4\beta_2\omega + \beta_1^2}{4\beta_2} + \omega}}{2 \sqrt{6} \sqrt{-\frac{4\beta_2\omega + \beta_1^2}{\lambda^2} - \omega}} \right) \right) \quad (90)$$

and

$$q_{4,8}(x, t) = q(x_-, t_-) = \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{-\frac{\beta_1 \left(\tanh \left(\frac{1}{8} \lambda \left(\sqrt{3} \sqrt{\frac{\beta_1^2}{\beta_2}} t \sqrt{-\frac{\beta_1^2}{\beta_2 \lambda^2}} + 4x \right) \right) + 1}{\beta_2}} \right)} \times \exp \left(i \left(\theta + t\omega - \frac{x \sqrt{3 \left(\frac{4\beta_2\omega + \beta_1^2}{4\beta_2} - \omega \right) - \frac{4\beta_2\omega + \beta_1^2}{4\beta_2} + \omega}}{2 \sqrt{6} \sqrt{-\frac{4\beta_2\omega + \beta_1^2}{\lambda^2} - \omega}} \right) \right) \quad (91)$$

3.3. Application of Generalized Tanh Method to Generalized Unstable-NLSE

3.3.1. Kerr Law Non-linearity

For Kerr law non-linearity, $F(q) = q$.

$$iq_t + \alpha q_{xx} + \beta q|q|^2 + \gamma q = 0 \quad (92)$$

Using eq. (8) in eq. (92) and separate the real and imaginary parts of the equation, we obtained.

The imaginary part as:

$$v = 2k\alpha. \quad (93)$$

The real part as:

$$-\alpha k^2 q + \alpha q'' + \beta q^3 + \gamma q - \omega q = 0 \quad (94)$$

Balancing q^3 and q'' eq. (94), we get $n = 1$. Substituting $n = 1$ in eq. (10), we obtain.

$$q(\xi) = a_0 + a_1 Q(\xi). \quad (95)$$

Where a_0, a_1 are constant to be determine.

Substituting eq. (95) and its derivatives into eq. (94), we obtain an over-determining expressions. Collecting the terms of Q^i

and conducting all the necessary computations, we obtain.

$$Q^0 : \quad a_0 (a_0^2 \beta + \gamma - \alpha k^2 - \omega) = 0. \quad (96)$$

$$Q^1 : \quad a_1 (3a_0^2 \beta + \gamma - \alpha k^2 + 2\alpha K - \omega) = 0. \quad (97)$$

$$Q^2 : \quad 3a_0 a_1^2 \beta = 0. \quad (98)$$

$$Q^3 : \quad 2\alpha a_1 + a_1^3 \beta = 0. \quad (99)$$

From Solving Eq. (96) - (99) we obtained the following Set values.

$$a_0 = 0; a_1 = \frac{i\sqrt{2}\sqrt{\alpha}}{\sqrt{\beta}}; k = \frac{\sqrt{\gamma + 2\alpha K - \omega}}{\sqrt{\alpha}}.$$

The solution of the obtained values are given as follows.

For $\sqrt{\gamma + 2\alpha K - \omega} > 0$ we have the following hyperbolic function solutions.

$$q_{5,1}(x, t) = -\frac{i\sqrt{2}\sqrt{\alpha}\sqrt{-K} \tanh(\sqrt{-K}(2\sqrt{\alpha}t\sqrt{\gamma + 2\alpha K - \omega} + x))}{\sqrt{\beta}} \times e^{i\left(\theta - \frac{x\sqrt{\gamma + 2\alpha K - \omega}}{\sqrt{\alpha}} + t\omega\right)}. \quad (100)$$

and

$$q_{5,2}(x, t) = -\frac{i\sqrt{2}\sqrt{\alpha}\sqrt{-K} \coth(\sqrt{-K}(2\sqrt{\alpha}t\sqrt{\gamma + 2\alpha K - \omega} + x))}{\sqrt{\beta}} \times e^{i\left(\theta - \frac{x\sqrt{\gamma + 2\alpha K - \omega}}{\sqrt{\alpha}} + t\omega\right)} \quad (101)$$

For $\sqrt{\gamma + 2\alpha K - \omega} < 0$ we have the following Trigonometric function solutions.

