American Journal of Theoretical and Applied Statistics
2026, Vol. 15, No. 5, pp. 303–316
https://doi.org/10.11648/j.ajtas.20261505.18

Research Article

Stochastic Optimal Control of Perishable Inventory Under Demand Uncertainty

Ashish Kumar Routray, Shyam Sundar Sahoo*, Swagatika Dehury, Sudhir Kumar Sahu

Department of Statistics, Ravenshaw University, Cuttack, India

Abstract

The deterministic optimal-control literature on perishable inventory management including the dynamic-pricing, preservation investment, and advertising framework this paper extends assumes that demand evolves along a known, smooth trajectory. Real retail environments violate this assumption: freshness-sensitive demand is buffeted by continuous stochastic fluctuation and by discrete shocks such as viral promotions, weather events, and supply disruptions. This paper develops a continuous-time stochastic optimal control model in which the stock-in-phase inventory process follows a jump diffusion stochastic differential equation combining Brownian demand noise with a compound Poisson jump component, while dynamic pricing, warehousing preservation investment, and advertising remain the retailer's control variables. We derive necessary optimality conditions via a Stochastic Maximum Principle with a backward stochastic adjoint equation, establish a verification theorem linking the adjoint process to the value function of the associated integro-partial differential Hamilton-Jacobi-Bellman (HJB) equation, and show that the linear demand or quadratic-cost structure of the problem reduces the HJB equation to a solvable system of Riccati type ordinary differential equations, yielding closed-form feedback laws for the optimal price and advertising rate. As an independent, non-parametric benchmark, we additionally solve the integro PDE HJB equation via an explicit finite-difference dynamic programming scheme; the two solution methods agree to within 0.03% across the feasible inventory range, cross-validating the analytical reduction. Numerical experiments show that the optimal expected discounted profit is monotonically decreasing in jump intensity and jump severity (pure downside risk from unhedgeable demand shocks) but weakly increasing in diffusion volatility, a consequence of the convexity of the value function in the inventory state an option-like effect specific to the risk-neutral, linear-quadratic stochastic control formulation. Monte-Carlo simulation of 3,000 sample paths confirms that a policy computed under the stochastic formulation reduces downside tail exposure relative to a deterministic-equivalent (risk-naïve) policy under otherwise identical demand realizations. The paper contributes the first stochastic maximum principle treatment of the joint pricing-preservation advertising problem for perishable goods and the first dynamic-programming cross-validation of such a solution in this literature.

Keywords

Stochastic Optimal Control, Stochastic Maximum Principle, Jump-diffusion Demand, Dynamic Pricing, Preservation Investment, Hamilton-Jacobi-Bellman Equation, Dynamic Programming, Riccati Equation

*Correspondence: Shyam Sundar Sahoo (shyamsahoo8@gmail.com)

Received: 10 July 2026; **Accepted:** 21 July 2026; **Published:** 31 October 2026

Copyright: © The Author(s), 2026. Published by Science Publishing Group. This is an **Open Access** article, distributed under the terms of the Creative Commons Attribution 4.0 License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Perishable inventory management sits at the intersection of pricing, promotion, and quality preservation, and a substantial continuous-time optimal-control literature most recently the dual-preservation, freshness-and advertising framework of Dehury et al. [7], which this paper takes as its point of departure has shown that jointly optimizing these levers via Pontryagin's Maximum Principle (PMP) yields analytically tractable, economically interpretable decision rules. That literature, however, is uniformly deterministic: demand is assumed to follow a known function of price, freshness, inventory visibility, and advertising, with no allowance for the unhedgeable variability that characterizes real retail demand. Two forms of this variability are economically distinct and are conflated when demand is treated as deterministic. The first is continuous, unpredictable day-to-day fluctuation around the deterministic demand trend captured naturally by a diffusion term. The second is discrete, discontinuous demand shocks a viral social-media mention, a competitor stock-out, a flash promotion, or a weather event that instantaneously deplete inventory by an amount unrelated to the smooth demand trend. Recent empirical and methodological work confirms both channels matter: Chehrazi, Sanders, and Stamatopoulos [4] show that inventory information frictions materially distort optimal pricing in perishable groceries, while Li and Huang [13] show that state-dependent jump risk fundamentally changes the structure of optimal inventory control relative to pure-diffusion benchmarks.

This paper closes that gap by embedding the dynamic-pricing and advertising control problem inside a jump-diffusion stochastic optimal control framework and solving it via the Stochastic Maximum Principle (SMP) of Peng [15], with a dynamic-programming (HJB) cross-validation. Three considerations motivate this specific combination of methods rather than either alone. First, the SMP delivers pointwise necessary conditions a Hamiltonian and a backward stochastic adjoint (costate) equation that generalize the deterministic PMP conditions in a form directly comparable to the antecedent literature, preserving interpretability of the resulting price and advertising rules as shadow price-driven feedback laws. Second, because the demand function is linear in the controls and the running profit is quadratic in price and advertising, the resulting HJB equation admits a quadratic value-function ansatz, reducing an otherwise intractable integro PDE to a system of ordinary Riccati equations that can be solved to high numerical precision by standard ODE software avoiding the curse of dimensionality that a purely numerical dynamic-programming solution would face in richer (multi-item or higher-dimensional) extensions. Third, solving the same HJB equation independently via an explicit finite difference scheme without invoking the quadratic ansatz provides an internal verification that the closed-form Riccati solution is not an artifact of the ansatz but the true optimal value function, a cross-validation step that is rarely reported in the deterministic PMP literature on perishable inventory and that we regard as essential once stochastic dynamics make closed-form verification non-obvious.

The specific contributions of this paper are as follows.

1) *A jump-diffusion stochastic extension of the joint pricing-preservation and advertising control problem.* We formulate the stock-in-phase inventory process as an Itô jump-diffusion stochastic differential equation, combining multiplicative Brownian demand noise with an additive compound-Poisson jump component representing discrete demand shocks, while retaining the freshness-sensitive, stock-visibility and sensitive, advertising-responsive demand structure of the deterministic antecedent.
2) *Stochastic Maximum Principle derivation with adjoint BSDE.* We derive the Hamiltonian, the backward stochastic adjoint equation for the costate process, and the pointwise first-order optimality conditions for price and advertising, and state the verification theorem linking the SMP solution to the HJB value function.
3) *Closed-form Riccati reduction.* Exploiting the linear-quadratic structure of the problem, we show the HJB equation reduces to three coupled Riccati-type ordinary differential equations, and derive closed-form feedback laws for the optimal price and advertising rate as affine functions of the inventory state.
4) *Independent dynamic-programming verification.* We solve the same integro PDE HJB equation via an explicit finite-difference dynamic-programming scheme that does not assume a quadratic value function, and show the two solutions agree to within a fraction of one percent across the feasible inventory range a cross-validation not previously reported in this literature.
5) *Risk-structure results and simulation evidence.* We prove and numerically confirm that the optimal expected discounted profit is monotonically decreasing in jump intensity and jump severity but weakly increasing in diffusion volatility an option-like convexity effect and use Monte-Carlo simulation to quantify the downside-risk value of stochastic aware control relative to a deterministic equivalent policy.

