
A Block-based Linear Multistep Formula for Directly Solving Nonlinear Fourth-order Initial Value Problems of ODEs

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To cite this article:

Duromola Monday Kolawole, Momoh Adelegan Lukuman, Akingbodi Oluwagbenga Joseph. (2025). A Block-based Linear Multistep Formula for Directly Solving Nonlinear Fourth-order Initial Value Problems of ODEs. *American Journal of Applied Mathematics*, 13(2), 103-116. <https://doi.org/10.11648/j.ajam.20251302.11>

Received: 13 January 2025; **Accepted:** 27 January 2025; **Published:** 27 February 2025

Abstract: This paper suggested a block-based linear multistep formula for directly solving nonlinear fourth-order initial value problems of ordinary differential equations (ODEs). The method was achieved by applying collocation and interpolation techniques to a first-kind Chebyshev polynomial. A continuous scheme was constructed through this procedure from where the proposed discrete formula was extracted. The extracted discrete formula was then implemented in block mode using the block matrix formulation and written explicitly as block equations. The proposed method is zero-stable, consistent, convergent, and p-stable, as demonstrated by the analysis of the basic properties of the derived scheme, with theoretical order eight. Six numerical examples were solved with the derived method to test its accuracy and effectiveness, all showing minimal error. A comparison with existing methods in the cited literature revealed that the proposed method offers good performance with minor errors.

Keywords: Linear Multi-step Method, Hybrid Points, Continuous Scheme, Discrete Scheme

1. Introduction

Differential equations are essential in various fields of science and engineering, as they are utilized to model and analyze dynamic systems, as reported in the works of [1, 2]. Fourth-order ordinary differential equations have gained significant attention due to their applications in structural mechanics, fluid dynamics, and quantum mechanics as discussed in the studies of [3, 4]. Researchers have introduced several numerical methods to solve fourth-order ordinary differential equations, one of which is the linear multistep method which has been extensively studied by [5]. As reported in the work of [6], Linear multistep methods are a type of numerical technique that approximates the solution of a differential equation by using a linear combination of previous function values and their derivatives. Linear multistep methods offer significant advantages, including higher-order accuracy than single-step methods like Runge-Kutta, as opined in the study of [7]. This feature makes them particularly effective

for solving higher-order differential equations, such as fourth-order ordinary ones as discussed in the work of [8].

This work aims at providing numerical solutions to the fourth-order problems where the lower derivatives are present. That is

$$\begin{aligned}y^{(4)}(x) &= f(x, y(x), y'(x), y''(x), y'''(x)), y(x_0) = \eta_0, \\y'(x_0) &= \eta_1, \quad y''(x_0) = \eta_2, \quad y'''(x_0) = \eta_3.\end{aligned}\quad (1)$$

It is assumed that f in (1) is a continuous real-value function [9]. The reduction approach was the first to be adopted for the equation of form (1) due to the availability of methods to handle the equivalent version of its first-order system, as reported in the work of [5, 10] and several others. As time passed, authors such as [7, 11–13] opined that due to the complexity encountered in the reduction approach and the increase in the scale of differential equations, the approach was considered not suitable for a larger system; hence the need for the direct approach proposed in the studies of [8, 14, 15] and

many others for handling (1).

The review of the articles [3, 16–21] shows that authors have independently made available numerical methods for solving fourth-order ODEs. The standard feature of their proposed methods is that they are implicit linear multistep methods implemented in block versions. Although most of these methods have an order of accuracy equal to and more significant than the one proposed in this article, they still do not perform better than the method. This might be due to the hybrid nature of the present method.

Specifically, [9] proposed a fully hybrid linear multistep formula to solve equations of type (1). The method directly solved the equation (1), but its accuracy in terms of error can still be improved. Therefore, this article examines the formulation and implementation of an efficient numerical algorithm to directly solve fourth-order ordinary differential equations. It also focuses on improving the accuracy of the existing methods.

2. Mathematical Formulation

This section aims to develop a linear multistep method for solving fourth-order ODEs of type (1) using Chebyshev polynomial of first kind denoted here as

$$y(x) = \sum_{j=0}^{(m+n)-1} a_j C_j(x) \quad (2)$$

where $y(x)$ is assumed to approximate (1) and x is continuously differentiable. Differentiating (2) four times yields:

$$y^{(4)}(x) = \sum_{j=4}^{(m+n)-1} a_j C_j^{(4)}(x). \quad (3)$$

Equating (1) and (3) gives the differential system

$$f(x, y(x), y'(x), y''(x), y'''(x)) = \sum_{j=4}^{(m+n)-1} a_j C_j^{(4)}(x). \quad (4)$$

Note that the parameters a_j 's are to be determined uniquely. Applying $x = x_{n+i}$, $i = \frac{1}{21} \left(\frac{1}{42} \right) \frac{1}{7}$ to equation (2) and $x = x_{n+i}$, $i = 0 \left(\frac{1}{42} \right) \frac{1}{6}$ to (3) yields the following system of equations:

$$y_{n+i} = \sum_{i=0}^{12} a_i C_i(x), \quad i = \frac{1}{21} \left(\frac{1}{42} \right) \frac{1}{7}, \quad (5)$$

$$f_{n+i} = \sum_{i=4}^{12} a_i C_i^{(4)}(x), \quad i = 0 \left(\frac{1}{42} \right) \frac{1}{6}. \quad (6)$$

Using the scaling function $x_{n+i} = x_n + ih$, the matrix representations of (5) and (6) were solved to get the coefficients a_j 's for $j = 0(1)12$. Substituting the a_j 's into (2), $x = x_n + th$ we get a continuous scheme

$$y(t) = \sum_{i=3}^6 \alpha_i(t) y_{n+\frac{i}{42}} + h^4 \sum_{i=0}^7 \beta_i(t) f_{n+\frac{i}{42}} \quad (7)$$

where, the coefficients that determine the continuous scheme (7) are listed below.

$$\begin{aligned} \alpha_1 &= -4 (24696t^3 - 4410t^2 + 259t - 5), & \alpha_2 &= 9 (32928t^3 - 5488t^2 + 294t - 5), \\ \alpha_3 &= -36 (8232t^3 - 1274t^2 + 63t - 1), & \alpha_4 &= 2 (49392t^3 - 7056t^2 + 329t - 5), \end{aligned}$$

$$\begin{aligned} \beta_0 &= \frac{h^4}{862562131200} (-10202571678015553536t^{11} + 5344204212293861376t^{10} - 1219411675424194560t^9 \\ &\quad + 159053696794460160t^8 - 13078650068047872t^7 + 704801434208256t^6 - 25048804290048t^5 \\ &\quad + 575041420800t^4 - 8075895212t^3 + 61969908t^2 - 202657t + 165), \\ \beta_1 &= \frac{h^4}{862562131200} (71418001746108874752t^{11} - 36073378432983564288t^{10} + 7820140092394291200t^9 \\ &\quad - 945801447009914880t^8 + 69031468404670464t^7 - 3016069250439168t^6 + 67624871086080t^5 \\ &\quad - 41772211096t^3 + 1058517306t^2 - 11206475t + 43725), \end{aligned}$$

