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Research Article

An Innovative Result on the Radius of Convexity and Uniform Convexity of Normalised Bessel Function

Natarajan Saraswathi*, Murugesan Kasthuri

Department of Mathematics, DKM College for Women (Autonomous), Vellore, India

Abstract

This paper investigates the geometric properties of normalised Bessel functions of the first kind, focusing on the radii of convexity and uniform convexity. Using analytic techniques such as logarithmic differentiation and properties of zeros of the Bessel function. Additionally, we provide new proofs related to the order α convexity and compare our results with existing results to reveal interesting relationships, and we provide a graphical interpretation that supports the analytical findings.

Keywords

Bessel Function, Convex Function, Uniformly Convex Function, Normalised Bessel Function

1. Introduction

Complex analysis is a well-established and continuously evolving area of mathematics with significant applications in physics, engineering, and applied sciences. Within this domain, geometric function theory plays a crucial role in understanding the qualitative behavior of analytic and univalent functions. In particular, the study of geometric properties such as *starlikeness*, *convexity*, and *uniform convexity* has attracted considerable attention due to their theoretical importance and practical applications [2, 7, 8].

Among special functions, the *Bessel functions of the first kind* occupy a central position because of their wide applicability in problems related to wave propagation, heat conduction, and solutions of differential equations arising in mathematical physics [14]. Consequently, the investigation of geometric properties of normalized Bessel functions has become an active research topic in recent years. Classical contributions have established foundational results on the radii of starlikeness and convexity for Bessel functions and their normalised forms [1, 3, 6, 7].

In recent years, several authors have significantly advanced this field. For instance, Cotîrlă, Kupán, and Szász obtained new results concerning the radius of convexity and uniform convexity of Bessel functions, providing sharper bounds and refined analytical techniques [15]. Similarly, Zayed, Kupán, and Szász investigated geometric properties of generalized Bessel functions, including convexity and uniform convexity, from a broader analytical perspective. Furthermore, Baricz, Kumar, and Singh derived asymptotic expansions and bounds for the radii associated with normalised Bessel functions, thereby deepening the theoretical understanding of these functions. Additional studies, such as those by Zhao, Shi, and Chu, have explored convexity properties of modified Bessel functions, emphasizing the continued relevance of this area of research.

Motivated by these developments, the present work focuses

*Correspondence: Natarajan Saraswathi (saraswathinanthan82@gmail.com)

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on determining the *radius of convexity* and *radius of uniform convexity* for a class of normalised Bessel functions corresponding to the parameter range $v \in (-2,-1)$. We also establish alternative proofs for the convexity of order α and compare the results obtained with existing findings to reveal new relationships and improvements. The structure of the paper is as follows. In the preliminary section, we recall essential definitions and lemmas required for our analysis. Subsequently, we present the main results concerning the radii of convexity and uniform convexity, along with rigorous proofs. Finally, we discuss the implications of our findings and suggest possible directions for future research.

Let $D_r = \{ z \in C: |z| < r\}$, where r>0. Consider an analytic function A defined in D_r, normalised by $A(0) = 0$ and $A'(0) = 1$

The function A of the form $A(z) = z + a_2 z^2 + \ldots$ (1)

The function is said to be convex in D_r, if it satisfies $Re\left(1 + \frac{zA''(z)}{A'(z)}\right) > 0, z \in D_r$ Similarly, the function is uniformly convex if $Re\left(1 + \frac{zA''(z)}{A'(z)}\right) > \left|\frac{zA''(z)}{A'(z)}\right|, z \in D_r$. The radius of convexity of order α for A is defined by the equality

$$r_A^c(\alpha)=\sup\left\{r \in (0,\infty): Re\left(1 + \frac{zA''(z)}{A'(z)}\right) > \alpha, z \in D_r\right\} \quad (2)$$

The concept of the radius of uniform convexity is

$$r_A^{uc}(\alpha)=\sup\left\{r \in (0,\infty): Re\left(1 + \frac{zA''(z)}{A'(z)}\right) > \left|\frac{zA''(z)}{A'(z)}\right|, z \in D_r\right\} \quad (3)$$

In this work, we focus on normalised Bessel functions and derive new bounds for these radii using analytic methods. These results obtained improve several known estimates and provide further insight into the geometric behaviour of these functions.