$$q_{5,3}(x, t) = \frac{i\sqrt{2}\sqrt{\alpha}\sqrt{K} \tan(\sqrt{K}(2\sqrt{\alpha}t\sqrt{\gamma + 2\alpha K - \omega} + x))}{\sqrt{\beta}} \times e^{i\left(\theta - \frac{x\sqrt{\gamma + 2\alpha K - \omega}}{\sqrt{\alpha}} + t\omega\right)}. \quad (102)$$

and

$$q_{5,4}(x, t) = -\frac{i\sqrt{2}\sqrt{\alpha}\sqrt{K} \cot(\sqrt{K}(2\sqrt{\alpha}t\sqrt{\gamma + 2\alpha K - \omega} + x))}{\sqrt{\beta}} \times e^{i\left(\theta - \frac{x\sqrt{\gamma + 2\alpha K - \omega}}{\sqrt{\alpha}} + t\omega\right)}. \quad (103)$$

3.3.2. Quadratic Cubic Law Non-linearity

For Quadratic Cubic law Non-linearity, $F(q) = \sqrt{q} + q$.

$$iq_t + \alpha q_{xx} + (\beta_1 |q| + \beta_2 |q|^2)q + \gamma q = 0. \quad (104)$$

Using eq. (8) in eq. (104) and separate the real and imaginary parts of the equation, we obtained.

The imaginary part as:

$$v = 2k\alpha. \quad (105)$$

The real part as:

$$-\alpha k^2 q + \alpha q'' + \beta_2 q^3 + \beta_1 q^2 + \gamma q - \omega q = 0 \quad (106)$$

Balancing q^3 and q'' eq. (106), we get $n = 1$. Substituting $n = 1$ in eq. (10), we obtain.

$$q(\xi) = a_0 + a_1 Q(\xi). \quad (107)$$

Where a_0, a_1 are constant to be determine. Substituting eq. (107) and its derivatives into eq. (106), we obtain an over-determining expressions. Collecting the terms of Q^i and conducting all the necessary computations, we obtain.

$$Q^0 : \quad a_0 (a_0 \beta_1 + a_0^2 \beta_2 + \gamma + \alpha (-k^2) - \omega) = 0. \quad (108)$$

$$Q^1 : \quad a_1 (2a_0\beta_1 + 3a_0^2\beta_2 + \gamma + \alpha(-k^2) + 2\alpha K - \omega) = 0. \quad (109)$$

$$Q^2 : \quad a_1^2 (3a_0\beta_2 + \beta_1) = 0. \quad (110)$$

$$Q^3 : \quad 2\alpha a_1 + a_1^3\beta_2 = 0. \quad (111)$$

From Solving Eq. (108) - (111) we obtained the following Set values.

Set 1:

$$a_0 = -\frac{\beta_1}{3\beta_2}; a_1 = \frac{ia_0}{\sqrt{K}}; K = \frac{\beta_1^2}{18\alpha\beta_2}; k = \frac{\sqrt{\gamma - 4\alpha K - \omega}}{\sqrt{\alpha}}.$$

The solution of the obtained values are given as follows.

For $\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega} > 0$ we have the following hyperbolic function solutions.

$$q_{6,1}(x, t) = \left(-\frac{\beta_1}{3\beta_2} + \frac{i\alpha\sqrt{-\frac{\beta_1^4}{\alpha^2\beta_2^2}} \tanh\left(\frac{\sqrt{-\frac{\beta_1^2}{\alpha\beta_2}}(2\sqrt{\alpha}t\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega} + x)}{3\sqrt{2}}\right)}{3\beta_1} \right) \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega}}{\sqrt{\alpha}}\right)\right). \quad (112)$$

and

$$q_{6,2}(x, t) = \left(-\frac{\beta_1}{3\beta_2} + \frac{i\alpha\sqrt{-\frac{\beta_1^4}{\alpha^2\beta_2^2}} \coth\left(\frac{\sqrt{-\frac{\beta_1^2}{\alpha\beta_2}}(2\sqrt{\alpha}t\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega} + x)}{3\sqrt{2}}\right)}{3\beta_1} \right) \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega}}{\sqrt{\alpha}}\right)\right). \quad (113)$$

For $\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega} < 0$ we have the following Trigonometric function solutions.