$$\begin{aligned}
\beta_2 &= \frac{h^4}{143760355200} (-35709000873054437376t^{11} + 17368663689955049472t^{10} - 3578708177875353600t^9 \\
&\quad + 403314731157381120t^8 - 26569811849720832t^7 + 990704361411072t^6 - 16906217771520t^5 \\
&\quad - 1176700602t^3 + 300341041t^2 - 7828065t + 62150), \\
\beta_3 &= \frac{h^4}{172512426240} (71418001746108874752t^{11} - 33401276326836633600t^{10} + 6547710518038609920t^9 \\
&\quad - 692451629973024768t^8 + 42089719763976192t^7 - 1426133392459776t^6 + 22541623695360t^5 \\
&\quad - 9857702008t^3 + 624844374t^2 - 21025361t + 251295), \\
\beta_4 &= \frac{h^4}{172512426240} (-71418001746108874752t^{11} + 32065225273763168256t^{10} - 5991022579257999360t^9 \\
&\quad + 599859656481964032t^8 - 34421059382814720t^7 + 1109047885811712t^6 - 16906217771520t^5 \\
&\quad - 1313019092t^3 + 586065480t^2 - 27392701t + 401445), \\
\beta_5 &= \frac{h^4}{287520710400} (71418001746108874752t^{11} - 30729174220689702912t^{10} + 5487352539408875520t^9 \\
&\quad - 525445248338841600t^8 + 28997544721711104t^7 - 906173272553472t^6 + 13524974217216t^5 \\
&\quad - 4124915256t^3 + 249129034t^2 - 10606491t + 161645), \\
\beta_6 &= \frac{h^4}{431281065600} (-35709000873054437376t^{11} + 14696561583808118784t^{10} - 2518350199245619200t^9 \\
&\quad + 232900056020459520t^8 - 12503838663816192t^7 + 382831909092864t^6 - 5635405923840t^5 \\
&\quad + 1088608598t^3 - 14890953t^2 + 266560t - 4125), \text{ and} \\
\beta_7 &= \frac{h^4}{862562131200} (10202571678015553536t^{11} - 4008153159220396032t^{10} + 662723736643584000t^9 \\
&\quad - 59645136297922560t^8 + 3137794018394112t^7 - 94674819520512t^6 + 1380099409920t^5 - 263717608t^3 \\
&\quad + 3431274t^2 - 54215t + 825).
\end{aligned}$$

Evaluating (7) at $t = \frac{1}{6}$ yields the main discrete formula of the proposed block-based linear multistep formulas. That is

$$\begin{aligned}
y_{n+\frac{1}{6}} - 4y_{n+\frac{1}{7}} + 6y_{n+\frac{5}{42}} - 4y_{n+\frac{2}{21}} + y_{n+\frac{1}{14}} &= \frac{-h^4}{47048843520} (5f_n - 40f_{n+\frac{1}{42}} + 135f_{n+\frac{1}{21}} \\
&\quad - 229f_{n+\frac{1}{14}} - 2329f_{n+\frac{2}{21}} - 10134f_{n+\frac{5}{42}} - 2539f_{n+\frac{1}{7}} + 11f_{n+\frac{1}{6}}).
\end{aligned} \tag{8}$$

2.1. The Additional Formulas

We start the implementation of the main formula by obtaining six additional formulas from the (7). The additional schemes are gotten by applying $t = 0, \frac{1}{42}, \frac{1}{21}$ to (7) and $t = 0$ to its first, second and third derivatives. These are obtained respectively as;

$$\begin{aligned}
y_n - 20y_{n+\frac{1}{14}} + 45y_{n+\frac{2}{21}} - 36y_{n+\frac{5}{42}} + 10y_{n+\frac{1}{7}} &= \frac{h^4}{5227649280} (f_n + 265f_{n+\frac{1}{42}} + 7615f_{n+\frac{1}{21}} \\
&\quad + 2260f_{n+\frac{1}{14}} + 12165f_{n+\frac{2}{21}} + 2939f_{n+\frac{5}{42}} + 5f_{n+\frac{1}{6}} - 50f_{n+\frac{1}{7}}),
\end{aligned} \tag{9}$$

$$\begin{aligned}
y_{n+\frac{1}{42}} - 10y_{n+\frac{1}{14}} + 20y_{n+\frac{2}{21}} - 15y_{n+\frac{5}{42}} + 4y_{n+\frac{1}{7}} &= \frac{h^4}{47048843520} (5f_n - 31f_{n+\frac{1}{42}} + 20570f_{n+\frac{1}{21}} \\
&\quad + 2475f_{n+\frac{1}{14}} + 42095f_{n+\frac{2}{21}} + 10665f_{n+\frac{5}{42}} - 199f_{n+\frac{1}{7}} + 20f_{n+\frac{1}{6}}),
\end{aligned} \tag{10}$$

$$\begin{aligned}
y_{n+\frac{1}{21}} - 4y_{n+\frac{1}{14}} + 6y_{n+\frac{2}{21}} - 4y_{n+\frac{5}{42}} + y_{n+\frac{1}{7}} &= \frac{h^4}{47048843520} (+5f_{n+\frac{1}{42}} + 2679f_{n+\frac{1}{21}} \\
&\quad - 51f_{n+\frac{1}{14}} + 9854f_{n+\frac{2}{21}} + 2679f_{n+\frac{5}{42}} - 51f_{n+\frac{1}{7}} + 5f_{n+\frac{1}{6}}),
\end{aligned} \tag{11}$$

$$\begin{aligned}
hy'_n - 1554y_{n+\frac{1}{14}} + 3969y_{n+\frac{2}{21}} - 3402y_{n+\frac{5}{42}} + 987y_{n+\frac{1}{7}} &= \frac{h^4}{82148774400} \left(28951f_n + 1600925f_{n+\frac{1}{42}} \right. \\
&\quad \left. + 6709770f_{n+\frac{1}{21}} + 15018115f_{n+\frac{1}{14}} + 19566215f_{n+\frac{2}{21}} + 4545639f_{n+\frac{5}{42}} + 7745f_{n+\frac{1}{6}} - 76160f_{n+\frac{1}{7}} \right),
\end{aligned} \tag{12}$$

$$h^2 y_n'' + 79380 y_{n+\frac{1}{14}} - 222264 y_{n+\frac{2}{21}} + 206388 y_{n+\frac{5}{42}} - 63504 y_{n+\frac{1}{7}} = \frac{h^4}{59270400} (19162 f_n + 327309 f_{n+\frac{1}{42}} + 557219 f_{n+\frac{1}{21}} + 966055 f_{n+\frac{1}{14}} + 906100 f_{n+\frac{2}{21}} + 231103 f_{n+\frac{5}{42}} - 9209 f_{n+\frac{1}{7}}), \quad (13)$$

and

$$h^3 y_n''' - 2000376 y_{n+\frac{1}{14}} + 6001128 y_{n+\frac{2}{21}} - 6001128 y_{n+\frac{5}{42}} + 2000376 y_{n+\frac{1}{7}} = \frac{-h^4}{2822400} \left(535111 f_n + 2767838 f_{n+\frac{1}{42}} + 467811 f_{n+\frac{1}{21}} + 3265870 f_{n+\frac{1}{14}} + 435005 f_{n+\frac{2}{21}} + 819954 f_{n+\frac{5}{42}} - 144263 f_{n+\frac{1}{7}} + 17474 f_{n+\frac{1}{6}} \right). \quad (14)$$