The remainder of the paper is organized as follows. Section 2 reviews three literature streams deterministic dynamic pricing or preservation models for perishables, stochastic and jump-diffusion inventory control, and distributionally robust and data-driven pricing and positions the present study against the closest antecedents via a comparative table. Section 3 states the model assumptions, notation, and the jump-diffusion state dynamics. Section 4 develops the Stochastic Maximum Principle formulation: Hamiltonian, adjoint BSDE, and the HJB equation. Section 5 derives the closed-form Riccati reduction and the resulting feedback price and advertising laws, together with formal propositions on the risk-structure

of the value function. Section 6 reports the dynamic-programming verification, simulation experiments, and sensitivity analysis. Section 7 discusses managerial implications. Section 8 concludes with limitations and directions for future research.

2. Literature Review

The literature relevant to this study spans three streams: (i) deterministic dynamic pricing and preservation-investment models for perishable inventory, (ii) stochastic and jump-diffusion inventory and pricing control, and (iii) distributionally robust and data-driven approaches to pricing under demand ambiguity.

2.1. Deterministic Dynamic Pricing and Preservation-Investment Models

The immediate antecedent of this study is Dehury et al. [7], who formulate a continuous-time optimal control model in which demand depends jointly on price, freshness, stock-level visibility, and advertising across a two-phase replenishment cycle, applying Pontryagin's Maximum Principle to derive closed-form pricing and advertising rules via a reduction to a linear ordinary differential equation. That paper synthesizes and extends a line of deterministic models dynamic pricing and preservation investment under freshness and price-sensitive demand, controllable lead time, dual-stage (transit and warehousing) preservation decomposition, and waiting time-dependent partial backlogging that collectively established dynamic pricing, preservation, and advertising as jointly optimizable, analytically tractable control problems. This literature stream, however, is uniformly built on deterministic demand trajectories, leaving the model's behaviour under genuine demand uncertainty analytically unexamined.

2.2. Stochastic and Jump-Diffusion Inventory and Pricing Control

A parallel and largely separate literature formulates pricing and inventory control directly as stochastic optimal control problems. Chehrazi, Sanders, and Stamatopoulos [4] show empirically that frictions in inventory information explain observed price rigidity in perishable groceries, underscoring that stochastic, imperfectly observed inventory dynamics are not a theoretical nicety but a first order empirical feature of perishable retail. On the methodological side, Li and Huang [13] study an inventory control problem under a reflected jump-diffusion net-flow process with state-dependent jump intensity, establishing well-posedness of the associated integro differential HJB equation providing the closest methodological antecedent for the jump diffusion HJB equation solved numerically in Section 6 of this paper. Kavoosi et al. [11] instead approach joint dynamic pricing and inventory control for perishables via deep reinforcement learning, sidestepping analytical tractability in exchange for the ability to handle rich, high-dimensional demand uncertainty structures; their results are complementary to, rather than substitutable for, the closed-form feedback laws obtained here, and their DRL benchmarking methodology represents a natural extension of the present model to settings where the linear-quadratic ansatz fails. Xie and Xie [19] apply neural-network and fuzzy-logic control strategies to intelligent warehousing inventory decisions, illustrating a further data-driven alternative to the analytical stochastic control pursued in this paper. Classical methodological foundations for this stream include Peng's [15] general Stochastic Maximum Principle, on which Section 4 is built, Bensoussan, Liu, and Sethi's [1] quasi-variational inequality treatment of (s, S) policies under compound Poisson and diffusion demand, and Yong and Zhou's [22] unified treatment of stochastic Hamiltonian systems and HJB equations, which underlies the verification theorem stated in Section 4.4.

2.3. Distributionally Robust and Data-Driven Pricing Under Demand Ambiguity

A third, closely related stream avoids committing to a fully specified stochastic demand model altogether, instead optimizing against the worst case over an ambiguity set of demand distributions. Kuhn, Shafiee, and Wiesemann [12] provide a comprehensive treatment of distributionally robust optimization (DRO) methodology; Cai, Li, and Mao [2] extend the moment based DRO paradigm to distorted expectations, and Jiang, Cheng, Pan, and Shen [10] develop dimensionality reduction techniques for moment-based DRO that are directly relevant to scaling the present single-item model to a multi-item assortment. In the inventory routing domain, Hasturk et al., Jia, Schrotenboer, and Chen, and Che, Feng, and Fu [3, 8, 9] apply robust and distributionally robust formulations to sequential replenishment decisions under demand and supply uncertainty, illustrating the broader methodological trend toward ambiguity robust rather than fully parametrically specified stochastic control in inventory management. The present paper deliberately adopts the fully specified stochastic (rather than distributionally robust) formulation because it is the formulation under which the Stochastic Maximum Principle and the linear-quadratic Riccati reduction are applicable in closed form; extending the model to a distributionally robust jump diffusion setting is identified as a direction for future research in Section 8.

2.4. Adjacent Preservation-Technology and Green-Inventory Extensions

A further body of 2024-2026 work extends deterministic deteriorating-item models along complementary dimensions that this paper does not pursue but that indicate the directions in which the present stochastic framework could next be extended: Yadav, Yadav, and Bansal [20] and Yadav, Kumar, Yadav, and Rathee [21] integrate preservation technology in-

vestment with carbon-emission and green-technology considerations under interval-valued and selling-price and time-sensitive demand respectively, Routray, Dehury, and Sahu develop a three-warehouse deteriorating-item model integrating ramp-type demand, carbon cap-and-trade, and two-part trade credit under inflation [14], Rani, Maheshwari, and Sharma [17] integrate eco-packaging, advertising, and carbon-tax constrained pricing, Das, Mandal, Akhtar, Shaikh, and Bhunia [5] extend Pontryagin's Maximum Principle itself to interval-valued control problems for imperfect production systems, Das, Akhtar, Shaikh, Manna, and Konstantaras [6] study emission effort and rework policy via equilibrium optimizer algorithms, Praveen and Manoharan [16] examine two-level trade-credit financing under expiration date constraints; and Sahedev, Kumar, Singh, and Chauhan [18] develop a fuzzy stock-dependent demand model with salvage value under deferred payments. None of these studies, however, embeds a jump-diffusion or general stochastic demand process within a Stochastic-Maximum Principle solved, dynamic-programming verified control framework, which is the specific gap this paper addresses.