$$q_{6,3}(x, t) = -\frac{i\beta_1 \left(\tan\left(\frac{\sqrt{\frac{\beta_1^2}{\alpha\beta_2}}(2\sqrt{\alpha}t\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega} + x)}{3\sqrt{2}}\right) - i \right)}{3\beta_2} \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega}}{\sqrt{\alpha}}\right)\right) \quad (114)$$

$$q_{6,4}(x, t) = \frac{i\beta_1 \left(\cot\left(\frac{\sqrt{\frac{\beta_1^2}{\alpha\beta_2}}(2\sqrt{\alpha}t\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega} + x)}{3\sqrt{2}}\right) + i \right)}{3\beta_2} \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{-\frac{2\beta_1^2}{9\beta_2} + \gamma - \omega}}{\sqrt{\alpha}}\right)\right) \quad (115)$$

3.3.3. Parabolic Law Non-linearity

For Quadratic Cubic law Non-linearity, $F(q) = q + q^2$.

$$iq_t + \alpha q_{xx} + (\beta_1 |q|^2 + \beta_2 |q|^4)q + \gamma q = 0. \quad (116)$$

Using eq. (8) in eq. (116) and separate the real and imaginary parts of the equation, we obtained.

The imaginary part as:

$$v = 2k\alpha. \quad (117)$$

The real part as:

$$-\alpha k^2 q + \alpha q'' + \beta_2 q^5 + \beta_1 q^3 + \gamma q - \omega q = 0 \quad (118)$$

Setting

$$q = V^{\frac{1}{2}} \quad (119)$$

eq. (118) is transform to

$$-4V^2 (-\gamma + \alpha k^2 + \omega) + 2\alpha VV'' - \alpha V'^2 + 4\beta_2 V^4 + 4\beta_1 V^3 = 0 \quad (120)$$

Balancing between highest order non linearity and highest order derivatives in eq. (120), we get $n = 1$. Substituting $n = 1$ in eq. (10), we obtain.

$$V(\xi) = a_0 + a_1 Q(\xi). \quad (121)$$

Where a_0, a_1 are constant to be determine.

Substituting eq. (121) and its derivatives into eq. (120), we obtain an over-determining expressions. Collecting the terms of Q^i and conducting all the necessary computations, we obtain.

$$Q^0 : \quad 4a_0^4\beta_2 + 4a_0^3\beta_1 - 4a_0^2(-\gamma + \alpha k^2 + \omega) - \alpha a_1^2 K^2 = 0. \quad (122)$$

$$Q^1 : \quad 4a_0a_1(3a_0\beta_1 + 4a_0^2\beta_2 + 2\gamma - 2\alpha k^2 + \alpha K - 2\omega) = 0. \quad (123)$$

$$Q^2 : \quad 2a_1^2(6a_0\beta_1 + 12a_0^2\beta_2 + 2\gamma - 2\alpha k^2 + \alpha K - 2\omega) = 0. \quad (124)$$

$$Q^3 : \quad 4Q(r)^3(a_0a_1(4a_1^2\beta_2 + \alpha) + a_1^3\beta_1) = 0. \quad (125)$$

$$Q^4 : \quad (3\alpha a_1^2 + 4a_1^4\beta_2) \quad (126)$$

From Solving Eq. (122) - (126) we obtained the following Set values.

Set 1:

$$a_0 = -\frac{3\beta_1}{8\beta_2}; K = \frac{3\beta_1^2}{16\alpha\beta_2}; a_1 = \frac{\sqrt{3}\sqrt{\alpha}a_0}{\sqrt{-2\gamma + 2\alpha k^2 - \alpha K + 2\omega}}; k = \frac{\sqrt{32\beta_2\gamma - 32\beta_2\omega - 9\beta_1^2 + 16\alpha\beta_2K}}{4\sqrt{2}\sqrt{\alpha}\sqrt{\beta_2}}.$$

The solution of the obtained values are given as follows.