Normalised form of the Bessel function

The Bessel function of the first kind of order v, denoted by $J_v(z)$, is defined by the series expansion [14]

$$J_v(z)=\sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{z}{2}\right)^{2n+v}}{n!\Gamma(n+v+1)},$$

To study geometric properties, it is convenient to consider normalised versions of this function. Accordingly, we define the following normalised forms [5] $g_v(z) = 2^v\Gamma(1+v)z^{1-v} J_v(z)$

$$h_v(z)= 2^v\Gamma(1+v)z^{1-\frac{v}{2}} J_v\left(z^{\frac{1}{2}}\right) \text{ and } f_v(z)=(2^v\Gamma(v+1)J_v(z))^{\frac{1}{v}}, v \neq 0 \quad (4)$$

Here, $v \in (-2,-1)$ and the functions g_v, h_v, f_v are analytic in the unit disk.

2. Preliminary Results

In this section, we present auxiliary results required for the main results.

Lemma 2.1 [4, 14]

Let $v \in (-2,-1)$. Then the zeros of $J_v(z)$ consists of infinitely many real zeros along with a pair of purely imaginary zeros. These zeros occur in conjugate pairs due to symmetry properties of the function. The zeros $z^{-v}J_v(z)$ are taken to be $\pm j_{v,n}$ where n ∈N. We may suppose, without restricting the generality, that $j_{v,1} =$ ia, a$> o$ and $0 <$ a $< j_{v,1} < j_{v,2} < j_{v,3} < \ldots < j_{v,n} < \ldots$

Lemma 2.2 [7, 14]

The following equality holds $\sum_{n=1}^{\infty} \frac{1}{j_{v,n}^2} = \frac{1}{4(v+1)}$

Lemma 2.3 [8]

Let $f_v(z)$ be defined as above. Then its logarithmic derivative admits the representation

$$\frac{zf_v'(z)}{f_v(z)} = 1 - \frac{2}{v}\sum_{n=1}^{\infty} \frac{z^2}{j_{v,n}^2 - z^2}$$

The series connects uniformly for any compact subset of C\ $\{\pm j_{v,n}: \text{n} \in \text{N}\}$

Lemma 2.4 [9]

If $v \in C$, $\delta \epsilon$ R and $\delta > \rho \geq |v|$, then $\left|\frac{v}{\delta - v}\right| \leq \frac{\rho}{\delta - \rho}$ and $\left|\frac{v}{(\delta - v)^2}\right| \leq \left|\frac{\rho}{(\delta - \rho)^2}\right|$

Lemma 2.5 [15]

If $v \in C$, $\delta, \gamma \in R$ and $\gamma \geq \delta > \rho \geq |v|$, then $\left|\frac{v^2}{(\delta+v)((\gamma-v)}\right| \leq \left|\frac{\rho^2}{(\delta-\rho)(\gamma+\rho)}\right|$

Lemma 2.6 [15]

If $v \in C, \delta, \gamma \in R$ and $\gamma \geq \delta > \rho \geq |v|$, then $\left|\frac{2v^2\,[2\gamma\delta+(\gamma-\delta)v]}{(\gamma-v)^2(\delta+v)^2}\right| \leq \left|\frac{2r^2\,[2\gamma\delta-(\gamma-\delta)\rho]}{(\gamma+\rho)^2(\delta-\rho)^2}\right|$

Lemma 2.7

If the function f_v are defined by (4) respectively, then

$$z\frac{f_v''(z)}{f_v'(z)} = z\left(\frac{1}{v} - 1\right)\frac{J_v'(z)}{J_v(z)} + \frac{(v^2-z^2)}{z}\frac{J_v(z)}{J_v'(z)} - 1 \quad (5)$$

Proof

$$f_v(z)=(2^v\Gamma(v+1)J_v(z))^{\frac{1}{v}}$$

Two times differentiation with respect to z. After multiplying by z, we get the following equality

$$z\frac{f_v''(z)}{f_v'(z)} = \frac{z^2(1-v)[J_v'(z)]^2 + z^2 v J_v(z) J_v''(z)}{z v J_v(z) J_v'(z)}$$

The function J_v is a solution of the Bessel differential equation. Thus, we can replace the function $z^2 J_v''$ by $z^2 J_v''(z) = (v^2 - z^2) J_v(z) - z J_v'(z)$ We obtain equation (5).

3. Main Results

General results on the radius of convexity for analytic functions can be found in [12], which we adapt to the present setting.

Theorem 3.1 [12]

If $\alpha \epsilon [0,1)$ and $v \in(-2,-1)$, Then the radius of convexity of order α for the mapping f_v is $r_v^c(\alpha) = r_1$, r_1 is the unique root of the equation [1, 7, 11, 15]

$$r\left(\frac{1}{v} - 1\right)\frac{I_v'(r)}{I_v(r)} + \frac{(v^2+r^2)}{r}\frac{I_v(r)}{I_v'(r)} = \alpha \quad (6)$$

in the interval (0, a).