2.2. Block Expression of the Derived Formula

In order to express (8) in block form, equations (8)-(14) are combined in form of matrix.

$$\bar{U}_0 \bar{Y} = \bar{U}_1 Y_n + h \bar{U}_2 Y_n' + h^2 \bar{U}_3 Y_n'' + h^3 \bar{U}_4 Y_n''' + h^4 (F_i \bar{F}_i), \quad (15)$$

where \bar{U}_i , $i = 0, \dots, 4$, F_i , $i = 0, 1$ are 7×7 matrices whose entries are coefficients of equations (8) and (8)-(14);

$$\begin{aligned} \bar{Y} &= \left(y_{n+\frac{1}{42}}, y_{n+\frac{1}{21}}, y_{n+\frac{1}{14}}, y_{n+\frac{2}{21}}, y_{n+\frac{5}{42}}, y_{n+\frac{1}{7}}, y_{n+\frac{1}{6}} \right), \\ Y_n &= \left(y_{n-\frac{1}{42}}, y_{n-\frac{1}{21}}, y_{n-\frac{1}{14}}, y_{n-\frac{2}{21}}, y_{n-\frac{5}{42}}, y_{n-\frac{1}{7}}, y_{n-\frac{1}{6}}, y_n \right), \\ Y_n' &= \left(y_{n-\frac{1}{42}}', y_{n-\frac{1}{21}}', y_{n-\frac{1}{14}}', y_{n-\frac{2}{21}}', y_{n-\frac{5}{42}}', y_{n-\frac{1}{7}}', y_{n-\frac{1}{6}}', y_n' \right), \\ Y_n'' &= \left(y_{n-\frac{1}{42}}'', y_{n-\frac{1}{21}}'', y_{n-\frac{1}{14}}'', y_{n-\frac{2}{21}}'', y_{n-\frac{5}{42}}'', y_{n-\frac{1}{7}}'', y_{n-\frac{1}{6}}'', y_n'' \right), \\ Y_n''' &= \left(y_{n-\frac{1}{42}}''', y_{n-\frac{1}{21}}''', y_{n-\frac{1}{14}}''', y_{n-\frac{2}{21}}''', y_{n-\frac{5}{42}}''', y_{n-\frac{1}{7}}''', y_{n-\frac{1}{6}}''', y_n''' \right), \\ \bar{F}_1 &= \left(f_{n+\frac{1}{42}}, f_{n+\frac{1}{21}}, f_{n+\frac{1}{14}}, f_{n+\frac{2}{21}}, f_{n+\frac{5}{42}}, f_{n+\frac{1}{7}}, f_{n+\frac{1}{6}} \right), \text{ and} \\ \bar{F}_0 &= \left(f_{n-\frac{1}{42}}, f_{n-\frac{1}{21}}, f_{n-\frac{1}{14}}, f_{n-\frac{2}{21}}, f_{n-\frac{5}{42}}, f_{n-\frac{1}{7}}, f_n \right). \end{aligned}$$

By matrix inversion, (15) becomes a matrix equation with;

$$\begin{aligned} \bar{U}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{U}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \end{pmatrix}, \\ \bar{U}_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3528} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{882}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{392}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{441}{25} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3528}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{98}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{72}{1} \end{pmatrix}, \quad \bar{U}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{444528} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{55566}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{16464}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{27783}{125} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{444528}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2058}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1296}{1} \end{pmatrix}, \end{aligned}$$

U_0 being a 7×7 identity matrix, F_0 and F_1 are also 7×7 matrices whose entries are omitted for the purpose of space.

Remark

Its should be noted that block formulas (15) is useful in obtaining the direct solution of fourth-order ODEs where lower derivatives are absent. However, (15) needs to be modified for solving general fourth-order ODEs by combining the first, second and third derivatives of (7) for which $t = \frac{i}{42}$, $i = 1(1)7$. Hence, we obtained

$$y_{n+\frac{1}{42}} = \frac{h^3 y_n'''}{444528} + \frac{h^2 y_n''}{3528} + \frac{1}{42} h y_n' + y_n + \frac{h^4}{372626840678400} \left(3102701 f_n + 4137616 f_{n+\frac{1}{42}} - 5122521 f_{n+\frac{1}{21}} \right. \\ \left. + 5415020 f_{n+\frac{1}{14}} - 3967805 f_{n+\frac{2}{21}} + 1878984 f_{n+\frac{5}{42}} - 517351 f_{n+\frac{1}{7}} + 62956 f_{n+\frac{1}{6}} \right) \quad (16)$$

$$y_{n+\frac{1}{21}} = \frac{h^3 y_n'''}{55566} + \frac{1}{882} h^2 y_n'' + \frac{1}{21} h y_n' + y_n + \frac{h^4}{1455573596400} \left(132526 f_n + 315461 f_{n+\frac{1}{42}} - 306810 f_{n+\frac{1}{21}} \right. \\ \left. + 320335 f_{n+\frac{1}{14}} - 233050 f_{n+\frac{2}{21}} + 109899 f_{n+\frac{5}{42}} - 30176 f_{n+\frac{1}{7}} + 3665 f_{n+\frac{1}{6}} \right) \quad (17)$$

$$y_{n+\frac{1}{14}} = \frac{h^3 y_n'''}{16464} + \frac{1}{392} h^2 y_n'' + \frac{1}{14} h y_n' + y_n + \frac{h^4}{170382643200} \left(58453 f_n + 179604 f_{n+\frac{1}{42}} - 132813 f_{n+\frac{1}{21}} \right. \\ \left. + 149800 f_{n+\frac{1}{14}} - 109485 f_{n+\frac{2}{21}} + 51732 f_{n+\frac{5}{42}} - 14219 f_{n+\frac{1}{7}} + 1728 f_{n+\frac{1}{6}} \right) \quad (18)$$

$$y_{n+\frac{2}{21}} = \frac{4}{27783} h^3 y_n''' + \frac{2}{441} h^2 y_n'' + \frac{2}{21} h y_n' + y_n + \frac{h^4}{90973349775} \left(78152 f_n + 274240 f_{n+\frac{1}{42}} - 154008 f_{n+\frac{1}{21}} \right. \\ \left. + 209600 f_{n+\frac{1}{14}} - 149870 f_{n+\frac{2}{21}} + 70848 f_{n+\frac{5}{42}} - 19480 f_{n+\frac{1}{7}} + 2368 f_{n+\frac{1}{6}} \right) \quad (19)$$

$$y_{n+\frac{5}{42}} = \frac{125}{444528} h^3 y_n''' + \frac{25}{3528} h^2 y_n'' + \frac{5}{42} h y_n' + y_n + \frac{h^4}{14905073627136} \left(25842625 f_n + 98195000 f_{n+\frac{1}{42}} \right. \\ \left. - 42733125 f_{n+\frac{1}{21}} + 74762500 f_{n+\frac{1}{14}} - 49290625 f_{n+\frac{2}{21}} + 23688000 f_{n+\frac{5}{42}} \right. \\ \left. - 6516875 f_{n+\frac{1}{7}} + 792500 f_{n+\frac{1}{6}} \right) \quad (20)$$