For $\sqrt{16\beta_2(\gamma - \omega) - 3\beta_1^2} > 0$ we have the following hyperbolic function solutions.

$$q_{7,1}(x, t) = \sqrt{-\frac{3\beta_1}{8\beta_2} - \frac{3\sqrt{\alpha}\sqrt{-\frac{\beta_1^2}{\beta_2}}\sqrt{-\frac{\beta_1^2}{\alpha\beta_2}} \tanh\left(\frac{1}{4}\sqrt{3}\sqrt{-\frac{\beta_1^2}{\alpha\beta_2}}\left(\frac{\sqrt{\alpha t}\sqrt{16\beta_2(\gamma - \omega) - 3\beta_1^2}}{2\sqrt{\beta_2}} + x\right)\right)}{8\beta_1}} \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{32\beta_2\gamma - 32\beta_2\omega - 6\beta_1^2}}{4\sqrt{2}\sqrt{\alpha}\sqrt{\beta_2}}\right)\right) \quad (127)$$

and

$$q_{7,2}(x, t) = \sqrt{-\frac{3\beta_1}{8\beta_2} - \frac{3\sqrt{\alpha}\sqrt{-\frac{\beta_1^2}{\beta_2}}\sqrt{-\frac{\beta_1^2}{\alpha\beta_2}} \coth\left(\frac{1}{4}\sqrt{3}\sqrt{-\frac{\beta_1^2}{\alpha\beta_2}}\left(\frac{\sqrt{\alpha}t\sqrt{16\beta_2(\gamma-\omega)-3\beta_1^2}}{2\sqrt{\beta_2}} + x\right)\right)}{8\beta_1}} \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{32\beta_2\gamma - 32\beta_2\omega - 6\beta_1^2}}{4\sqrt{2}\sqrt{\alpha}\sqrt{\beta_2}}\right)\right) \quad (128)$$

For $\sqrt{16\beta_2(\gamma-\omega)-3\beta_1^2} < 0$ we have the following hyperbolic function solutions.

$$q_{7,3}(x, t) = \sqrt{\frac{3\sqrt{\alpha}\sqrt{-\frac{\beta_1^2}{\beta_2}}\sqrt{\frac{\beta_1^2}{\alpha\beta_2}} \tanh\left(\frac{1}{4}\sqrt{3}\sqrt{\frac{\beta_1^2}{\alpha\beta_2}}\left(\frac{\sqrt{\alpha}t\sqrt{16\beta_2(\gamma-\omega)-3\beta_1^2}}{2\sqrt{\beta_2}} + x\right)\right)}{8\beta_1} - \frac{3\beta_1}{8\beta_2}} \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{32\beta_2\gamma - 32\beta_2\omega - 6\beta_1^2}}{4\sqrt{2}\sqrt{\alpha}\sqrt{\beta_2}}\right)\right) \quad (129)$$

$$q_{7,4}(x, t) = \sqrt{-\frac{3\beta_1}{8\beta_2} - \frac{3\sqrt{\alpha}\sqrt{-\frac{\beta_1^2}{\beta_2}}\sqrt{\frac{\beta_1^2}{\alpha\beta_2}} \cot\left(\frac{1}{4}\sqrt{3}\sqrt{\frac{\beta_1^2}{\alpha\beta_2}}\left(\frac{\sqrt{\alpha}t\sqrt{16\beta_2(\gamma-\omega)-3\beta_1^2}}{2\sqrt{\beta_2}} + x\right)\right)}{8\beta_1}} \times \exp\left(i\left(\theta + t\omega - \frac{x\sqrt{32\beta_2\gamma - 32\beta_2\omega - 6\beta_1^2}}{4\sqrt{2}\sqrt{\alpha}\sqrt{\beta_2}}\right)\right) \quad (130)$$

4. Analysis of Conservation Laws

In this section, we will study the conservation laws (CLs) of the unstable nonlinear Schrödinger equation using the direct method. This method involves using multipliers. To do this, we first convert the equation into a system of nonlinear partial differential equations (NLPDEs) using the following steps. transformation [10, 15]:

$$q(x, t) = u(x, t) + iv(x, t), \quad (131)$$

where $u(x, t)$ and $v(x, t)$ are functions. Substituting Eq. (131) into Eq. (1) and separating the real and imaginary parts, we get:

$$\begin{aligned} -u_t + \alpha u_{xx} + \beta(u^2 + v^2)u + \gamma u &= 0 \\ v_t + \alpha v_{xx} + \beta(u^2 + v^2)v + \gamma v &= 0 \end{aligned} \quad (132)$$

Next, we will provide a brief description of the techniques and then apply the concepts to the generalized unstable nonlinear Schrödinger equation.