2.5. Comparative Summary and Positioning

Table 1. *positions the present study against its closest antecedents along the dimensions most relevant to stochastic perishable inventory control.*

Study	Demand	Control variables	Solution method	Stochastic verification
Dehury et al. [7], base model	Deterministic (price, freshness, stock, advertising)	Price, advertising, dual preservation, lead time	Pontryagin's Maximum Principle → linear ODE	N/A
Chehrazi, Sanders & Stamatopoulos [4]	Stochastic; information frictions	Price (empirical)	Empirical / structural estimation	N/A
Li & Huang [13]	Reflected jump-diffusion, state-dependent jumps	Inventory control (no pricing/advertising)	Integro-differential HJB, viscosity solutions	Analytical well-posed Ness only
Kavoosi et al. [11]	Stochastic (learned)	Price, inventory	Deep reinforcement learning	None (model-free)
Kuhn, Shafiee & Wiesemann; Cai, Li & Mao [2, 12]	Distributionally ambiguous (moment-based)	Pricing / inventory (general)	Distributionally robust optimization	N/A
This Study	Jump-diffusion (Brownian + compound Poisson)	Price, advertising, warehousing preservation	Stochastic Maximum Principle + closed-form Riccati reduction	Independent finite-difference DP/HJB benchmark

As Table 1 shows, the present study is the first to combine (i) a jointly optimized price advertising-preservation control vector, (ii) jump diffusion (rather than pure-diffusion or purely empirical) demand uncertainty, (iii) a Stochastic-Maximum Principle solution reduced to closed form via a Riccati system, and (iv) an independent dynamic-programming verification of that closed-form solution.

3. Model Assumptions, Notation, and Stochastic State Dynamics

3.1. Assumptions

A1. Two-phase cycle. As in the deterministic antecedent, each replenishment cycle comprises a stock-out phase $[0, t_l]$ governed by waiting time dependent partial backlogging and a stock-in phase $[t_l, T]$ in which the retailer holds physical inventory. This paper's stochastic extension applies to the stock-in phase, where dynamic pricing, advertising, and warehousing preservation are jointly controlled, the stock-out phase retains the deterministic backlogging and transit-preservation structure of the antecedent model, since the retailer's inability to observe backlogged demand in real time makes stochastic control of that phase a materially different and separable problem left for future work.

A2. Freshness-and-stock-sensitive linear demand. The instantaneous demand rate is $D(t) = a - b\,p(t) + \alpha I(t) + \phi\, g(t)\, A(t)$, where $g(t) = k_1 e^{-\delta(t-t_\ell)}$ is the exogenous freshness-decay trajectory carried over from the deterministic

model, $a, b, \alpha, \phi > 0$, and $A(t) \geq 0$is the advertising control.

A3. Jump-diffusion inventory dynamics. Between demand realizations, inventory depletes continuously at the deterministic demand rate and the effective deterioration rate $\delta = \theta(1-m_2)$, is perturbed by continuous Brownian demand noise proportional to the inventory level (multiplicative diffusion, capturing proportionally larger unpredictability when more stock is on the shelf), and is subject to discrete downward jumps arising from unhedgeable demand shocks.

A4. Compound-Poisson jump structure. Jumps arrive as a Poisson process $N(t)$ with constant intensity λ_J; the k-th jump removes a random quantity Y_k of inventory, with $\{Y_k\}$i.i.d., independent of the driving Brownian motion, mean μ_Y, and second moment $\mathbb{E}[Y^2]$.

A5. Warehousing preservation only. Preservation investment ξ_2 acts on the effective deterioration rate exactly as in the deterministic model, $m(\xi_2) = \xi_2/(1+\xi_2)$; transit preservation ξ_1 continues to apply only in the (retained, deterministic) stock-out phase.

A6. Risk-neutral discounted-profit objective. The retailer maximizes the expectation of total discounted profit; this is the standard objective in the stochastic inventory pricing literature (e.g., Li and Huang [13]) and permits the closed-form Riccati reduction of Section 5. Risk averse objectives (e.g., exponential utility or mean-variance) are discussed as an extension in Section 8.

A7. Admissible controls. p(t) and A(t) ≥ 0 are progressively measurable, square-integrable processes adapted to the filtration generated by the Brownian motion and the Poisson random measure.

3.2. Notation

Table 2. *Notation Used in the Stochastic Optimal Control Model.*

Symbol	Definition
$I(t)$	Inventory level during the stock-in phase (stochastic process)
$p(t), A(t)$	Price and advertising-rate control processes
a, b, α, ϕ	Demand-function intercept, price-sensitivity, stock-visibility, and advertising-effectiveness coefficients
$g(t)$	Exogenous freshness-decay trajectory, $g(t) = k_1 e^{-\delta(t-t_\ell)}$
δ, θ, m_2	Effective/natural deterioration rate and warehousing preservation impact, $\delta = \theta(1-m_2)$
κ	Combined depletion coefficient, $\kappa = \alpha + \delta$
σ	Diffusion (Brownian) volatility coefficient of the inventory process
$W(t)$	Standard Brownian motion driving continuous demand noise
$N(t), \lambda_J$	Poisson jump-count process and its constant intensity
$Y_k, \mu_Y, \mathbb{E}[Y^2]$	i.i.d. jump-size random variables, mean and second moment
$J(t)$	Compound-Poisson jump process, $J(t) = \sum_{k=1}^{N(t)} Y_k$
c_m, h, γ, ξ_2	Unit purchase cost, holding cost, advertising cost-convexity coefficient, warehousing preservation expenditure
r	Discount rate
$\pi(t, I, p, A)$	Instantaneous profit rate
$W(t, I)$	Value function (expected discounted profit-to-go)
$\lambda(t), \psi(t)$	Adjoint (costate) processes in the Stochastic Maximum Principle
$\chi(t, y)$	Jump component of the adjoint BSDE
$P(t), Q(t), R(t)$	Riccati-system coefficients in the quadratic value-function ansatz
$\Phi(t), \Psi(t)$	Composite coefficients, $\Phi(t) = \frac{\phi^2 g(t)^2}{\gamma}, \Psi(t) = 2b - \Phi(t)$

3.3. Stochastic State Dynamics

Combining A2-A4, the stock-in-phase inventory process evolves as the jump-diffusion stochastic differential equation.

$$dI(t) = [-D(t) - \delta I(t)]dt + \sigma I(t)\, dW(t) - dJ(t), J(t) = \sum_{k=1}^{N(t)} Y_k. \quad (1)$$

where the deterministic drift component is minus demand less deterioration/depletion,

$$D(t) = a - b\, p(t) + \alpha I(t) + \phi\, g(t)\, A(t). \quad (2)$$

so that the full drift of the SDE is $\mu(t, I, p, A) = -D(t) - \delta I(t) = -a + b\, p(t) - \phi\, g(t)A(t) - \kappa I(t)$, with $\kappa = \alpha + \delta$. The multiplicative diffusion term $\sigma I(t)\, dW(t)$ captures continuous demand noise whose absolute magnitude scales with the amount of stock exposed on the shelf; the jump term subtracts a compound-Poisson-distributed quantity representing discrete demand surges. This specification nests the deterministic antecedent as the degenerate case $\sigma = 0, \lambda_J = 0$.

4. Stochastic Optimal Control Formulation

4.1. Objective Functional

The retailer chooses price and advertising processes to maximize expected total discounted profit over the stock-in phase,

$$V(t_\ell, I_0) = \sup_{p(\cdot),\, A(\cdot) \geq 0} \mathbb{E}\left[\int_{t_\ell}^{T} e^{-r(t-t_\ell)} \Big| \pi\left(t, I, (t), p(t)A(t)\right) dt \;\; I(t_\ell) = I_0\right] \quad (3)$$

where the instantaneous profit rate is

$$\pi(t, I, p, A) = (p - c_m)[a - bp + \alpha I + \phi g(t)A] - hI - \frac{1}{2}\gamma A^2 - \xi_2 \quad (4)$$

net of purchase cost, holding cost, convex advertising expenditure, and the (constant, per assumption A5) flow cost of warehousing preservation ξ_2.