$$y_{n+\frac{1}{7}} = \frac{h^3 y_n'''}{2058} + \frac{1}{98} h^2 y_n'' + \frac{1}{7} h y_n' + y_n + \frac{h^4}{665557200} \left(2038 f_n + 8163 f_{n+\frac{1}{42}} - 2808 f_{n+\frac{1}{21}} \right. \\ \left. + 6385 f_{n+\frac{1}{14}} - 3690 f_{n+\frac{2}{21}} + 1917 f_{n+\frac{5}{42}} - 518 f_{n+\frac{1}{7}} + 63 f_{n+\frac{1}{6}} \right) \quad (21)$$

$$y_{n+\frac{1}{6}} = \frac{h^3 y_n'''}{1296} + \frac{1}{72} h^2 y_n'' + \frac{1}{6} h y_n' + y_n + \frac{h^4}{22170931200} \left(109493 f_n + 455308 g f_{n+\frac{1}{42}} - 125685 f_{n+\frac{1}{21}} \right. \\ \left. + 368480 f_{n+\frac{1}{14}} - 182525 f_{n+\frac{2}{21}} + 111132 f_{n+\frac{5}{42}} - 26803 f_{n+\frac{1}{7}} + 3400 f_{n+\frac{1}{6}} \right) \quad (22)$$

$$y'_{n+\frac{1}{42}} = \frac{h^2 y_n'''}{3528} + \frac{1}{42} h y_n'' + y_n' + \frac{h^3}{268850534400} \left(335799 f_n + 562618 f_{n+\frac{1}{42}} - 662757 f_{n+\frac{1}{21}} + 694230 f_{n+\frac{1}{14}} \right. \\ \left. - 506675 f_{n+\frac{2}{21}} + 239406 f_{n+\frac{5}{42}} - 65823 f_{n+\frac{1}{7}} + 8002 f_{n+\frac{1}{6}} \right) \quad (23)$$

$$y'_{n+\frac{1}{21}} = \frac{1}{882} h^2 y_n''' + \frac{1}{21} h y_n'' + y_n' + \frac{h^3}{2100394800} \left(13376 f_n + 38762 f_{n+\frac{1}{42}} - 32823 f_{n+\frac{1}{21}} + 34730 f_{n+\frac{1}{14}} \right. \\ \left. - 25300 f_{n+\frac{2}{21}} + 11934 f_{n+\frac{5}{42}} - 3277 f_{n+\frac{1}{7}} + 398 f_{n+\frac{1}{6}} \right) \quad (24)$$

$$y'_{n+\frac{1}{14}} = \frac{1}{392} h^2 y_n''' + \frac{1}{14} h y_n'' + y_n' + \frac{h^3}{368793600} \left(5703 f_n + 20406 f_{n+\frac{1}{42}} - 11709 f_{n+\frac{1}{21}} + 15050 f_{n+\frac{1}{14}} \right. \\ \left. - 10995 f_{n+\frac{2}{21}} + 5202 f_{n+\frac{5}{42}} - 1431 f_{n+\frac{1}{7}} + 174 f_{n+\frac{1}{6}} \right) \quad (25)$$

$$y'_{n+\frac{2}{21}} = \frac{2}{441} h^2 y_n''' + \frac{2}{21} h y_n'' + y_n' + \frac{h^3}{131274675} \left(3747 d f_n + 14804 f_{n+\frac{1}{42}} - 5826 f_{n+\frac{1}{21}} + 10860 f_{n+\frac{1}{14}} \right. \\ \left. - 7315 f_{n+\frac{2}{21}} + 3468 f_{n+\frac{5}{42}} - 954 f_{n+\frac{1}{7}} + 116 f_{n+\frac{1}{6}} \right) \quad (26)$$

$$y'_{n+\frac{5}{42}} = \frac{25}{3528} h^2 y_n''' + \frac{5}{42} h y_n'' + y_n' + \frac{h^3}{10754021376} \left(490375 f_n + 2052250 f_{n+\frac{1}{42}} - 577125 f_{n+\frac{1}{21}} \right. \\ \left. + 1603750 f_{n+\frac{1}{14}} - 891875 f_{n+\frac{2}{21}} - 125375 f_{n+\frac{1}{7}} + 15250 f_{n+\frac{1}{6}} + 456750 f_{n+\frac{5}{42}} \right) \quad (27)$$

$$y'_{n+\frac{1}{7}} = \frac{1}{98} h^2 y_n''' + \frac{1}{7} h y_n'' + y_n' + \frac{h^3}{2881200} \left(192 f_n + 834 f_{n+\frac{1}{42}} - 171 f_{n+\frac{1}{21}} + 690 f_{n+\frac{1}{14}} - 300 f_{n+\frac{2}{21}} \right. \\ \left. + 198 f_{n+\frac{5}{42}} - 49 f_{n+\frac{1}{7}} + 6 m f_{n+\frac{1}{6}} \right) \quad (28)$$

$$y'_{n+\frac{1}{6}} = \frac{1}{72} h^2 y_n''' + \frac{1}{6} h y_n'' + y_n' + \frac{h^3}{111974400} \left(10263 f_n + 45766 f_{n+\frac{1}{42}} - 6909 f_{n+\frac{1}{21}} + 39690 f_{n+\frac{1}{14}} \right. \\ \left. - 13475 f_{n+\frac{2}{21}} + 12642 f_{n+\frac{5}{42}} - 1911 f_{n+\frac{1}{7}} + 334 f_{n+\frac{1}{6}} \right) \quad (29)$$

$$y''_{n+\frac{1}{42}} = \frac{1}{42}hy_n''' + y_n'' + \frac{h^2}{3200601600} \left(416173f_n + 950684f_{n+\frac{1}{42}} - 1025097f_{n+\frac{1}{21}} + 1059430f_{n+\frac{1}{14}} - 768805f_{n+\frac{2}{21}} + 362112f_{n+\frac{5}{42}} - 99359f_{n+\frac{1}{7}} + 12062mf_{n+\frac{1}{6}} \right) \quad (30)$$

$$y''_{n+\frac{1}{21}} = \frac{1}{21}hy_n''' + y_n'' + \frac{h^2}{50009400} \left(14939f_n + 55642f_{n+\frac{1}{42}} - 34986f_{n+\frac{1}{21}} + 39950f_{n+\frac{1}{14}} - 29405f_{n+\frac{2}{21}} + 13926f_{n+\frac{5}{42}} - 3832f_{n+\frac{1}{7}} + 466mf_{n+\frac{1}{6}} \right) \quad (31)$$

$$y''_{n+\frac{1}{14}} = \frac{1}{14}hy_n''' + y_n'' + \frac{h^2}{13171200} \left(6133f_n + 26094f_{n+\frac{1}{42}} - 8037f_{n+\frac{1}{21}} + 17220f_{n+\frac{1}{14}} - 12165f_{n+\frac{2}{21}} + 5742f_{n+\frac{5}{42}} - 1579f_{n+\frac{1}{7}} + 192f_{n+\frac{1}{6}} \right) \quad (32)$$