4.1. The Multiplier Approach

Let $x = (x_1, x_2, \dots, x_n)$ represent n independent variables, and $u = (u^1, u^2, \dots, u^m)$ represent m dependent variables. We consider a system of r PDEs of k^{th} -order represented by [15]:

$$Q_\alpha[u] = Q_\alpha(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}), \quad \alpha = 1, 2, \dots, r, \quad (133)$$

with $u_{(1)} = \{u_{(i)}^\alpha\}$, $u_{(2)} = \{u_{(ij)}^\alpha\}$, $\{u_{(i)}^\alpha\} = \frac{\partial u_i^\alpha}{\partial x_i}$, $\{u_{(ij)}^\alpha\} = \frac{\partial^2 u_i^\alpha}{\partial x_i \partial x_j}, \dots$. We let $V = (V^1, V^2, \dots, V^N)$ denote arbitrary functions of the independent variables x and denote partial derivatives $\frac{\partial}{\partial x_i}$ by subscripts i [15], i.e., $V_i^\sigma = \frac{\partial V^\sigma}{\partial x_i}$, $V_{ij}^\sigma = \frac{\partial^2 V^\sigma}{\partial x_i \partial x_j}$, etc.

1. The local conservation laws Multiplier is given in the form

$$\Gamma = \Gamma(x, t, u, v) \quad (134)$$

2.

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u_i^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_i^\alpha} + u_{ijk}^\alpha \frac{\partial}{\partial u_{jk}^\alpha} + \dots \quad (135)$$

where $i, j, k, \dots = 1, 2, \dots, m$.

3. Multipliers for system Eq. (133) are a set of functions $\{\Gamma^\alpha[V]\}$ satisfying:

$$\Gamma^\alpha[V]Q_\alpha[V] = D_i W^i[V], \quad (136)$$

for some functions $W^i[V]$. If $V^\sigma = V^\sigma(x)$ is the solution of PDE Eq. (132), from Eq. (136), we obtain the CLs [15]:

$$D_i W^i[V] = 0 \quad (137)$$

of Eq. (133) and for each i , $W^i[V]$ is a flux.

4. The standard Euler operators with respect to the differential function V^j and the derivatives V_j^i , $V_{i_1 i_2}^j$, etc., are defined by:

5.

$$E_V^j = \frac{\partial}{\partial V^j} - D_i \frac{\partial}{\partial V_i^j} + \dots + (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial V_{i_1 \dots i_s}^j} \quad (138)$$

for each $j = 1, 2, \dots, m$. $\{\Gamma^\alpha[V]\}$ yields a set of multipliers for the CLs of Eq. (133) if each Euler operator in Eq. (138) annihilates the left side of Eq. (136):

$$E_V^j(\Gamma^\alpha[V]Q_\alpha[V]) \equiv 0, \quad j = 1, \dots, n \quad (139)$$

for arbitrary V, V_i, V_{ij}, \dots , etc.

4.2. Application of the Multiplier Approach to Generalized Unstable NLSE

Substituting Eq. (132) in Eq. (139) multiplied by Γ yield

$$E_u(\Gamma^1(x, t, u, v)(-u_t + \alpha u_{xx} + \beta(u^2 + v^2)u + \gamma u + \Gamma^2(x, t, u, v)(v_t + \alpha v_{xx} + \beta(u^2 + v^2)v + \gamma v))) = 0 \quad (140)$$

After Expansion, with respect to different combinations of derivatives of u and v yield the following overdetermined system for the multipliers Γ^1 , and Γ^2 :

$$\begin{aligned} \Gamma_t^1 = 0, \Gamma_t^2 = 0, \Gamma_x^1 = 0, \Gamma_x^2 = 0, \Gamma_u^1 = 0, \Gamma_u^2 = 0, \Gamma_v^1 = 0, \Gamma_v^2 = 0, \Gamma_{u_t}^1 = 0, \Gamma_{u_t}^2 = \frac{\Gamma^2 v_x + \Gamma^1 u_x}{u_t v_x - u_x v_t}, \\ \Gamma_{v_t}^1 = \frac{-\Gamma^2 v_x - \Gamma^1 u_x}{u_t v_x - u_x v_t}, \Gamma_{v_t}^2 = 0, \Gamma_{u_x}^1 = 0, \Gamma_{u_x}^2 = \frac{-\Gamma^2 v_t + \Gamma^1 u_t}{u_t v_x - u_x v_t}, \Gamma_{v_t}^1 = \frac{\Gamma^2 v_t + \Gamma^1 u_t}{u_t v_x - u_x v_t}, \Gamma_{v_x}^2 = 0 \end{aligned} \quad (141)$$