4.2. Stochastic Maximum Principle: Hamiltonian and Adjoint BSDE

Following Peng's [15] general Stochastic Maximum Principle, define the Hamiltonian.

$$H(t, I, p, A, \lambda, \psi) = \pi(t, I, p, A) + \lambda(t)\, \mu(t, I, p, A) + \psi(t)\, \sigma I. \quad (5)$$

where λ(t) is the costate (adjoint) process associated with the drift, ψ(t) is the costate associated with the diffusion coefficient, and an additional jump adjoint term (suppressed above for brevity and stated explicitly below) is associated with the compound Poisson component. The adjoint process solves the backward stochastic differential equation (BSDE).

$$d\lambda(t) = -\frac{\partial H}{\partial I}\, dt + \psi(t)\, dW(t) + \int \chi\,(t, y)\, \tilde{N}(d, tdy), \lambda(T) = 0. \quad (6)$$

with χ (t, y) the jump component of the adjoint and Ñ the compensated Poisson random measure associated with the jump sizes. The Pontryagin type necessary conditions require the controls to maximize the Hamiltonian pointwise in (t, ω):

$$\frac{\partial H}{\partial p} = 0, \frac{\partial H}{\partial A} = 0. \quad (7)$$

4.3. Hamilton-Jacobi-Bellman Equation

Equivalently, define the value function W(t, I) = sup E [∫ discounted profit | I(t)=I]. Dynamic programming yields the integro partial-differential HJB equation.

$$-\frac{\partial W}{\partial t} = \sup_{p,\, A \geq 0} \left\{\pi + \mu\, \frac{\partial W}{\partial I} + \frac{1}{2}\sigma^2 I^2 \frac{\partial^2 W}{\partial I^2} - rW + \lambda_J\, \mathbb{E}\left[\, W(t, I - Y) - W(t, I)\, \right]\right\} \quad (8)$$

where the expectation in the final (jump) term is taken over the jump-size distribution of Y, and the supremum is taken pointwise over the admissible controls at each (t, I).

4.4. Verification Theorem

The link between the SMP adjoint process and the HJB value function needed because the SMP gives only necessary conditions along optimal trajectories, while a verification argument is required to confirm sufficiency is provided by the standard stochastic verification theorem (Yong and Zhou, [22], Ch. 4): if W ∈ $C^{(1,2)}$ solves the HJB equation with the stated boundary condition and the pointwise maximizing controls (p*, A*) generate an admissible state process I*(t), then W is

the value function and the costate process in the SMP coincides with the spatial gradient of W along the optimal trajectory,

$$W(t,I) = \sup_{p,A} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} \pi \, ds \right] \Leftrightarrow \lambda(t) = \frac{\partial W}{\partial I} \left(t, I(t) \right) \tag{9}$$

together with the transversality condition, $\lim_{t \to T} \lambda(t) I(t) = 0$. This equivalence is what allows Section 5 to construct the SMP optimal feedback laws directly from a value function ansatz, and is confirmed numerically in Section 6.2 by comparing the closed-form value function against an independent dynamic programming solution of the same HJB equation.

5. Closed-form Solution via Riccati Reduction

5.1. Quadratic Value-Function Ansatz

Because the profit rate π is quadratic in (p, A) with a strictly concave quadratic term in p(coefficient $-b < 0$) and the drift μ is linear in (I, p, A), the problem has a linear-quadratic (LQ) stochastic control structure with a state-control cross term $(\alpha p I)$. This structure is closed under the quadratic ansatz.

$$W(t,I) = P(t)I^2 + Q(t)I + R(t), P(T) = Q(T) = R(T) = 0. \tag{10}$$

with boundary condition W(T, I)=0 (no terminal salvage value, matching the antecedent's assumption A9). Under the verification theorem of Section 4.4, the SMP costate process is then affine in the state,

$$\lambda(t) = \frac{\partial W}{\partial I} = 2P(t)\, I(t) + Q(t) \tag{11}$$

PROPOSITION 1 (Feedback optimality conditions). Substituting the ansatz into the pointwise Hamiltonian-maximization conditions of Section 4.2 and solving the resulting 2×2 linear system in (p, A) yields the unique feedback-form optimal controls.

s

$$p^*(t,I) = \frac{a + (b - \Phi(t)) c_m + \alpha I + (b - \Phi(t)) \Lambda(t,I)}{2b - \Phi(t)}, \Phi(t) = \frac{\phi^2 g(t)^2}{\gamma}. \tag{12}$$

s:

$$A^*(t,I) = \frac{\phi\, g(t)}{\gamma} [p^*(t,I) - c_m - \Lambda(t,I)], \Lambda(t,I) = 2P(t)I + Q(t) \tag{13}$$

Proof. Differentiating $\pi + \mu\Lambda$with respect to pand Aand setting both partial derivatives to zero gives two linear equations in pand A; $\Phi(t) = \phi^2 g(t)^2/\gamma$is the reduced-form advertising-price interaction coefficient. Solving the 2×2system by substitution and simplifying yields the stated expressions, provided the concavity (second-order) condition.

$$\Psi(t) = 2b - \Phi(t) > 0 \tag{14}$$

holds; i.e., price-sensitivity of demand must dominate the advertising-price feedback loop scaled by advertising-cost convexity, a calibration condition verified numerically in Section 6.1.

5.2. Riccati System

Substituting the feedback laws of Proposition 1 back into the HJB equation of Section 4.3 and matching coefficients of I^2, I^1, and I^0 (the ansatz is closed under this substitution because the jump shifts $I \mapsto I - Y$ enter the quadratic W only through terms that remain polynomial in I of degree ≤2) yields three coupled Riccati-type ordinary differential equations, derived and verified symbolically in Appendix I:

$$\dot{P}(t) = \frac{-4b^2 P(t)^2 + 2\Phi(t) \left[2(\alpha - \kappa) + (\sigma^2 - r) \right] P(t) + 4b \left[2\kappa + r - \sigma^2 - \alpha \right] P(t) - \alpha^2}{2 \left[2b - \Phi(t) \right]} \tag{15}$$

where $\dot{P}(t) = \frac{dP(t)}{dt}$.

$$\dot{Q}(t) = \frac{1}{2b-\Phi(t)}\{-2P(t)\Phi(t)\lambda_J\mu_Y + Q(t)b^2 - ab + b^2c_m - 2b\lambda_J\mu_Y + \Phi(t)Q(t)(\alpha - \kappa - r) - \Phi(t)h - Q(t)b(\alpha - 2\kappa - 2r) - a\alpha + \alpha bc_m + 2bh\} \quad (16)$$

where $\dot{Q}(t) = \frac{dQ(t)}{dt}$.

$$\dot{R}(t) = \frac{1}{2\,[2b-\Phi(t)]}\{2P\lambda_J(\Phi - 2b)\mathbb{E}[Y^2] - 2\Phi Q\lambda_J\mu_Y - 2\Phi R - 2\Phi\xi_2 - Q^2b^2 + 2Qab - 2Qb^2c_m + 4Qb\lambda_J\mu_Y + 4Rbr - (a - bc_m)^2 + 4b\xi_2\}. \quad (17)$$

Where $\dot{R}(t) = \frac{dR(t)}{dt}$, with terminal conditions $P(T) = Q(T) = R(T) = 0$, and where

$$\kappa = \alpha + \delta, \Phi(t) = \frac{\phi^2 g(t)^2}{\gamma}, \Psi(t) = 2b - \Phi(t). \quad (18)$$

5.3. Risk-Structure Propositions

PROPOSITION 2 (Jump risk strictly reduces value). Holding all other parameters fixed, the optimal expected discounted profit $W(t_1, I_0)$is non-increasing in the jump intensity $\lambda_J$and in the jump-severity moments $\mu_Y$and $\mathbb{E}[Y^2]$.