Proof

By Lemma (3)

$$\frac{z f_v'(z)}{f_v(z)} = 1 - \frac{2}{v}\sum_{n=1}^{\infty}\frac{z^2}{j_{v,n}^2 - z^2} \quad (7)$$

Replacing $j_{v,1} = ia$, then (7) becomes

$$z\frac{f_v'(z)}{f_v(z)} = 1 + \frac{2a^2 z^2}{2a^2(a^2+z^2)} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{z^2}{j_{v,n}^2 - z^2} \quad (8)$$

By Lemma (2)

$$\frac{1}{a^2} = \sum_{n=2}^{\infty}\frac{1}{j_{v,n}^2} - \frac{1}{4(v+1)} \quad (9)$$

Using (9) in (8) then we get

$$\frac{z f_v'(z)}{f_v(z)} = 1 - \frac{a^2}{2v(v+1)}\frac{z^2}{(a^2+z^2)} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\frac{z^4}{(a^2+z^2)(j_{v,n}^2 - z^2)} \quad (10)$$

Taking Logarithmic differentiation of this equality leads to

$$1 + \frac{z f_v''(z)}{f_v'(z)} = 1 - \frac{a^2}{2v(v+1)}\frac{z^2}{a^2+z^2} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\frac{z^4}{(a^2+z^2)(j_{v,n}^2 - a^2)} - \frac{-\frac{a^2}{v(v+1)}\frac{a^2 z^2}{(a^2+z^2)^2} - \frac{4z^4}{v}\sum_{n=2}^{\infty}\frac{a^3+j_{v,n}^2}{j_{v,n}^2}\frac{(2a^2 j_{v,n}^2 + z^2(j_{v,n}^2 - a^2))}{(a^2+z^2)^2(j_{v,n}^2 - z^2)^2}}{1 - \frac{a^2}{2v(v+1)}\frac{z^2}{(a^2+z^2)} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\frac{z^4}{(a^2+z^2)(j_{v,n}^2 - z^2)}} \quad (11)$$

As a function f_v, the smallest root of the equation is the radius of starlikeness, denoted as $r_{f_v}^*$, $1 + \frac{a^2}{2v(v+1)}\frac{r^2}{(a^2-r^2)} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\frac{r^4}{(a^2-r^2)(j_{v,n}^2 + r^2)} = \frac{ir f_v'(ir)}{f_v(ir)} = 0$ in (0, a). Thus, we have

$$0 < r_{f_v}^* < a < j_{v,2} < j_{v,3} < \cdots < j_{v,n} < \cdots \quad (12)$$

Taking $v(v+1) < 0$.Then the equation (11) suggests the inequality shown below.

$Re(1 + \frac{z f_v''(z)}{f_v'(z)}) = 1 + \frac{a^2}{2v(v+1)}\left|\frac{z^2}{a^2+z^2}\right| - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{v,n}^2 - a^2)}\right| - \frac{-\frac{a^2}{v(v+1)}\left|\frac{a^2 z^2}{(a^2+z^2)^2}\right| - \frac{4z^4}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\left|\frac{(2a^2 j_{v,n}^2 + z^2(j_{v,n}^2 - a^2))}{(a^2+z^2)^2(j_{v,n}^2 - z^2)^2}\right|}{1 - \frac{a^2}{2v(v+1)}\left|\frac{z^2}{(a^2+z^2)}\right| - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{v,n}^2 - z^2)}\right|}$ for every z∈D (r_v^*)

Using $\delta = a^2$,$\rho = r^2$ and v= z^2 in Lemma 4,

$$\frac{a^2}{2(1+v)}\left|\frac{z^2}{a^2+z^2}\right| \geq \frac{a^2}{2(1+v)}\frac{r^2}{a^2-r^2} \text{ and } \frac{a^2}{2(1+v)}\left|\frac{z^2}{(a^2+z^2)^2}\right| \geq \frac{a^2}{2(1+v)}\frac{r^2}{(a^2-r^2)^2} \quad (13)$$