$$y''_{n+\frac{2}{21}} = \frac{2}{21}hy_n''' + y_n'' + \frac{h^2}{6251175} \left(3956f_n + 17788f_{n+\frac{1}{42}} - 2874f_{n+\frac{1}{21}} + 14180f_{n+\frac{1}{14}} - 7490f_{n+\frac{2}{21}} + 3684f_{n+\frac{5}{42}} - 1018f_{n+\frac{1}{7}} + 124f_{n+\frac{1}{6}} \right) \quad (33)$$

$$y''_{n+\frac{5}{42}} = \frac{5}{42}hy_n''' + y_n'' + \frac{h^2}{128024064} \left(102425f_n + 475000f_{n+\frac{1}{42}} - 40125f_{n+\frac{1}{21}} + 421250f_{n+\frac{1}{14}} - 130625f_{n+\frac{2}{21}} + 102900f_{n+\frac{5}{42}} - 26875f_{n+\frac{1}{7}} + 3250f_{n+\frac{1}{6}} \right) \quad (34)$$

$$y''_{n+\frac{1}{7}} = \frac{1}{7}hy_n''' + y_n'' + \frac{h^2}{205800} \left(199f_n + 942f_{n+\frac{1}{42}} - 36f_{n+\frac{1}{21}} + 890f_{n+\frac{1}{14}} - 165f_{n+\frac{2}{21}} + 306f_{n+\frac{5}{42}} - 42f_{n+\frac{1}{7}} + 6f_{n+\frac{1}{6}} \right) \quad (35)$$

$$y''_{n+\frac{1}{6}} = 7\frac{1}{6}hy_n''' + y_n'' + \frac{h^2}{9331200} \left(10597f_n + 50666f_{n+\frac{1}{42}} + 147f_{n+\frac{1}{21}} + 49000f_{n+\frac{1}{14}} - 4165f_{n+\frac{2}{21}} + 19698f_{n+\frac{5}{42}} + 2989f_{n+\frac{1}{7}} + 668f_{n+\frac{1}{6}} \right) \quad (36)$$

$$y'''_{n+\frac{1}{42}} = y_n''' + \frac{h}{5080320} \left(36799f_n + 139849f_{n+\frac{1}{42}} - 121797f_{n+\frac{1}{21}} + 123133f_{n+\frac{1}{14}} - 88547f_{n+\frac{2}{21}} + 41499f_{n+\frac{5}{42}} - 11351f_{n+\frac{1}{7}} + 1375f_{n+\frac{1}{6}} \right) \quad (37)$$

$$y'''_{n+\frac{1}{21}} = y_n''' + \frac{h}{158760} \left(1107f_n + 5864f_{n+\frac{1}{42}} - 639f_{n+\frac{1}{21}} + 2448f_{n+\frac{1}{14}} - 1927f_{n+\frac{2}{21}} + 936f_{n+\frac{5}{42}} - 261f_{n+\frac{1}{7}} + 32f_{n+\frac{1}{6}} \right) \quad (38)$$

$$y'''_{n+\frac{1}{14}} = y_n''' + \frac{h}{188160} \left(1325f_n + 6795f_{n+\frac{1}{42}} + 1377f_{n+\frac{1}{21}} + 5927f_{n+\frac{1}{14}} - 3033f_{n+\frac{2}{21}} + 1377f_{n+\frac{5}{42}} - 373f_{n+\frac{1}{7}} + 45f_{n+\frac{1}{6}} \right) \quad (39)$$

$$y'''_{n+\frac{2}{21}} = y_n''' + \frac{h}{19845} \left(139f_n + 724f_{n+\frac{1}{42}} + 108f_{n+\frac{1}{21}} + 892f_{n+\frac{1}{14}} - 53f_{n+\frac{2}{21}} + 108f_{n+\frac{5}{42}} - 32f_{n+\frac{1}{7}} + 4f_{n+\frac{1}{6}} \right) \quad (40)$$

$$y'''_{n+\frac{5}{42}} = y_n''' + \frac{h}{1016064} \left(7155f_n + 36725f_{n+\frac{1}{42}} + 6975f_{n+\frac{1}{21}} + 41625f_{n+\frac{1}{14}} + 13625f_{n+\frac{2}{21}} + 17055f_{n+\frac{5}{42}} - 2475f_{n+\frac{1}{7}} + 275f_{n+\frac{1}{6}} \right) \quad (41)$$

$$y'''_{n+\frac{1}{7}} = y_n''' + \frac{h}{5880} \left(41f_n + 216f_{n+\frac{1}{42}} + 27f_{n+\frac{1}{21}} + 272f_{n+\frac{1}{14}} + 27f_{n+\frac{2}{21}} + 216f_{n+\frac{5}{42}} + 41f_{n+\frac{1}{7}} \right) \quad (42)$$

$$y'''_{n+\frac{1}{6}} = y_n''' + \frac{h}{103680} \left(751f_n + 3577f_{n+\frac{1}{42}} + 1323f_{n+\frac{1}{21}} + 2989f_{n+\frac{1}{14}} + 2989f_{n+\frac{2}{21}} + 1323f_{n+\frac{5}{42}} + 3577f_{n+\frac{1}{7}} + 751f_{n+\frac{1}{6}} \right) \quad (43)$$

3. Analysis of the Properties of the Derived Method

This section delved into the analysis of the basic properties of the presented method.

3.1. Error Constant, Local Truncation Error, and Order of the Derived Method

Proposition 1

If $y(x)$ is continuously differentiable and assumed to be $s(x)$ then, the Local Truncation Error (LTE) of each formulas the proposed take the form; $T_{\frac{r}{42}}\{s(x) : h\} = c_{n+13}s^{(13)}(x_n)h^{13} + O(h^{14})$.

Proof

Let begin by defining the LTE of the formulas in (16)-(21) as

$$T_{\frac{r}{42}}\{s(x) : h\} = s\left(x_n + \frac{r}{42}h\right) - \left\{ \sum_{b=0}^3 \alpha_b s^{(b)}(x) h^b - h^4 \sum_{r=0}^7 \beta_r s^{(4)}\left(x + \frac{r}{42}h\right) \right\}. \quad (44)$$

whose Taylor series expansion about the point x yields

$$\begin{aligned} T_{\frac{r}{42}}\{s(x) : h\} &= c_0 s(x) + c_1 h s'(x) + c_2 h^2 s''(x) + \dots + c_{p+3} h^{p+3} s^{(p+3)}(x) + c_{p+4} h^{p+4} s^{(p+4)}(x), \\ &= c_{p+4} h^{p+4} s^{(p+4)}(x_n) + O(h^{p+5}). \end{aligned} \quad (45)$$

Here the term c_{p+4} is the error constant. The LTE associated with the formulas (16)-(21) are:

$$\begin{aligned} T_{\frac{1}{42}}\{s(x) : h\} &= \frac{25h^{12}s^{(12)}(x_n)}{1457784251641574351437824} + O(h^{13}), \\ T_{\frac{1}{21}}\{s(x) : h\} &= \frac{733h^{12}s^{(12)}(x_n)}{1657298758500000000000} + O(h^{13}), \\ T_{\frac{1}{14}}\{s(x) : h\} &= \frac{h^{12}s^{(12)}(x_n)}{561330000000000000} + O(h^{13}), \\ T_{\frac{2}{21}}\{s(x) : h\} &= \frac{37h^{12}s^{(12)}(x_n)}{8092279094238281250} + O(h^{13}), \\ T_{\frac{5}{42}}\{s(x) : h\} &= \frac{317h^{12}s^{(12)}(x_n)}{33941478574080000000} + O(h^{13}), \\ T_{\frac{1}{7}}\{s(x) : h\} &= \frac{h^{12}s^{(12)}(x_n)}{601425000000000000} + O(h^{13}), \\ T_{\frac{1}{6}}\{s(x) : h\} &= \frac{h^{12}s^{(12)}(x_n)}{601425000000000000} + O(h^{13}) \end{aligned}$$

The order, error constant and LTE of (23)-(43) were obtained in similar manner. Since $c_0 = c_1 = \dots = c_{p+3} = 0$, $c_{p+4} \neq 0$, then (16)-(43) are uniformly of order $p = 8$, (see [5, 9]).

3.2. Consistency of the Method

Definition 1 (see [5, 9]) “The linear multistep method is said to be consistent if it has order $p \geq 1$. It is obvious that the present method is consistent.”

3.3. Zero Stability of the Block Method

As claimed by [7, 22] zero stability of a numerical method imitates the dynamics of the methods as $h \rightarrow 0$. This required

setting h to zero in (15) which reduces to

$$\bar{U}_0 \bar{Y} = \bar{U}_1 Y_n \quad (46)$$

where, \bar{U}_0 and \bar{U}_1 are as defined before.

Definition 2 (see [5]) “The linear multistep method (15) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root with modulus one is simple.”

The characteristic polynomial of (46) is,

$$\text{Det}(\lambda \bar{U}_0 - \bar{U}_1) = 0 \quad (47)$$

$$\text{Det} \left(\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right) = 0 \quad (48)$$

This gives $\lambda^7(\lambda - 1) = 0$, that is $\lambda = (0, 0, 0, 0, 0, 0, 0, 1)$. Furthermore, other results indicated that the roots of the characteristics polynomial of the derived method are all equal to one (i.e not exceeding the order of the differential equation) and simple, hence by Def 2, the method is zero stable.

3.4. Convergence

We further the analysis by stating the fundamental Dahlquist theorem.

Theorem 3.1. (See [5, 9]) “The necessary and sufficient conditions for a linear multi- step method to be convergent are that it be consistent and zero-stable”.

Having shown that the proposed method is consistent and zero stable, hence, it is also convergent.

3.5. Region of Absolute Stability of the Method

Definition 3 [5, 22] “The block method is P -stable if the periodicity interval of the method $(0, +\infty)$ ”

Proposition The derived block method is P -stable.

This proposition can be established by considering the first and second characteristics polynomial of (15) whose stability polynomial is

$$\Pi(r, z) = p(r) + zq(r) \quad (49)$$

which is gotten by applying (15) to the decay function

$$y^{(4)}(x) = -\lambda^4 y(x). \quad (50)$$

where, $z = \lambda^4 h^4$ and $\lambda = \frac{\partial f}{\partial y}$. Solving (49) for $\Pi(r, z) = 0$ with $r = e^{I\theta}$ gives

$$z = \frac{18223519635182674907136 \times 10^{27} (e^{\frac{1}{6}\theta} - 1)}{18662400e^{\frac{1}{6}\theta} - 15611903689201}. \quad (51)$$

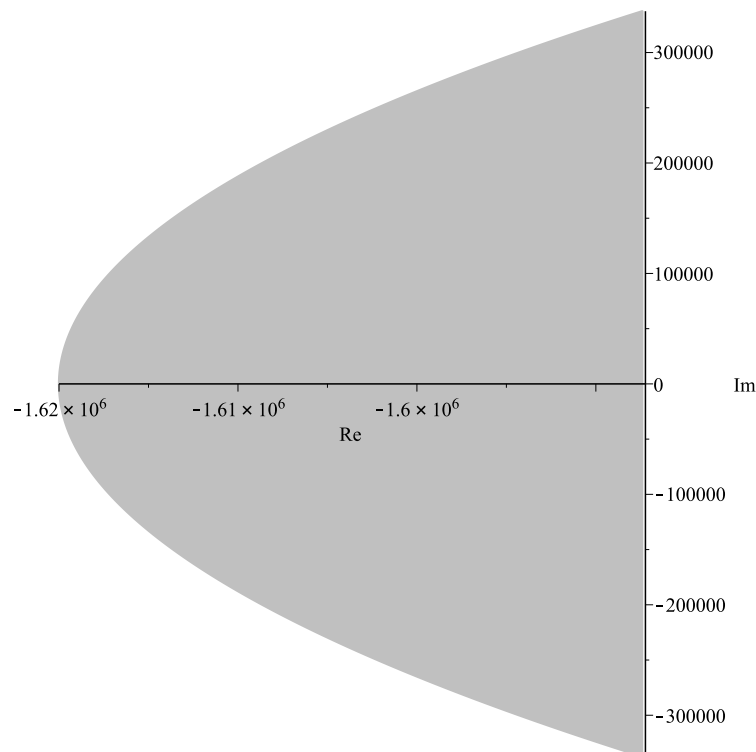


Figure 1. Region of absolute stability of the method.

The region of absolute stability or the suggested method is depicted in Figure 1; its unstable areas are shaded. This verified that the stability region of the suggested method includes the whole RHS of the complex plane i.e $(0, +\infty)$. The developed method is hence p -stable.

4. Numerical Experiments

In this section, we present numerical experiment with six numerical examples to test the efficiency of the derived numerical method. We denoted the exact solution with $y(x)$ and the approximate solution with y_n . Hence, the absolute error is measured by $|y(x) - y_n|$ to confirm the accuracy of the derived method as compared with some cited method in literature.

4.1. Problem 1

We start the numerical experiment by considering a more general fourth-order ODEs of the form

$$y^{(4)}(x) = y'''(x) + y''(x) + y'(x) + 2y(x),$$
$$y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 30$$
(52)

which has appeared in [17]. The analytical solution is given in the ref [17] as $y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x$. The interval of solution for this example is $[0, 2]$ with ten iterations and the results are presented Tables 1 and 3. It is clear from Table 2 that the derived method compares well with methods in cited refs [12], [17] and [23].

Table 1. Computed Results of Problem 1.

x	yapprox	yExact	yError
0.2	0.0421714	0.0421714	7.8410×10^{-16}
0.4	0.3579	0.3579	1.4877×10^{-14}
0.6	1.2904	1.2904	7.5939×10^{-14}
1.6	38.9718	38.9718	7.7876×10^{-12}
1.8	62.9237	62.9237	1.5014×10^{-11}
2.	99.0875	99.0875	2.7782×10^{-11}

Table 2. Comparison of results of problem 1.

x	AE in [23]	AE in [12]	Error in [17]	Error in new method
0.103125	4.83×10^{-17}	2.11×10^{-13}	2.27×10^{-19}	2.17×10^{-19}
0.206250	1.39×10^{-16}	5.70×10^{-12}	4.56×10^{-19}	8.67×10^{-19}
0.306250	6.69×10^{-16}	6.80×10^{-10}	3.89×10^{-19}	4.34×10^{-19}
0.406250	2.01×10^{-15}	2.21×10^{-9}	6.04×10^{-19}	8.67×10^{-19}
0.506250	4.74×10^{-15}	1.27×10^{-8}	9.73×10^{-19}	0.00
0.603125	9.19×10^{-15}	3.46×10^{-6}	8.90×10^{-19}	0.00
0.703125	1.60×10^{-13}	6.55×10^{-6}	8.84×10^{-19}	8.67×10^{-19}
0.803125	2.54×10^{-13}	9.59×10^{-6}	6.69×10^{-19}	3.47×10^{-18}
0.903125	3.81×10^{-13}	1.05×10^{-6}	2.26×10^{-18}	1.74×10^{-18}
1.003125	5.41×10^{-13}	5.70×10^{-6}	1.65×10^{-18}	0.00

Table 3. Comparison of results of problem 1.

h	Method	E (x _N) at x = 2
0.1	New Method	$3.22e^{-12}$
	[17]	$8.07e^{-10}$
	[16]	$1.74e^{-8}$
0.05	New Method	$3.22e^{-12}$
	[17]	$3.22e^{-12}$
	[16]	$8.45e^{-11}$
0.025	New Method	$3.22e^{-12}$
	[17]	$7.30e^{-16}$
	[16]	$3.69e^{-13}$

4.2. Problem 2

The second experimental problem considered is the nonhomogeneous, non linear fourth-order ODEs of the type

$$y^{(4)}(x) = -4x^2 + e^x (x^2 - 4x + 1) - y(x)y'(x) + (y'(x))^2, y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0$$

which has featured in [20]. The exact solution according to [20] is $y(x) = x^2 + e^x$. We integrated the problem over $[0, 0.2]$ for 10 iterations. Table 4 showcases the numerical convergence of method.

Table 4. Computed Results of Problem 2.

x	yapprox	yExact	yError
0.02	1.0206	1.0206	$5.369349409534152 \times 10^{-11}$
0.04	1.04241	1.04241	$1.7300092469696438 \times 10^{-9}$
0.06	1.06544	1.06544	$1.3229241258017055 \times 10^{-8}$
0.14	1.16987	1.16987	$9.420090574252526 \times 10^{-7}$
0.16	1.19911	1.19911	$1.8505389298706376 \times 10^{-6}$
0.18	1.22961	1.22962	$3.3604098703676755 \times 10^{-6}$
0.2	1.2614	1.2614	$5.735328080458402 \times 10^{-6}$

4.3. Problem 3

We also considered a fourth-order ODEs without lower derivatives

$$y^{(4)}(x) = -y''(x), y(0) = 0, y'(0) = -\frac{1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi}, y'''(0) = \frac{1.2}{144 - 100\pi}$$

whose exact solution is reported as $y(t) = \frac{-t - 1.2 \sin(t) - \cos(t) + 1}{144 - 100\pi}$ according to [17]. We solved this problem over $[0, 0.2]$ for ten iterations and are compared with [17] and [23] with [12] and [16], as presented in Table 5 . It is clear from Table 6 that methods shows good performance is up to at fifteen decimal digits.

Table 5. Comparison of results of Problem 3.

h	Method	E (x _N) at x = 1.01325
0.003125	New Method	$8.13e^{-20}$
	[17]	$1.65e^{-18}$
	[23]	$1.58e^{-7}$
0.103125	New Method	$4.34e^{-19}$
	[17]	$2.95e^{-17}$
	[12]	$5.69e^{-6}$

Table 6. Comparison of results of Problem 3.

x	Error in [17]	Error in [16]	Error in New Method
0.2	3.512997×10^{-13}	2.318519×10^{-13}	7.84095×10^{-16}
0.4	4.183300×10^{-12}	2.260324×10^{-12}	1.4877×10^{-14}
0.6	1.430233×10^{-11}	1.965140×10^{-11}	7.59393×10^{-14}
0.8	3.592435×10^{-11}	9.914494×10^{-11}	2.64233×10^{-13}
1.0	7.276201×10^{-11}	3.311345×10^{-10}	7.31859×10^{-13}
1.2	1.336016×10^{-10}	9.000018×10^{-10}	1.75504×10^{-12}
1.4	2.234540×10^{-10}	2.117600×10^{-9}	3.83338×10^{-12}
1.6	3.579606×10^{-10}	4.550582×10^{-9}	7.78755×10^{-12}
1.8	5.433495×10^{-10}	9.117964×10^{-9}	1.50138×10^{-11}
2.0	8.079574×10^{-10}	1.740907×10^{-8}	2.77822×10^{-11}

4.4. Problem 4

The experimental problem

$$y^{(4)}(x) = \frac{3Sin(y(x))(3 + 2Sin^2(y(x)))}{Cos^7(y(x))}, y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1$$

which can be found in [15] is the fourth example considered. The problem is integrated in the interval $[0, \frac{9}{10}]$. The exact solution is given by $y(x) = arcsin(x)$.

Table 7. Comparison of results of Problem 3.

x	yapprox	yExact	yError
0.1	0.100167	0.100167	1.3878×10^{-17}
0.2	0.201358	0.201358	2.7756×10^{-17}
0.3	0.304693	0.304693	1.6653×10^{-16}
0.4	0.411517	0.411517	1.6653×10^{-16}
0.5	0.523599	0.523599	2.2205×10^{-16}
0.6	0.643501	0.643501	2.2205×10^{-16}
0.7	0.775397	0.775397	3.3307×10^{-16}
0.8	0.927295	0.927295	3.3307×10^{-16}
0.9	1.11977	1.11977	6.46150×10^{-14}

Table 8. Comparison of results of Problem 3.

x	yapprox	yExact	NDSolve	yError	NDSolveError
0.1	0.100167	0.100167	0.100167	1.3878×10^{-17}	5.6969×10^{-10}
0.2	0.201358	0.201358	0.201358	2.7756×10^{-17}	1.4339×10^{-9}
0.3	0.304693	0.304693	0.304693	1.6653×10^{-16}	2.6689×10^{-9}
0.4	0.411517	0.411517	0.411517	1.6653×10^{-16}	4.5226×10^{-9}
0.5	0.523599	0.523599	0.523599	2.2205×10^{-16}	7.3306×10^{-9}
0.6	0.643501	0.643501	0.643501	2.2205×10^{-16}	1.1850×10^{-8}
0.7	0.775397	0.775397	0.775398	3.3307×10^{-16}	1.9447×10^{-8}
0.8	0.927295	0.927295	0.927295	3.3307×10^{-16}	3.2945×10^{-8}

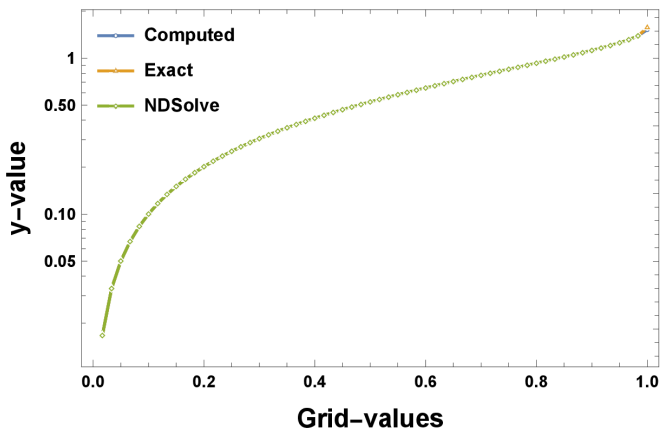


Figure 2. Numerical and exact solution of problem 4.

4.5. Problem 5

Another sample problem considered in this work is

$$y^{(4)}(x) = y^2(x) + cos^2(x) + sin(x) - 1, y(0) = 0,$$

$$y'(0) = 1, y''(0) = 0, y'''(0) = -1$$

which has also appeared in [15]. The interval $[0, 1]$ is where the problem is integrated. The formula for the precise solution is $y(x) = sin(x)$. The numerical results for the test problem are shown in Table 9. These outcomes show how accurate the suggested approach is, as they match the analytical solutions to sixteen decimal places. Figure 3 compares the absolute errors of Method with NDSolve and clearly underscores its superior performance.

Table 9. Computed results of Problem 5.

x	yapprox	yExact	NDSolve	yError	NDSolveError
0.1	0.0998334	0.0998334	0.0998334	1.3878×10^{-17}	1.578×10^{-8}
0.2	0.198669	0.198669	0.198669	2.7756×10^{-17}	1.7804×10^{-8}
0.3	0.29552	0.29552	0.29552	5.5511×10^{-17}	1.8635×10^{-8}
0.4	0.389418	0.389418	0.389418	1.6653×10^{-16}	1.9466×10^{-8}
0.5	0.479426	0.479426	0.479426	5.5511×10^{-17}	2.3483×10^{-8}
0.7	0.644218	0.644218	0.644218	1.1102×10^{-16}	2.4403×10^{-8}
0.8	0.717356	0.717356	0.717356	2.2205×10^{-16}	2.4967×10^{-8}
0.9	0.783327	0.783327	0.783327	3.3307×10^{-16}	2.4949×10^{-8}
1.	0.841471	0.841471	0.841471	4.4409×10^{-16}	2.3560×10^{-8}

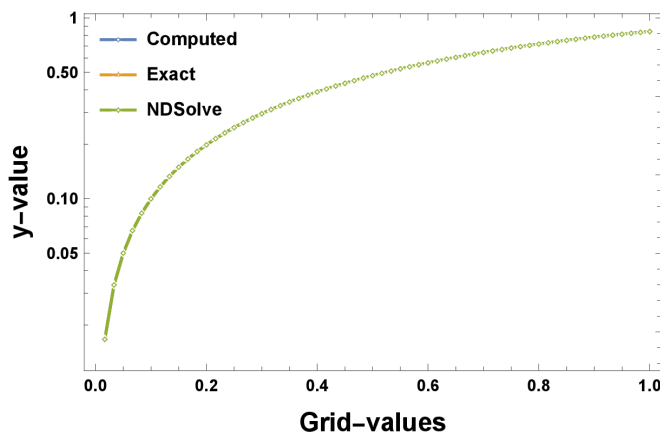


Figure 3. Numerical and exact solution of problem 5.

4.6. Problem 6

The last sample problem considered in this work is

$$y^{(4)}(x) = 1 - y(x)$$

which has featured in [15]. The problem is integrated in the over $[0, 10]$. The exact solution is given by $y(x) = -\frac{1}{2}e^{-\frac{x}{\sqrt{2}}} \left(-2e^{\frac{x}{\sqrt{2}}} + e^{\sqrt{2}x} \cos\left(\frac{x}{\sqrt{2}}\right) + \cos\left(\frac{x}{\sqrt{2}}\right) \right)$. $[0, \frac{1}{6}]$ is the interval on which this problem is specified. The numerical results are shown in Table 10 and match the analytical answer to eighteen decimal places, demonstrating the great accuracy of the approach. The method's higher performance is seen when comparing its absolute errors with NDSolve in Figure 4.

Table 10. Computed results of Problem 6.

x	yapprox	yExact	NDSolve	yError	NDSolveError
0.1	4.166666×10^{-6}	4.1667×10^{-6}	4.1721×10^{-6}	7.5886×10^{-18}	5.4695×10^{-9}
0.2	0.0000667	0.0000667	0.0000667	8.3633×10^{-17}	1.7101×10^{-10}
0.3	0.0003374	0.0003374	0.000337497	8.3809×10^{-17}	1.4258×10^{-9}
0.4	0.00106665	0.00106665	0.00106665	4.5537×10^{-17}	2.6946×10^{-9}
0.5	0.00260407	0.00260407	0.00260407	7.4159×10^{-17}	4.178×10^{-9}
0.6	0.00539958	0.00539958	0.00539958	5.6376×10^{-17}	5.9626×10^{-9}
0.7	0.0100027	0.0100027	0.0100027	6.5920×10^{-17}	9.1246×10^{-9}
0.8	0.0170625	0.0170625	0.0170625	8.3267×10^{-17}	1.1504×10^{-8}

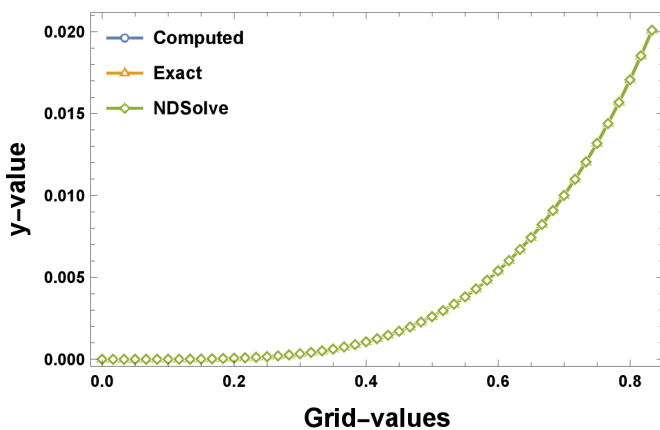


Figure 4. Numerical and exact solution of problem 6.

5. Conclusion

This study provided an order-eight Chebyshev-induced hybrid block method that solves initial value problems of fourth-order ODEs without the need to reduce to a set of ordinary differential equations of the first order. The derived method is p- and zero stable, consistent, and convergent. Six numerical examples were used to test the accuracy and usability of the proposed scheme. The derived method is efficient because it produces minimal errors and, as a result, has a higher accuracy for handling the direct solution of the fourth-order initial value problem of ordinary differential equations.

Abbreviations

ODEs	Ordinary Differential Equations
LTE	Local Truncation Error Mathematica
RHS	Right Hand Side
NDSolve	Numerical Solver in Wolfram
AE	Absolute Errors

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Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments. A. L. M. thanks H. Ramos for his help.

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Akingbodi Oluwagbenga Joseph: Methodology, Writing - original draft

Funding

This research received no external funding.

Conflicts of Interest

The authors declare no conflicts of interest.

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