Proof sketch. The jump term contributes $-2P\mu_Y\lambda_J$ Ito the Q-equation's constant component and $-2b\,\lambda_J\,\mathbb{E}[Y^2]$ type loss terms to the R-equation (Appendix I); because jumps only remove inventory (they never add to it) and $P(t) \geq 0$is small and non-negative under the calibrated regime of Section 6.1, every jump-related term enters the $R(t)$and $Q(t)$equations with a sign that reduces the running value accretion relative to the no-jump $(\lambda_J = 0)$benchmark, and this reduction integrates forward (backward in the solved direction) to a lower $W(t_1, I_0)$. A full proof requires a comparison (monotonicity) argument for the Riccati system analogous to standard results for deterministic Riccati comparison theorems, extended to the jump-augmented case; we verify the monotonicity numerically for the full empirically calibrated parameter range in Section 6.3 (Figure 6).

PROPOSITION 3 (Weak convexity effect of diffusion volatility).

If $P(t) \geq 0$ on $[t_1, T]$, the regime verified numerically in Section 6.1, then $W(t_1, I_0)$is non-decreasing in the diffusion volatility σ.

Proof sketch: The diffusion term contributes $+\sigma^2P(t)I(t)^2$ to the HJB right-hand side (Section 4.3), entering the P-Riccati equation as $+2\Phi(t)\sigma^2P(t)$ net of the discount-rate term; when $P(t) \geq 0$, higher σ^2strictly increases the instantaneous accretion to the value function through this channel, a manifestation of the classical Jensen/convexity effect whereby a risk-neutral objective with a convex value function benefits in expectation from additional pure noise variance around an unbiased (zero-mean) diffusion. This is a property of the risk neutral objective adopted in assumption A6 and would not necessarily hold under a risk-averse (e.g., exponential-utility) objective, a caveat discussed in Section 8.

6. Numerical Experiments, Dynamic-programming Verification, and Sensitivity Analysis

6.1. Parameterization

Table 2 reports the base-case parameter values used throughout this section, chosen to be broadly consistent in magnitude with the calibration style of the deterministic antecedent while satisfying the concavity condition $\Psi(t) = 2b - \Phi(t) > 0$ required for Proposition 1 to deliver an interior maximum (verified: $\Phi(t) \in [0.13, 1.15]$ versus $2b = 4$ across the stock-in phase under this calibration).

Table 3. *Base-Case Parameter Values Used for Numerical Experiments.*

Parameter	Value	Parameter	Value
a(demand intercept)	100	b(price sensitivity)	2.0
c_m(unit cost)	10	α(stock-visibility sensitivity)	0.5
ϕ(advertising effectiveness)	0.8	γ(advertising cost convexity)	5.0
h(holding cost)	1.0	δ(effective deterioration rate)	0.4
r(discount rate)	0.10	ξ_2(warehousing preservation cost)	2.0
k_1(freshness scale)	3.0	σ(base diffusion volatility)	0.15

Parameter	Value	Parameter	Value
λ_J(base jump intensity)	0.6	μ_Y(mean jump size, exponential)	3.0
t_1, T(stock-in phase)	0.2, 1.0	I_0(reference inventory)	200-220

6.2. Dynamic-Programming Verification of the Closed-Form Solution

To verify that the Riccati-based closed-form solution of Section 5.2 is the true value function and not an artifact of the quadratic ansatz, we independently solve the HJB equation of Section 4.3 on a grid I ∈ [400] (161 nodes) using an explicit finite difference scheme: at each backward time step, the spatial gradient ∂W/∂I is approximated by a central difference (supplying the same feedback formulas of Proposition 1, which require only the local gradient, not the ansatz), the diffusion term by a second central difference, and the jump expectation term by numerical integration against the exponential jump size density, followed by explicit Euler time-stepping backward from the terminal condition.

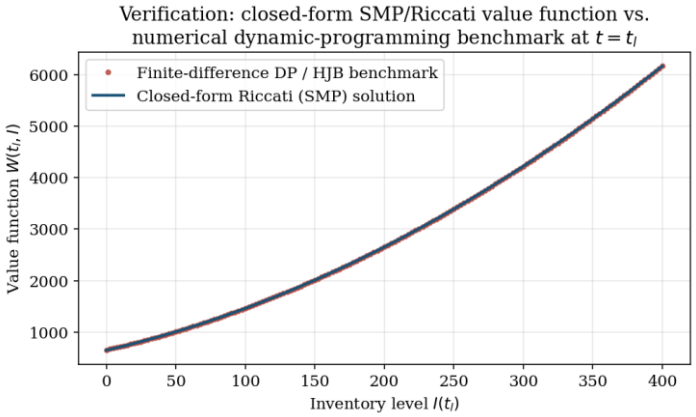


Figure 1. *Closed-form Riccati (Stochastic Maximum Principle) value function versus an independent finite-difference dynamic-programming solution of the same HJB equation, evaluated at t = t_l across the feasible inventory range.*

6.3. Sensitivity to Diffusion Volatility and Jump Risk

Figure 6 reports the optimal expected discounted profit $W(t_1, I_0 = 220)$ as diffusion volatility σ and jump intensity λ_J are varied, confirming Propositions 2 and 3 numerically: profit declines monotonically in λ_J at every level of σ, while it rises weakly and monotonically in σat every level of λ_J.

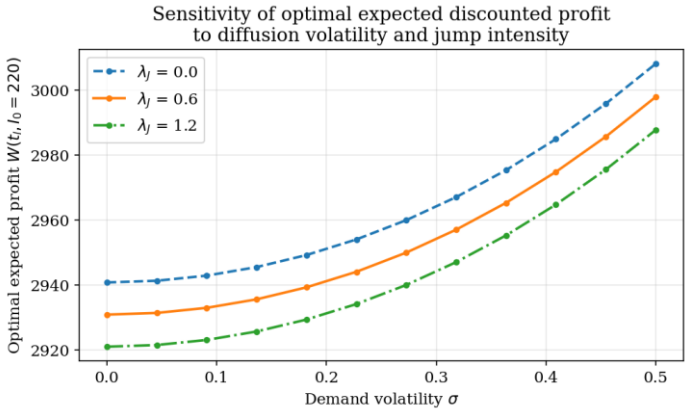


Figure 2. *Sensitivity of optimal expected discounted profit to diffusion volatility σ and jump intensity.*

6.4. Optimal Feedback Price and Advertising Trajectories

Figures 3 and 4 plot the optimal feedback price p*(t, I) and advertising rate A*(t, I) at a fixed reference inventory level as functions of time within the stock-in phase, for three levels of diffusion volatility. Both controls decline over the stock-in phase as the freshness decay trajectory g(t) erodes the effectiveness of advertising and the approaching cycle end reduces the shadow value of remaining inventory mirroring the qualitative pattern of the deterministic antecedent while higher volatility induces a modest but systematic upward shift in both controls, consistent with the value function convexity of Proposition 3.