Similarly, Lemma (5) and Lemma (6) imply that

$$\left|\frac{z^4}{(a^2+z^2)(j_{v,n}^2 - z^2)}\right| \leq \frac{z^4}{(a^2-r^2)(j_{v,n}^2 + r^2)} \text{ and } \left|\frac{2z^4[2a^2 j_{v,n}^2 + z^2(j_{v,n}^2 - a^2)]}{(a^2+z^2)^2\,(j_{v,n}^2 - z^2)^2}\right| \leq \frac{2r^4[2a^2 j_{v,n}^2 + z^2(j_{v,n}^2 - a^2)]}{(a^2-r^2)^2\,(j_{v,n}^2 + r^2)^2} \quad (14)$$

$$\mathrm{Re}(1 + \frac{z f_v''(z)}{f_v'(z)}) \geq 1 + \frac{a^2}{2v(v+1)}\frac{r^2}{a^2-r^2} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\frac{r^4}{(a^2-r^2)(j_{v,n}^2 + r^2)} - \frac{-\frac{a^2}{v(v+1)}\frac{a^2 r^2}{(a^2-r^2)^2} - \frac{4r^4}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\frac{(2a^2 j_{v,n}^2 - r^2(j_{v,n}^2 - a^2))}{(a^2-r^2)^2(j_{v,n}^2 + r)^2}}{1 - \frac{a^2}{2v(v+1)}\frac{r^2}{(a^2-r^2)} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j_{v,n}^2}{j_{v,n}^2}\frac{r^4}{(a^2-r^2)(j_{v,n}^2 + r^2)}} = 1 + ir\frac{f_v''(ir)}{f_v'(ir)} = \varphi(r) \quad (15)$$

Provided that $a > r^*_{f_v} > |z|$, where $r^*_{f_v}$ verifies the inequalities (11)

The following equalities hold: $\varphi(0)=1$ and $\lim_{r \to r^*_{f_v}} \varphi(r) = \infty$. Consequently, the equation

$1 + ir\frac{f_v''(ir)}{f_v'(ir)} = \alpha$ has a real root in $(0,\ r^*_{f_v})$.The minimal positive solution of the corresponding functional equation derived from the normalised form $1 + ir\frac{f_v''(ir)}{f_v'(ir)} = \alpha$ is denoted by $r^c_{f_v}(\alpha)$, and this root is the radius of convexity of order α of the function f_v. The first equality of Lemma (7) and the equality $J_v(\mathrm{iz}) = i^v I_v(\mathrm{z})$ implies that the equation $1 + ir\frac{f_v''(ir)}{f_v'(ir)} = \alpha$ is equivalent to (6). Thus, the obtained radius satisfies the required conditions. These results are consistent with earlier findings [11, 13].

In the following theorem, we find the mapping f_v radius of uniform convexity.

Theorem 3.2

Under the same assumptions of Theorem 1, the radius of uniform convexity for the mapping f_v is $r^*_v(\alpha) = r_2$, where r_2 is the smallest positive root of the equation [1, 15]

$$\frac{2r(1-v)}{v}\frac{I_v'(z)}{I_v(z)} + \frac{v^2+r^2}{r}\frac{I_v}{I_v'} = 0 \qquad (16)$$

in $(0,r^*_v)$

Proof

$$\left|\frac{zf_v''(z)}{f_v'(z)}\right| \le -\frac{a^2}{2v(v+1)}\left|\frac{z^2}{a^2+z^2}\right| - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j^2_{v,n}}{j^2_{v,n}}\left|\frac{z^4}{(a^2+z^2)(j^2_{v,n}-a^2)}\right| - \frac{-\frac{a^2}{v(v+1)}\left|\frac{a^2z^2}{(a^2+z^2)^2}\right| - \frac{4z^4}{v}\sum_{n=2}^{\infty}\frac{a^3+j^2_{v,n}}{j^2_{v,n}}\left|\frac{(2a^2j^2_{v,n}+z^2(j^2_{v,n}-a^2))}{(a^2+z^2)^2(j^2_{v,n}-z^2)^2}\right|}{1-\frac{a^2}{2v(v+1)}\left|\frac{z^2}{(a^2+z^2)}\right| - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j^2_{v,n}}{j^2_{v,n}}\left|\frac{z^4}{(a^2+z^2)(j^2_{v,n}-z^2)}\right|}$$