Solving the determining system of PDEs Eq. (141), we obtain the following zeroth-order multipliers $\Gamma^1(x, t, u, v, u_t, v_t, u_x, v_x)$, and $\Gamma^2(x, t, u, v, u_t, v_t, u_x, v_x)$ for the model, which are given by:

$$\begin{aligned} \Gamma^1 &= -v_t C_2 - v_x C_1, \\ \Gamma^2 &= u_t C_2 + u_x C_1 \end{aligned} \quad (142)$$

where C_1 , and C_2 are constants.

Using Eq(140) and Eq(142) we obtained the following fluxes equations:

$$\text{flux}_t = -\frac{1}{4}, u^4 \beta C_2 - \frac{1}{2} v^2 u^2 \beta C_2 - \frac{1}{2}, u^2 \gamma C_2 + \frac{1}{2} \alpha v v_{xx} C_2 - v^3 \beta v_x C_1 t - \gamma v v_x C_1 t - u v_x C_1 - \frac{1}{2} \alpha u_{xx} C_2 \quad (143)$$

$$\begin{aligned} \text{flux}_x &= -\frac{1}{4}, u^4 \beta C_1 - \frac{1}{2} u^2 v^2 \beta C_1, -\frac{1}{2}, \gamma u^2 C_1 + v^3 v_t \beta C_1 t + \gamma v v_t C_1 t - v_t v_x \alpha C_2 \\ &\quad + u v_t C_1 + \frac{1}{2} u_{xx} \alpha C_1 - \frac{1}{2} v_{xx} \alpha C_1 + u_t u_x \alpha C_2 \end{aligned} \quad (144)$$

From the obtained flux we get the following conserved vectors.

1. If $c_1 = 1$; $c_2 = 0$; then we have the following conserved vectors:

$$\Gamma^1 = v_x, \quad \Gamma^2 = u_x$$

$$Z^t = -v^3 v_x \beta t - v v_x \gamma t - u v_x \quad (145)$$

$$Z^x = -\frac{1}{4}\beta u^4 - \frac{1}{2}\beta v^2 u^2 - \frac{1}{2}\gamma u^2 + v^3 v_t \beta t + \gamma v v_t t + v_t u + \frac{1}{2}\alpha u_{xx} - \frac{1}{2}\alpha v_{xx} \quad (146)$$

2. If $c_1 = 0$; $c_2 = 1$; then we have the following conserved vectors:

$$\Gamma^1 = v_t, \quad \Gamma^2 = u_t$$

$$Z^t = -\frac{1}{4}\beta u^4 - \frac{1}{2}\beta v^2 u^2 - \frac{1}{2}\gamma u^2 + \frac{1}{2}\alpha v_{xx} - \frac{1}{2}\alpha u_{xx} \quad (147)$$

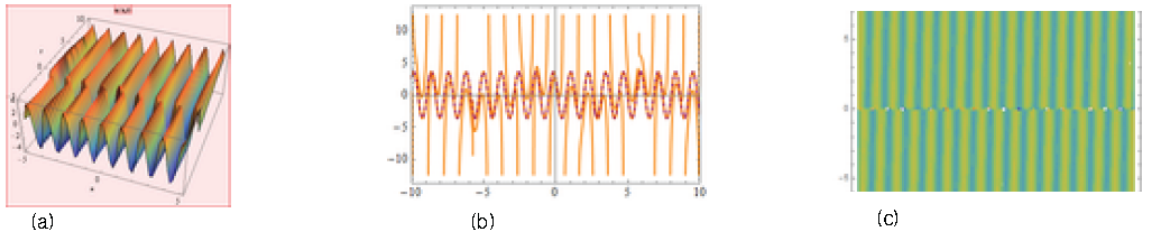
$$Z^x = u_t u_x \alpha - v_t v_x \alpha \quad (148)$$

5. Results and Discussions

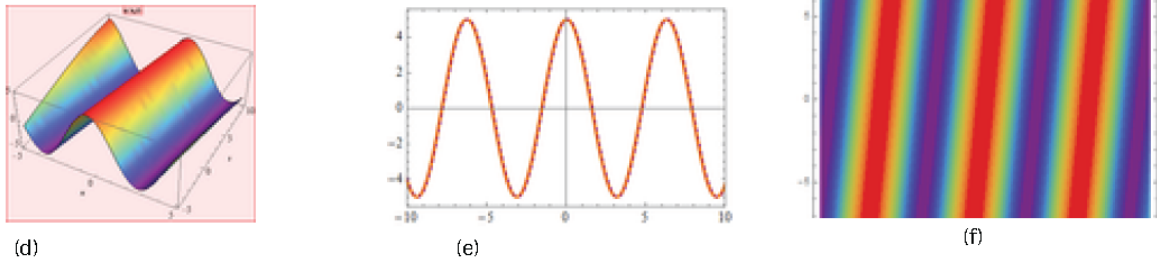
This section presents graphical depictions of various solutions and explains their outcomes. The solutions in Equation (36) show dark solitons, while those in Equations (40) and (43) show bright-dark optical solitons using the complex envelope function ansatz method. Additionally, the Bernoulli sub-ODE method and generalized Tanh method

reveal different types of solitons, including dark, bright, and singular solitons. The following analysis provides a detailed understanding of the nonlinear optical phenomena being studied.

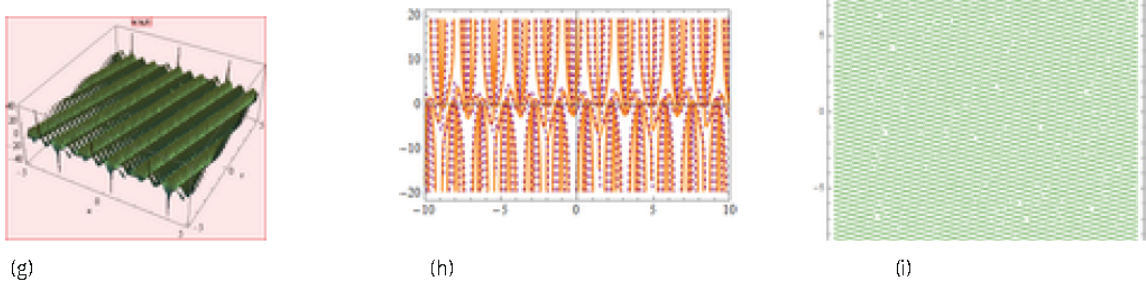
To gain a thorough understanding of the physical properties of these solitons, we use specific parameter values to create 3D, 2D, and contour plots for some of the dark and singular soliton solutions.



This sketch is a graphical depiction of the solution (36) for the values of $\omega = 0.1, \lambda = 3, \alpha = 1, \beta = 2, \Omega = 2, \gamma = 1, \theta = 3\pi/2$ on an intervals $-5 \leq x \leq 5$ and $10 \leq t \leq 10$ and $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for the 2D graph.

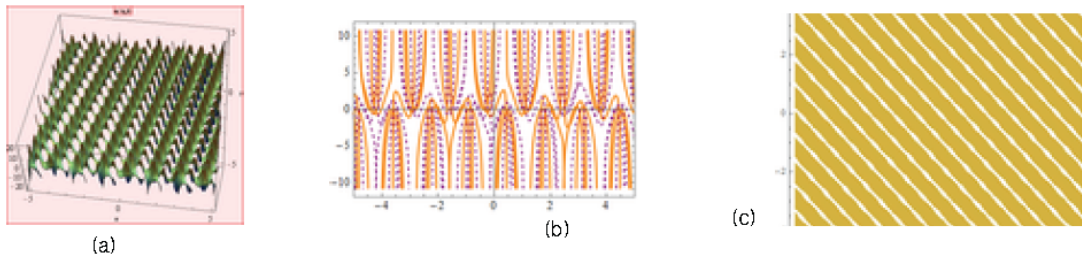


This sketch is a graphical depiction of the solution (40) for the values of $\omega = 0.1, \rho = 3, k = 1, \Omega = 2, \theta = 3\pi/2$ on an intervals $-5 \leq x \leq 5$ and $10 \leq t \leq 10$ and $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for the 2D graph.

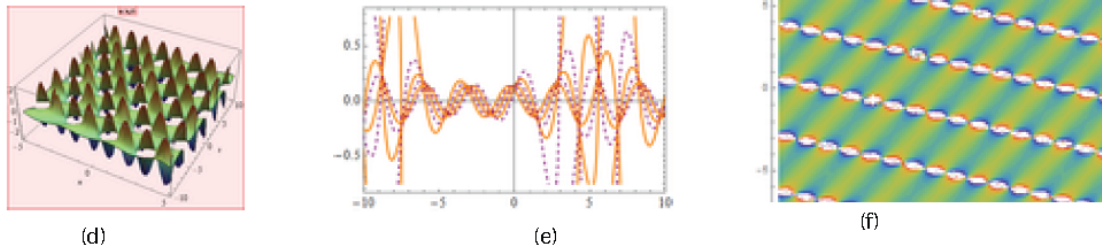


This sketch is a graphical depiction of the solution (43) for the values of $\omega = 1, \lambda = 5, \alpha = 0.5, \beta = 0.3, \gamma = 0.1, \theta = 3\pi/2$ on an intervals $-10 \leq x \leq 10$ and $10 \leq t \leq 10$ and $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for the 2D graph.

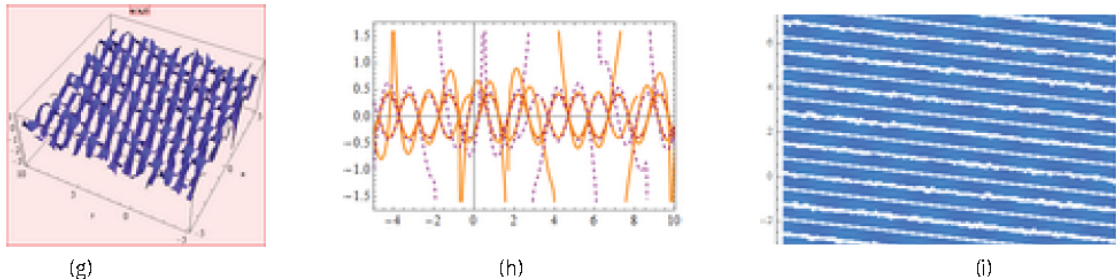
Figure 1. Three-dimensional (3D), two-dimensional (2D), and contour plots of the unstable nonlinear Schrödinger equation (UNSE) illustrating the dynamics of the model. The plots demonstrate dark and dark-bright soliton solutions.



This sketch is a graphical depiction of the solution (103) for the values of $\omega = 1$, $k = 15$, $\alpha = 0.1$, $\beta = 0.2$, $\gamma = 0.2$, $\theta = 3\pi/2$ on an intervals $-5 \leq x \leq 5$ and $5 \leq t \leq 10$ and $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for the 2D graph.



This sketch is a graphical depiction of the solution (112) for the values of $\omega = 1$, $k = 1$, $\alpha = 1$, $\beta = 2.5$, $\gamma = 10$, $\theta = \pi/4$ on an intervals $-5 \leq x \leq 5$ and $5 \leq t \leq 10$ and $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for the 2D graph.



This sketch is a graphical depiction of the solution (127) for the values of $\omega = 1$, $k = 1$, $\alpha = 1$, $\beta = 2.5$, $\gamma = 10$, $\theta = \pi/4$ on an intervals $-5 \leq x \leq 5$ and $5 \leq t \leq 10$ and $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for the 2D graph.

Figure 2. Three-dimensional (3D), two-dimensional (2D), and contour plots of the unstable nonlinear Schrödinger equation (UNSE) illustrating the dynamics of the model. The plots demonstrate dark, bright and singular soliton solutions.

6. Conclusions

In conclusion, research on Generalized Unstable Nonlinear Schrödinger Equations is a broad and exciting field that could lead to important discoveries and useful applications. We present a wide range of wave phenomena, such as solitons, rogue waves, and nonlinear effects, through thorough research and analysis. The inherent complexity of wave propagation in nonlinear systems is brought to light by these equation.

Finally, we obtained conservation laws of the equation using multiplier approaches.

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Authors Contribution

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Conflicts of Interest

The authors declare no conflicts of interest.

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