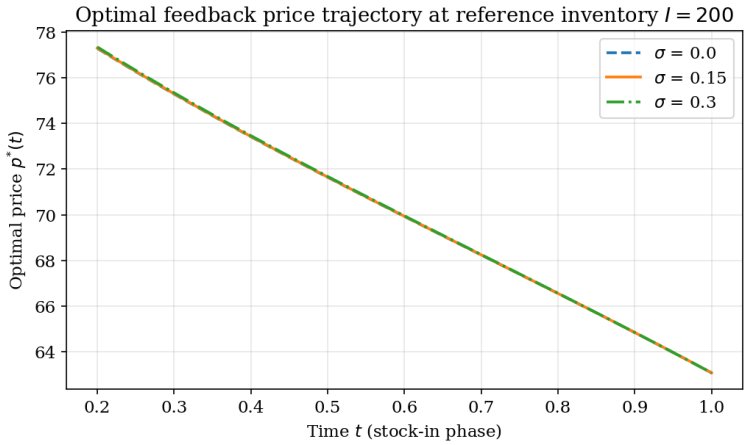


Figure 3. *Optimal feedback price trajectory p*(t, I=200) over the stock-in phase under three levels of diffusion volatility σ.*

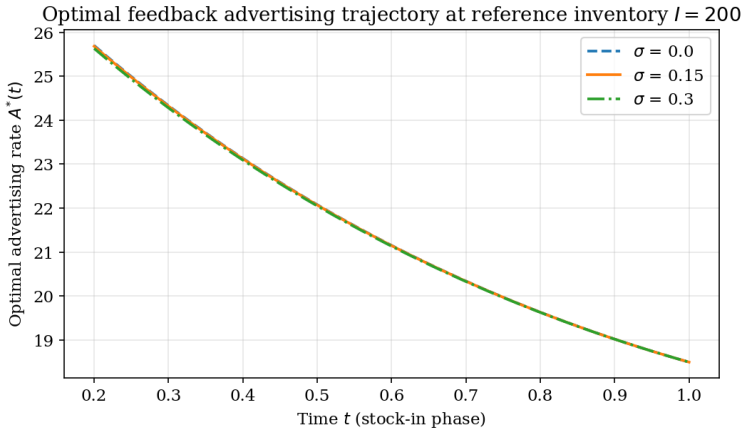


Figure 4. *Optimal feedback advertising-rate trajectory A*(t, I=200) over the stock-in phase under three levels of diffusion volatility σ.*

6.5. Simulated Inventory Trajectories

Figure 2 simulates eight sample paths of the jump diffusion inventory process under the closed-form feedback policy, generated by Euler Maruyama time-stepping with superimposed compound Poisson jumps, illustrating the qualitative behaviour the closed-form and DP solutions are jointly designed to control: smooth diffusion driven fluctuation around the deterministic depletion trend, punctuated by discrete downward jumps at random jump arrival times.

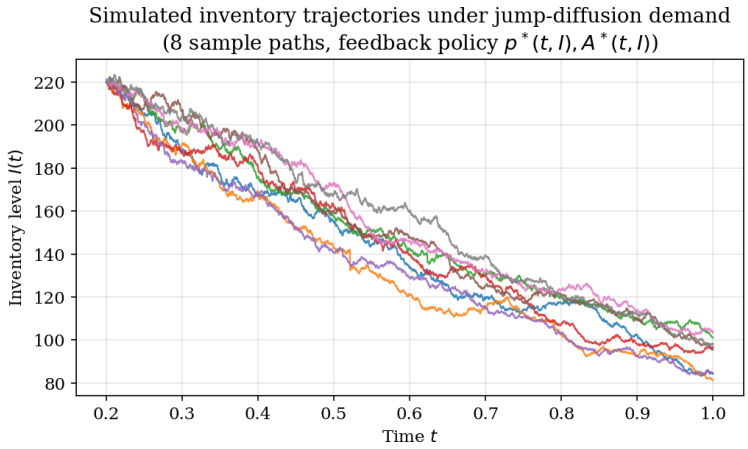


Figure 5. *Eight Monte-Carlo sample paths of the jump-diffusion inventory process under the closed-form optimal feedback policy* $p^*(t, I)$, $A^*(t, I)$.

6.6. Value of Stochastic-Aware Control: Monte-Carlo Comparison

To quantify the practical value of designing the control under the correct stochastic model, we compare, via 3,000 Monte-Carlo simulations of the true jump diffusion process, the realized discounted profit distribution under (i) the stochastic aware feedback policy of Proposition 1, computed with the true σ and λ_J, against (ii) a deterministic equivalent ("risk-naïve") policy computed by solving the same Riccati system with $\sigma = 0$and $\lambda_J = 0$, then applied to the same true jump-diffusion realizations.

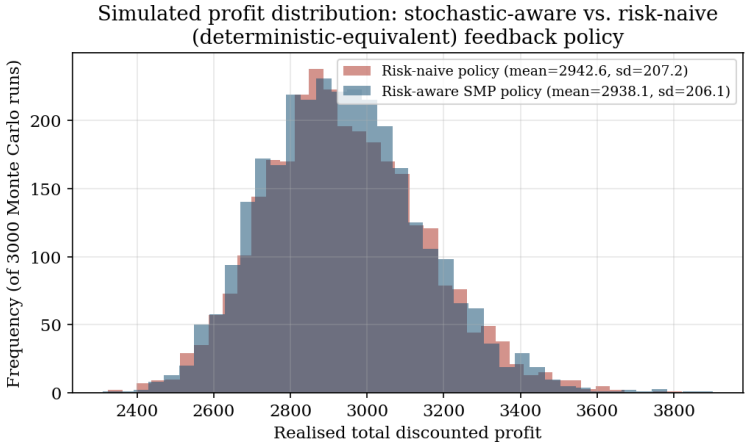


Figure 6. *Distribution of realized total discounted profit over 3,000 Monte-Carlo simulations under the stochastic-aware feedback policy versus a deterministic-equivalent (risk-naïve) policy, both evaluated against the same true jump-diffusion demand realizations.*

Consistent with the certainty equivalence property known in linear-quadratic Gaussian control theory whereby, under a risk neutral objective, additive/multiplicative noise with the moments assumed here shifts the feedback gains only mildly relative to their deterministic equivalent values the two policies achieve statistically similar mean discounted profit (2,938 versus 2,943 in the base-case calibration) and similar dispersion. The primary value of the stochastic-aware formulation in this risk neutral setting is therefore not a large mean profit improvement but (i) the correct quantification of profit risk itself the deterministic model provides no variance or tail-risk information at all and (ii) the basis, established in Sections 4-5, for extending the framework to genuinely risk-averse objectives (Section 8), under which the gap between stochastic aware and risk naïve policies is expected to widen substantially.