Again, using inequalities (13) and (14), and in combination with the above inequality

$$\left|\frac{zf_v''(z)}{f_v'(z)}\right| \le -\frac{a^2}{2v(v+1)}\frac{r^2}{a^2-r^2} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j^2_{v,n}}{j^2_{v,n}}\frac{r^4}{(a^2-r^2)(j^2_{v,n}+r^2)} - \frac{-\frac{a^2}{v(v+1)}\frac{a^2r^2}{(a^2-r^2)^2} - \frac{4z^4}{v}\sum_{n=2}^{\infty}\frac{a^2+j^2_{v,n}}{j^2_{v,n}}\frac{(2a^2j^2_{v,n}-r^2(j^2_{v,n}-a^2))}{(a^2-r^2)^2(j^2_{v,n}+r)^2}}{1-\frac{a^2}{2v(v+1)}\frac{r^2}{(a^2-r^2)} - \frac{2}{v}\sum_{n=2}^{\infty}\frac{a^2+j^2_{v,n}}{j^2_{v,n}}\left|\frac{r^4}{(a^2-r^2)(j^2_{v,n}+r^2)}\right.} = -ir\frac{f_v''(ir)}{f_v'(ir)} \qquad (17)$$

Inequalities (15) and (17) imply

$$\mathrm{Re}\left(1 + \frac{zf_v''(z)}{f_v'(z)}\right) - \left|\frac{\mathrm{Z}\,\mathrm{A}''(z)}{\mathrm{A}'(z)}\right| = 1 + 2ir\frac{f_v''(ir)}{f_v'(ir)}, \mathrm{z} \in D(r^*_v) \quad (18)$$

The minimum non-negative of the equation $\psi(r) = 1 + 2ir\frac{f_v''(ir)}{f_v'(ir)} = 0$ in the interval $(0,r^*_v)$ is denoted by r^{uc}_v. According to (18), it is the biggest with the property that

$\mathrm{Re}\left(1 + \frac{zf_v''(z)}{f_v'(z)}\right) - \left|\frac{\mathrm{ZA}''(z)}{\mathrm{A}'(z)}\right| > 0, \mathrm{z} \in D(r^{uc}_v)$ Lemma 7 and the equality $J_v(\mathrm{iz}) = i^v I_v(\mathrm{z})$ implies that the equation $1 + 2ir\frac{f_v''(ir)}{f_v'(ir)} = 0$ is equal to (16). Complete the proof.

4. Corollary

From the above formulation, both convexity and uniform convexity of order $\frac{1}{2}$ are characterised by the same equation $1 + 2ir\frac{f_v''(ir)}{f_v'(ir)} = 0$. Similar relationships have been observed in [1, 3]

5. Graphical Interpretation

The graphical analysis supports the theoretical results obtained in Theorems 1 and 2. The function $\phi(r)$ is continuous and strictly increasing on $(0, r^*_{f_v})$, ensuring the existence of a unique solution to $\phi(r) = \alpha$, which determines the radius of convexity. Similarly, $\psi(r)$ admits a unique zero in $(0, r^*_v)$, confirming the existence of the radius of uniform convexity. These visualisations validate the analytical findings and demonstrate the geometric behaviour of the normalised Bessel function [9, 10]

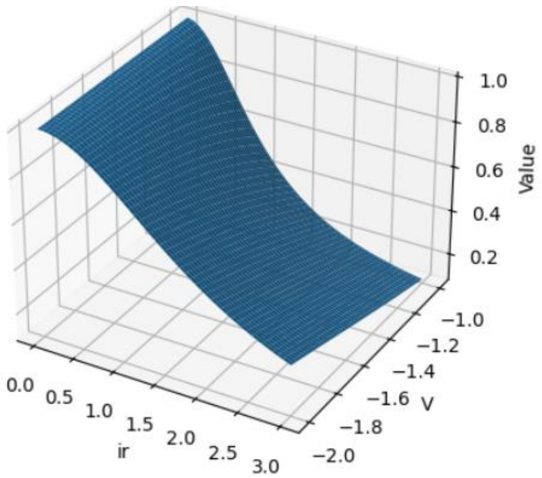

Figure 1. *Comparison of the Radius of Convexity and Uniform Convexity Equation.*

6. Conclusion

In this study, we present novel results on the geometric properties of Bessel functions, specifically focusing on their radius of convexity and uniform convexity. By employing analytical techniques and properties of special functions, we have established new bounds and criteria that improve upon existing results in the literature. These findings not only enhance our understanding of the geometric behaviour of Bessel functions in the complex domain but also contribute to the broader theory of univalent and convex functions. The results may have further implications in applied mathematics, particularly in areas involving wave propagation, heat conduction, and other physical phenomena modelled by Bessel functions.

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Author Contributions

Natarajan Saraswathi: Conceptualization, Formal Analysis, Methodology, Writing – original draft

Murugesan Kasthuri: Supervision, Validation, Writing – review & editing

Conflicts of Interest

The authors declare that they have no conflicts of interest related to this publication.

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