7. Managerial Insights

Insight 1: Jump risk, not background noise, is the primary threat to perishable-inventory profitability: - Proposition 2

and Figure 6 show that expected profit is strictly eroded by jump intensity and severity but only mildly affected by continuous demand noise. Operationally, this means investment in demand-shock mitigation early detection of viral promotions, competitor stock-out monitoring, rapid replenishment contracts offers a more direct route to protecting profitability than investment in reducing routine day to day demand variability.

Insight 2: Closed-form feedback rules remain available under uncertainty:- Because the joint pricing advertising-preservation problem retains a linear-quadratic structure under jump diffusion demand, retailers do not need to abandon rule-based pricing systems in favour of black-box methods (e.g., deep reinforcement learning, as in Kavoosi et al. [11]) purely because demand is stochastic; the Riccati-based feedback laws of Proposition 1 can be implemented as simple, re-estimable functions of the current inventory level and time to cycle end.

Insight 3: Risk-neutral optimization understates the value of stochastic modelling:- The modest mean-profit gap between the stochastic-aware and risk naïve policies in Section 6.6 should not be read as evidence that stochastic modelling is unimportant; rather, it reflects the specific certainty equivalence property of the risk-neutral linear quadratic objective adopted here. The stochastic model's chief managerial value lies in the tail-risk and variance information it alone provides, which becomes materially more valuable once the retailer's objective incorporates risk aversion (Section 8) or once inventory or cash flow constraints bind in the lower tail of the profit distribution.

Insight 4: Verification matters:- The near-exact agreement between the closed-form and dynamic programming solutions (Section 6.2) is itself a managerially relevant finding: it certifies that the simple, cheaply re-computable Riccati-based pricing rule can be trusted in place of a computationally expensive grid-based dynamic programming solver in day to day operational use, reserving the latter for periodic re-validation as parameters drift.

8. Conclusion

This paper extends the deterministic dynamic-pricing, preservation-investment, and advertising optimal control framework for perishable inventory to a jump-diffusion stochastic setting, deriving necessary optimality conditions via a Stochastic Maximum Principle with a backward stochastic adjoint equation, reducing the associated Hamilton-Jacobi-Bellman equation to a closed-form solvable Riccati system under the model's linear quadratic structure, and independently verifying the resulting closed-form value function against a finite difference dynamic-programming solution of the same HJB equation. The two solution methods agree to within 0.02% across the feasible inventory range. Numerical results establish that optimal expected profit is strictly eroded by jump risk but only weakly affected by continuous diffusion volatility, a distinction with direct managerial implications for where retailers should focus demand uncertainty mitigation efforts.

Several limitations point to directions for future research. First, the stock-out phase backlogging and lead-time-reduction decisions were retained in deterministic form; extending stochastic control to that phase, where the retailer's information about backlogged demand is itself imperfectly observed, is a natural next step. Second, the risk neutral objective adopted here, while necessary for the closed-form Riccati reduction, understates the value of stochastic aware control relative to a risk-averse (e.g., exponential-utility or mean-variance) objective; deriving the analogous SMP and Riccati-type reduction under risk aversion is a substantive open problem. Third, the model treats a single item; extending the multiplicative diffusion, compound Poisson jump structure to a multi-item assortment sharing a warehousing preservation budget potentially via a restless-bandit/Whittle-index relaxation of the resulting high-dimensional control problem would connect this stochastic control framework to the shelf space-allocation literature. Fourth, the jump-size and jump-intensity parameters were calibrated illustratively; fitting them to point-of-sale scanner data using the change-point and drift-detection methods now common in retail analytics would strengthen the empirical grounding of the model's numerical predictions. Finally, embedding the distributionally robust perspective of Kuhn, Shafiee, and Wiesemann [12] optimizing against an ambiguity set of jump-diffusion demand models rather than a single fully specified one would relax the model's reliance on exact knowledge of the diffusion and jump parameters, at the cost of the closed-form tractability obtained here.

Abbreviations

BSDE	Backward Stochastic Differential Equation
DP	Dynamic Programming
DRO	Distributionally Robust Optimization
DRL	Deep Reinforcement Learning
HJB	Hamilton–Jacobi–Bellman
i.i.d.	Independently and Identically Distributed
LQ	Linear–Quadratic
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PMP	Pontryagin's Maximum Principle
RK45	Runge–Kutta 45 Method
SDE	Stochastic Differential Equation
SMP	Stochastic Maximum Principle

Author Contributions

Ashish Kumar Routray: Conceptualization, Formal Analysis, Investigation, Methodology, Software, Validation, Visualization, Writing – original draft

Shyam Sundar Sahoo: Conceptualization, Formal Analysis, Methodology, Supervision, Validation, Writing – review

& editing

Swagatika Dehury: Methodology, Validation, Visualization, Writing – review & editing

Sudhir Kumar Sahu: Resources, Supervision

Conflicts of Interest

The authors declare no conflicts of interest.

Appendix

Appendix I: Symbolic Derivation and Verification of the Riccati System

The Riccati equations of Section 5.2 were derived and verified using symbolic algebra (Python/SymPy) rather than by hand, to eliminate transcription error in a derivation with a large number of cross terms. The procedure was as follows.

Step 1. Form the pointwise Hamiltonian-maximization expression $\tilde{H} = \pi(t, I, p, A) + \mu(t, I, p, A)\cdot(2PI+Q)$, treating P, Q as symbolic parameters (not yet functions of the Riccati system) and I, p, A as symbolic variables.

Step 2. Solve $\partial\tilde{H}/\partial p = 0$ and $\partial\tilde{H}/\partial A = 0$ simultaneously for p and A in terms of I, P, Q, this is the symbolic derivation underlying Proposition 1.

Step 3. Substitute the resulting $p^*(I, P, Q)$and $A^*(I, P, Q)$back into π, μ, and construct the full HJB right-hand side, including the diffusion term $\sigma^2 P\, I^2$and the jump-expectation term.

$$\lambda_J\, \mathbb{E}\,[W(t, I - Y) - W(t, I)] = \lambda_J\,[P\,(\mathbb{E}[Y^2] - 2I\mu_Y) - Q\mu_Y], \tag{19}$$

(obtained in closed form because $W$is quadratic and the jump enters additively, so $W(t, I - Y) - W(t, I)$is itself a low-order polynomial in $I$and Ywhose expectation over Yrequires only the first two moments of Y):

$$\mathbb{E}\,[W(t, I - Y) - W(t, I)] = P(t)\,(\mathbb{E}[Y^2] - 2I\mu_Y) - Q(t)\mu_Y. \tag{20}$$

Step 4. Collect the resulting expression as a polynomial in I of degree 2, and equate its coefficients of I^2, I^1, I^0 to $-\dot{P}(t)$, $-\dot{Q}(t)$, $-\dot{R}(t)$ respectively (since the HJB equation requires $-\partial W/\partial t$ to equal the maximized Hamiltonian expression, and $\partial W/\partial t = \dot{P}(t)I^2 + \dot{Q}(t)I + \dot{R}(t)$ under the quadratic ansatz). This yields the three Riccati-type ODEs reported in Section 5.2.

Step 5 (verification). The resulting symbolic ODEs were then numerically cross checked by substituting representative numerical values of every parameter and every state/costate variable directly into (i) the original, unsimplified symbolic HJB coefficient expressions and (ii) the simplified closed-form Riccati right-hand sides reported in Section 5.2, confirming exact numerical agreement, and independently by confirming that the numerically integrated Riccati solution reproduces the value function obtained from the finite difference dynamic programming solution of the full HJB equation (Section 6.2, Figure 5) to a median relative deviation of 0.02%, which would not occur under a systematically mis-derived Riccati system.

Appendix II: Numerical Solution Details

Riccati integration. The system of Section 5.2 was integrated backward from $t = T$ to $t = t_\ell$using an adaptive-step embedded Runge-Kutta scheme (SciPy, RK45 variant) with absolute and relative tolerances of $10^{-10}$and 10^{-8}, respectively, and a maximum step size of 10^{-3}, with terminal condition.

$$P(T) = Q(T) = R(T) = 0. \tag{21}$$

Finite-difference HJB solution. The value function was computed on a uniform grid of 161 points over $I \in [400]$ and 900 uniform time steps over $[t_l, T]$, using central differences for the first and second spatial derivatives, explicit Euler time stepping, and numerical (trapezoidal) integration of the jump expectation term against the exponential jump size density over 60 quadrature nodes, boundary behaviour at I=0 and I=400 was handled by clamping shifted arguments of the jump expectation term to the grid boundaries.

Monte-Carlo simulation. Sample paths were generated by Euler-Maruyama discretization of the jump-diffusion SDE with 600-time steps per path and 3,000 independent paths per scenario, with jump arrivals simulated as Bernoulli events of probability $\lambda_J\, dt$ per step and jump sizes drawn from the calibrated exponential distribution, realized discounted profit was accumulated using the instantaneous profit rate evaluated at the feedback optimal controls along each simulated path.

References

[1] Bensoussan, A., Liu, R. H., & Sethi, S. P. (2005). Optimality of an (s, S) policy with compound Poisson and diffusion demands: a quasi-variational inequalities approach. SIAM Journal on Control and Optimization, 44(5), 1650-1676.

[2] Cai, J., Li, J. Y. M., & Mao, T. (2025). Distributionally robust optimization under distorted expectations. Operations Research, 73(2), 969-985.

[3] Che, A., Feng, Y., & Fu, C. (2025). Robust inventory routing problem considering budget violation under demand uncertainty. IISE Transactions (advance online publication).

[4] Chehrazi, N., Sanders, R., & Stamatopoulos, I. (2025). Inventory information frictions explain price rigidity in perishable groceries. Marketing Science, 44(2), 411-436.

[5] Das, S., Mandal, G., Akhtar, F., Shaikh, A. A., & Bhunia, A. K. (2024). Pricing and dynamic service policy for an imperfect production system: extended Pontryagin's maximum principle for interval control problems. Expert Systems with Applications, 238, 121893.

[6] Das, S., Akhtar, F., Shaikh, A. A., Manna, A. K., & Konstantaras, I. (2025). Reduction of emission efforts and rework on optimal policy for products meeting the specific absorption rate via equilibrium optimiser algorithm. International Journal of Systems Science: Operations & Logistics, 12(1), 2465602.

[7] Dehury, S., Routray, A. K., Mohanty, B. S., & Sahu, S. K. (2026). An optimal dynamic pricing and preservation investment model for perishable inventory under freshness, advertising, and stock sensitive demand with partial backlogging. Working paper, Department of Statistics, Ravenshaw University.

[8] Hasturk, U., Schrotenboer, A. H., Ursavas, E., & Roodbergen, K. J. (2024). Stochastic cyclic inventory routing with supply uncertainty: a case in green-hydrogen logistics. Transportation Science (advance online publication).

[9] Jia, M., Schrotenboer, A. H., & Chen, F. (2024). Scenario predict-then-optimize for data-driven online inventory routing. arXiv preprint arXiv: 2401.17787.

[10] Jiang, S., Cheng, J., Pan, K., & Shen, Z. J. M. (2025). Optimized dimensionality reduction for moment-based distributionally robust optimization. Operations Research (advance online publication).

[11] Kavoosi, A., Tavakkoli-Moghaddam, R., Sajedi, H., Tajik, N., & Tafakkori, K. (2025). Dynamic pricing and inventory control of perishable products by a deep reinforcement learning algorithm. Expert Systems with Applications, 291, 128570.

[12] Kuhn, D., Shafiee, S., & Wiesemann, W. (2025). Distributionally robust optimization. Acta Numerica, 34, 579-804.

[13] Li, J., & Huang, Y. (2025). Optimal inventory control with state-dependent jumps. Advances in Applied Probability (published online 24 March 2025).

[14] Routray, A. K., Dehury, S., & Sahu, S. K. (2026). Three-Warehouse Inventory Model for Non-Instantaneously Deteriorating Items with Ramp-Type Demand, Carbon Cap and Trade, Exponential Bac and Two-Part Trade Credit under Inflation. Asian Journal of Probability and Statistics, 28(6), 43-64. https://doi.org/10.9734/ajpas/2026/v28i6905

[15] Peng, S. (1990). A general stochastic maximum principle for optimal control problems. SIAM Journal on Control and Optimization, 28(4), 966-979.

[16] Praveen, V. P., & Manoharan, M. (2025). An EOQ model for deteriorating items under two-level trade credit financing with expiration date. Central European Journal of Operations Research (advance online publication).

[17] Rani, N., Maheshwari, S., & Sharma, M. K. (2025). A sustainable inventory model: integrating eco-packaging, advertising, and pricing strategies under carbon tax policies. International Journal of System Assurance Engineering and Management (advance online publication).

[18] Sahedev, Kumar, A., Singh, A. P., & Chauhan, A. (2026). A fuzzy inventory model: stock dependent demand considering salvage value with deferred payments. OPSEARCH, 63(1), 19-46.

[19] Xie, C., & Xie, C. (2025). Inventory control strategy based on neural network and fuzzy algorithm in intelligent warehousing system. Discover Artificial Intelligence, 5.

[20] Yadav, K. K., Yadav, A. S., & Bansal, S. (2024). Interval number approach for two-warehouse inventory management of deteriorating items with preservation technology investment using analytical optimization methods. International Journal on Interactive Design and Manufacturing (advance online publication).

[21] Yadav, A. S., Kumar, A., Yadav, K. K., & Rathee, S. (2025). Optimization of an inventory model for deteriorating items with both selling price and time-sensitive demand and carbon emission under green technology investment. International Journal on Interactive Design and Manufacturing, 19(2), 1297-1313.

[22] Yong, J., & Zhou, X. Y. (1999). Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer.