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On k-prime Ideal of a Finite Commutative Ring

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Abstract: Let $\mathfrak{B}$ be a finite commutative ring with unity. Let I be the k-prime ideal of $\mathfrak{B}$ which is a generalization of prime ideal. For any fixed integer $k \geq 3$, an ideal I of a ring $\mathfrak{B}$ is said to be k-prime whenever for any set G of nonzero, distinct, and non-unit elements of $\mathfrak{B}$, the product $g_1.g_2.g_3...g_k$ belongs to I implies that the product of the elements of a proper subset of G is in I. This paper investigates the algebraic structure of k-prime ideal in finite commutative ring Z_n and studies their behaviour under ideal operations. Initially it is proved that every prime ideal of a ring $\mathfrak{B}$ is k-prime and also every non-zero ideal is k_M-prime. The behaviour of k-prime ideal under sum, product and union is then examined. Also, some ring theoretic properties related to k-prime ideals are discussed with suitable examples. Furthermore, it is proved that if I is a k-prime ideal of $\mathfrak{B}$, then $rad(I)$ is a k-prime ideal of $\mathfrak{B}$. Moreover the number of radical ideals in a finite commutative ring $\mathfrak{B}$ is generalized. The study is further extended to minimal prime ideals over a given ideal and their relationship with k-prime ideals. A generalized formula for the length of the ideal is established. Finally, the relationship among k-prime ideals, minimal primes over an ideal and length of the ideal are illustrated through suitable examples.

Keywords: Zero-divisor, k-zero-divisor, Radical Ideal, Minimal Primes Over I, Chain

1. Introduction

The concept of zero-divisor graph $\Gamma(\mathfrak{B})$ of a commutative ring $\mathfrak{B}$ was first introduced by I.Beck [1988] [5] , where all elements of $\mathfrak{B}$ were taken as vertices and he was mainly interested in colorings. Subsequently, Anderson and Livingston [7] redefined this graph by restricting the vertex set to the nonzero zero-divisors of $\mathfrak{B}$ and they investigated its structural properties. They also examined the interaction between ring theoretic properties and of a commutative ring and graph theoretic properties of zero-divisor graph. The concept of zero-divisor graph of a commutative ring has been extensively studied by many authors.

In 2007, Eslahchi and Rahimi [6] extended the concept of a zero-divisor of a commutative ring $\mathfrak{B}$ to that of a k-zero-divisor and examine the relationship between the algebraic properties of $\mathfrak{B}$ and the graph theoretic properties of the associated k-uniform hypergraph $H_k(\mathfrak{B})$. They studied some examples of k-zero-divisor. Let $\mathfrak{B}$ be a finite commutative ring. For a fixed integer $k \geq 2$, a nonzero nonunit element $\mathfrak{g}_1 \in \mathfrak{B}$ is called k-zero-divisor if there exist $k-1$ distinct nonunit elements $g_2, g_3, ..., g_k \in \mathfrak{B}$, all different from $\mathfrak{g}_1$ such that $\mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_\mathfrak{k} = 0$ and the product of no proper subset of $\{\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_\mathfrak{k}\}$ is zero. Using k-zero-divisor, a new definition called k-prime ideal is defined. The prime ideals play a very important role in commutative ring theory so it has been generalized and studied in several directions.

In 2012, Amor Haouaoui and Ali Benhissi [1] redefined the definition of k-zero-divisor and k-prime ideal. An element $\mathfrak{g}_1$ in $\mathfrak{B}$ is said to be k-zero-divisor in $\mathfrak{B}$ if there exist $k-1$ elements $\mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_\mathfrak{k}$ in $\mathfrak{B}$ such that $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_k = 0$ and for all i $\in 1, 2, ..., k$, the product $\hat{\mathfrak{g}}_\mathfrak{i} = \mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_{\mathfrak{i}-1}.\mathfrak{g}_{\mathfrak{i}+1}...\mathfrak{g}_\mathfrak{k}$ is non-zero. They studied the properties of k-zero-divisor and they give some examples of k-uniform hypergraph of a commutative ring $\mathfrak{B}$. They also characterize the k-zero-divisor in power ring series. They introduced a new definition of a k-prime ideal for $k \geq 2$. For any fixed integer $k \geq 2$, an ideal I of a ring $\mathfrak{B}$ is said to be k-prime whenever for any $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_\mathfrak{k}$ in $\mathfrak{B}$, the product $\mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_\mathfrak{k} \in I$ implies there are $i_1, i_2...i_{k-1} \in \{1, 2, ..., k\}$, all are distinct, then the product $\mathfrak{g}_{\mathfrak{i}_1}.\mathfrak{g}_{\mathfrak{i}_2}...\mathfrak{g}_{\mathfrak{i}_{\mathfrak{k}-1}} \in I$, also they proved for an ideal I in the ring $\mathfrak{B}$, $\mathfrak{B}/I$ is a k-integral domain if I is a k-prime ideal. They have extended the concept only to k-zero-divisor hypergraph.

Later in 2023, Elham Mehdi-Nezhad and Amir M. Rahimi [9] characterized the connectedness, diameter, dominating sets, and domination number by using the k-zero-divisor hypergraph $H_k(\mathfrak{B})$ of a finite direct product of integral domains and a class of commutative Artinian rings $\mathfrak{B}$. The authors showed that the k-zero-divisor hypergraph associated with the direct product of $k \geq 3$ integral domains (respectively, commutative Artinian rings that are direct products of $k \geq 3$ local rings) is connected, with diameter at most 3 and domination number at most 2. They also provided several examples for illustrating these results. I referred to all these hypergraph papers [13-15].

In 2024, C.V.Mythily and D.Kalamani [11] introduced the study on S-prime ideal as nilpotent ideal. Let S be a multiplicative subset of ring $\mathfrak{B}$ and I_s as an S-prime ideal disjoint from S. This paper explores the sum, union and intersection of two S-prime ideals and proved that nilradical is S-prime. They also applied Zorn's lemma to state that the S-prime ideal is unique in a local ring. Finally they classified the S-prime ideal in semilocal ring and generalizes the S-prime in finite rings.

Recently in 2025, N.B Nalawade, M.S Bapat, S.G Jakkewad, G.A Dhanorkar and D.J Bhosale [12] they investigated the structural properties of the zero-divisor hypergraph and the zero-divisor superhypergraph over the ring Z_n. They analyzed the similarities and differences by examining parameters such as order,size,diameter and girth.They shown that both the hypergraph and superhypergraph have the same order.These findings contribute to a deeper understanding of the relationship between hypergraphs and superhypergraphs associated with Z_n. These paper provide the basic idea for the concept of k-prime ideal of a finite commutative ring [2]-[4].

Recent work on k-zero-divisor hypergraphs has considered several structural and topological aspects. More recently, Selvakumar, Beautlin Jemi and Moniri Hamzekolaee studied a topological aspect of the 3-zero-divisor hypergraph, characterizing finite commutative nonlocal rings whose associated hypergraphs have prescribed genus and crosscap [16]. In addition, Selvakumar and Subajini introduced and studied the ideal-based k-zero-divisor hypergraph, establishing results concerning connectivity, diameter, chromatic number, and planarity for suitable classes of commutative rings and ideals [17]. These developments demonstrate the continued interest in investigating algebraic properties of commutative rings through structural, domination and ideal-based properties of their associated k-zero-divisor hypergraphs.

Most authors have extended only k-zero divisor in their research. Using the k-prime definition of Eslahchi and Rahimi, I have extended the k-prime ideal to some ideal properties. We assume throughout that all rings are finite commutative with unity. Suppose that $\mathfrak{B}$ is a ring. Then I is an ideal of $\mathfrak{B}$, $Z(\mathfrak{B})^*$ denotes the set of nonzero zero-divisors of $\mathfrak{B}$ and $Rad(I)$ denotes the radical ideal of I. For any fixed integer $k \geq 3$, an ideal I of a ring $\mathfrak{B}$ is said to be k-prime whenever for any set $G = \{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3 ..., \mathfrak{g}_\mathfrak{k}\}$ of nonzero, distinct, and nonunit elements of $\mathfrak{B}$, the product $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_\mathfrak{k} \in I$ implies that the product of the elements of a proper subset of G is in I. This paper investigates the algebraic structure of k-prime ideal and studies their behaviour under ideal operations and some ring theoretic properties like minimal prime ideal over I and the length of each ideal is discussed.

2. Preliminaries

In this section, some fundamental definitions related to the concept of this paper are given and they have been studied in Abstract Algebra by Dummit and Foote [2004] [8] and Contemporary Abstract Algebra by J. Gallian [2017] [10].

Definition 2.1. A prime ideal P of a commutative ring $\mathfrak{B}$ is a proper ideal of $\mathfrak{B}$ such that $\mathfrak{s}, \mathfrak{t} \in \mathfrak{B}$ and $\mathfrak{st} \in P$ imply $\mathfrak{s} \in P$ or $\mathfrak{t} \in P$.

Definition 2.2. A prime ideal P of a commutative ring $\mathfrak{B}$ is a proper ideal of $\mathfrak{B}$ such that $\mathfrak{s}, \mathfrak{t} \in \mathfrak{B}$ and $\mathfrak{st} \in P$ imply $\mathfrak{s} \in P$ or $\mathfrak{t} \in P$.

Definition 2.3. Let $\mathfrak{B}$ be a ring. A non-zero element $\mathfrak{s}$ of $\mathfrak{B}$ is called a zero divisor if there is a nonzero element $\mathfrak{t}$ in $\mathfrak{B}$ such that either $\mathfrak{st}$ = 0 or $\mathfrak{ts}$ = 0.

Definition 2.4. Let A be an ideal of the commutative ring $\mathfrak{B}$ and $radA = \{r \in \mathfrak{B} \mid r^s \in I$ for some $s \in \mathbb{Z}^+\}$ is called the radical of I.

Definition 2.5. An ideal A of the commutative ring $\mathfrak{B}$ is called radical ideal if $radA = A$

Definition 2.6. Let C and D be ideals of $\mathfrak{B}$ then the sum of C and D is defined by $C + D = \{\mathfrak{s} + \mathfrak{t} \mid \mathfrak{s} \in C, \mathfrak{t} \in D\}$.

Definition 2.7. Let C and D be ideals of $\mathfrak{B}$ then the product of C and D, denoted by CD, be the set of all finite sums of elements of the form $\mathfrak{st}$ with $\mathfrak{s} \in C$ and $\mathfrak{t} \in D$.

Definition 2.8. A prime ideal P is minimal over I if $I \subseteq P$ and there is no smaller prime ideals between I and P.

Definition 2.9. The length of the ideal I is the number of strict inclusions in a chain

3. Main Results

Let $\mathfrak{B}$ be a ring of order n. Let G be the set of nonzero zero-divisors of $\mathfrak{B}$ and I be an ideal of $\mathfrak{B}$.

Definition 3.1. For any fixed integer $k \geq 3$, an ideal I of a ring $\mathfrak{B}$ is said to be k-prime whenever for any set $G = \{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_k\}$ of nonzero, distinct, and nonunit elements of $\mathfrak{B}$, $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_\mathfrak{k} \in I$ implies that the product of the elements of a proper subset of G is in I.

Throughout this paper, we denote $k_m = \min k$ and $k_M = \max k$. It is clear that $k_m \geq 3$ and $k_M = |Z(\mathfrak{B})^*|$. The set G must contain atleast 3 nonzero zero-divisors of $\mathfrak{B}$. If I is a k-prime ideal, then k takes all values from k_m to k_M.

Remarks

1. The rings Z_4 and Z_9 are not k-prime ideals because the number of nonzero zero-divisors in these rings is less than 3.

2. As there is no zero-divisors in the field, it has no k-prime ideals.

Example 3.1. Let the ring $\mathfrak{B}$ be Z_{30}. The zero-divisors of $\mathfrak{B}$ are $0, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28$. The ideals of the ring $\mathfrak{B}$ are $< 1 >, < 2 >, < 3 >, < 5 >, < 6 >, < 10 >, < 15 >$ and $< 0 >$. Consider the prime ideal $I =< 2 >$. Assume the set G = $\{2, 3, 5\}$ where $k_m = 3$ and the product 2.3.5 = 0 $\in < 2 >$, 2.3 = 6 $\in < 2 >$ and 2 $\in < 2 >$. Therefore $< 2 >$ is 3-prime. In general , if G contains 3 or more elements $< 2 >$ is 3,4,5,...,21-prime. In the same way, the ideals $< 1 >, < 3 >, < 5 >$ are k-prime for all $3 \leq k \leq 21$. Consider the ideal $I = < 6 >$. Assume the set G = $\{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25\}$. Here the set G satisfies the condition of k-prime ideal. Therefore, $k_m = 18$. Hence, the ideal $< 6 >$ is 18,19,20,21-prime. Similarly, the ideal $< 10 >$ is 20,21-prime and $< 15 >$ is only 21-prime. The singleton subset do not satisfy the k-prime condition because the element $g \notin < 0 >$ for all $g \in G$.

Theorem 3.1. Every prime ideal of a ring $\mathfrak{B}$ is k-prime where $3 \leq k \leq k_M$.

Proof. Let P be the prime ideal of $\mathfrak{B}$. Let $g_1, g_2, ..., g_{k_M}$ be distinct elements in $Z(\mathfrak{B})^*$. The result is proved by mathematical induction on k.

If k = 3, then $G = \{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3\}$. Consider the product $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3 \in$ P.

As P is prime, either $\mathfrak{g}_1.\mathfrak{g}_2 \in$ P or $\mathfrak{g}_3 \in$ P. If $\mathfrak{g}_1.\mathfrak{g}_2 \in$ P then $\mathfrak{g}_1 \in$ P or $\mathfrak{g}_2 \in$ P.

If $\mathfrak{g}_1.\mathfrak{g}_2 \notin$ P then $\mathfrak{g}_3 \in$ P. If $\mathfrak{g}_3 \in$ P, $\mathfrak{g}_1.\mathfrak{g}_3 \in$ P and $\mathfrak{g}_2.\mathfrak{g}_3 \in$ P as P an ideal.

In either cases, $\mathfrak{g}_i.\mathfrak{g}_j \in$ P and $\mathfrak{g}_i \in$ P for $i \neq j$ and i, j = 1,2,3. Therefore, P is an 3-prime ideal.

By induction hypothesis, assume that P is k-prime where $k < k_M$.

Let G = $\left\{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3 ..., \mathfrak{g}_{k_M}\right\}$ Consider the product $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_{k_M} \in$ P As P is prime, $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_{k_{M-1}} \in$ P or $\mathfrak{g}_{k_M} \in$ P.If $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_{k_{M-1}} \in$ P. P is k_{M-1} prime by our assumption. If $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_{k_{M-1}} \notin$ P then $\mathfrak{g}_{k_M} \in$ P. As P is an ideal, the product of atleast one of the proper subset which contains $\mathfrak{g}_{k_M} \in$ P. Therefore, P is a k-prime ideal where $3 \leq k \leq k_M$.

From the above example, the converse of the theorem is not true.

Theorem 3.2. Every nonzero ideal of $\mathfrak{B}$ is k_M-prime.

Proof. Let I be the nonzero ideal of $\mathfrak{B}$. Let $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_{k_M}$ be distinct nonzero zero-divisors of $\mathfrak{B}$. Consider $G = \left\{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_{k_M}\right\}$

Since the set G contains all the nonzero zero-divisors $\mathfrak{B}$, atleast one of the $\mathfrak{g}_i$ belongs to I. Let it be $\mathfrak{g}_1$. The product of atleast one of the proper subset which contains $\mathfrak{g}_1$ of G belongs to I. Hence I is k_M-prime. Therefore, every nonzero ideal of $\mathfrak{B}$ is k_M-prime.

Theorem 3.3. Let $\mathfrak{B}$ be a ring of order n. If n = $2^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r}$ and $p_i's$ are distinct primes then $\left\langle \frac{n}{2} \right\rangle$ is only k_M-prime if

(1) $\alpha_1 \geq 3$, $\alpha_i = 0, 2 \leq i \leq r$ (or)
(2) $\alpha_1 \geq 1$, $\alpha_i > 0$ for some i.

Proof. Let $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_{k_M}$ be distinct nonzero zero-divisors of $\mathfrak{B}$. Let $I = \left\langle \frac{n}{2} \right\rangle$. The ideal $\left\langle \frac{n}{2} \right\rangle$ contains only the elements 0 and $\frac{n}{2}$. If n = $2^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r}$ where $p_2, p_3, ...p_r$ are different primes and $\alpha_i, i = 1, 2, ..., r$ are integers.

Case 1: $\alpha_1 \geq 3$, $\alpha_i = 0, 2 \leq i \leq r$

Consider the set $G = \left\{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_k\right\}$ where $\mathfrak{g}_i$'s nonzero zero-divisors, $k < k_M$, $\mathfrak{g}_i \neq \frac{n}{2}$ for all i Assume that the product $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_\mathfrak{k} \in I$.The proper subset containing the single element $\{\mathfrak{g}_i\}$ is not in I. Therefore, I is not k-prime. In this case, $I = \left\langle \frac{n}{2} \right\rangle$ is only k_M-prime.

Case 2: $\alpha_1 \geq 1$, $\alpha_i > 0$ for some i.

Case 2.1: $\alpha_1 = 1$, $\alpha_2 = 1$ if $\alpha_i = 0, 3 \leq i \leq r$

n is in the form $2p$ where p is prime and $p > 2$. The divisors of n are $1, 2, p$ and $2p$. Hence the nonzero proper ideals of $2p$ are $\langle 2\rangle$ and $\langle p\rangle$. Consider the set $G = \left\{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3...\mathfrak{g}_k\right\}$ where $\mathfrak{g}_i = p$ for some i. The product $\mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_k \in I$ as it is a multiple of p. In the same way, the product of a proper subset containing p is a multiple of p and hence it belongs to I. Therefore, I is k-prime for all $k < k_M$. Combining with theorem 3.1, $I = \left\langle \frac{n}{2} \right\rangle$ is k-prime for all k. Case 2.2: $\alpha_1 \geq 1$, $\alpha_2 \geq 1$ and $\alpha_i \geq$ 0,3 $\leq i \leq r$ atleast one must be in strict inequality.

Choose the set $G = \left\{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_k\right\}$ as in case (i). The proof follows from case(i). Therefore, I is only k_M-prime.

The properties of k-prime ideals such as sum, product, union and intersection are discussed in the following theorems.

Theorem 3.4. The sum of k-prime ideals of $\mathfrak{B}$ is k-prime for a fixed k.

Proof. Let k be any positive integer greater than or equal to 3. Let I and J be distinct k-prime ideals of $\mathfrak{B}$.

Consider the set $G = \left\{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_k\right\}$ where $\mathfrak{g}_i$'s nonzero zero- divisors, i = 1,2,...,k. Let $\mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_k \in I + J$. Case(i): $I \subseteq J$ or $J \subseteq I$

If $I \subseteq J$, then $I + J = J$. Since J is k-prime, the product of atleast one of the proper subset of G belongs to J and hence in $I + J$. Similar result is true for $J \subseteq I$. Case(ii): $I \nsubseteq J$ or $J \nsubseteq I$

$I + J = (I \cup J) \cup (I \cup J)'$ $(I \cup J)'$ contains either primes or product of r-primes where $r < k$. This implies that the product of $\mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_k$ is not in $(I \cup J)'$. It gives that $\mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_k$ is in $I \cup J$. This means that the product is either in I or in J. Assume that $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_k \in I$. Since I is k-prime, the product of atleast one of the proper subset is in I and hence the product is in $I + J$.

In both the cases, $I + J$ is k-prime ideal.

Therefore,the sum of k-prime ideals of $\mathfrak{B}$ is k-prime for a fixed k.

We now provide an example for the above theorem.

Example 3.2. Let the ring $\mathfrak{B}$= $\mathbb{Z}_{24}$. Consider the ideals $I =< 4 >$ and $J =< 6 >$ which are 14-prime ideals. Then $I + J$ = $< 2 >$ is also 14-prime.

Corollary 3.1. If the nonzero product IJ of two k-prime ideals I and J of $\mathfrak{B}$ is a k-prime ideal of $\mathfrak{B}$.

Corollary 3.2. Let I and J be k-prime ideals of $\mathfrak{B}$, then $I \cap J$ is a k-prime if $I \cap J \neq 0$.

Example 3.3. Let the ring $\mathfrak{B}= \mathbb{Z}_{36}$. Consider the ideals $I =< 2 >$, $J =< 6 >$ and $K =< 18 >$ which are 23-prime ideals. Then $I \cup J \cup K =< 2 >$ is also 23-prime.

Theorem 3.5. The ideal $I \neq 0$ of $\mathfrak{B}$ is a k-prime ideal where $k_m = \begin{cases} 3 & if \quad I\ is\ prime\ or\ I\ = \mathfrak{B} \\ n - \phi(n) - s + 1 & otherwise \end{cases}$. where s is the cardinality of the ideal I.

Proof. Let I be the nonzero proper ideal of $\mathfrak{B}$.

Consider the set $G = \{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_k\}$ where $\mathfrak{g}_i$'s , $i = 1, 2, ..., k$ are nonzero zero-divisors. Case(i): $I = \mathfrak{B}$ Choose $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_k \in G$ there exist the product, $\mathfrak{g}_1.\mathfrak{g}_2...\mathfrak{g}_k \in I$ where, $3 \leq k \leq k_M$ then product of all the proper subset of G belongs to I. Hence I is k-prime for all k which lies between 3 and k_M. Assume I is prime By theorem 3.1, every prime ideal is k-prime ideal of $\mathfrak{B}$ where $k \in [3, k_M]$ Hence I is k-prime for $3 \leq k \leq k_M$.

Therefore, k_m is 3. Case(ii): I is non prime proper ideal

Assume that the set G is disjoint from I. If the product $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_k \in I$ then the product of proper subset of G containing the single element $\mathfrak{g}_i$ is not in I. This implies that I is not a k-prime ideal. For I to be an k-prime ideal, G must contain atleast one element from I. G may contain atleast $(n - \phi(n) - 1) - (I^* - 1)$. If G has $n - \phi(n) - s + 1$ elements then I is k-prime.

Therefore, $k_m = n - \phi(n) - s + 1$.

Theorem 3.6. If I is a k-prime ideal of $\mathfrak{B}$, then $rad(I)$ is a k-prime ideal of $\mathfrak{B}$.

Proof. Let I be the nonzero proper ideal of $\mathfrak{B}$.

Consider the set $G = \{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_k\}$ where $\mathfrak{g}_i$'s , $i = 1, 2, ..., k$ are nonzero zero-divisors. Obviously, the ideals $\mathfrak{B}$ and $\langle p\rangle$ are both radical and k-prime ideal for all k where $k \in [3, k_M]$. Let I be the non-prime k-prime ideal of $\mathfrak{B}$. Suppose $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_k \in rad(I)$.

Since I is non-prime ideal, $|I| \leq \frac{k_M+1}{2}$ $k_m = n - \phi(n) - |I| + 1 \geq \frac{k_M}{2} + \frac{3}{2} > \frac{k_M}{2}$ Since $k > \frac{k_M}{2}$, the product of all nonzero zero-divisors of G is 0 which is an element of $rad(I)$. $\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_k = 0 \in rad(I)$

$\mathfrak{g}_1.\mathfrak{g}_2.\mathfrak{g}_3...\mathfrak{g}_k = 0 \in I$ Since I is k-prime, the product of atleast one of the proper subset of $\{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ..., \mathfrak{g}_{k_M}\} \in I \subseteq rad(I)$. Therefore, $rad(I)$ is a k-prime ideal.

The next theorem provides a generalization for the number of radical ideals in a finite commutative ring with unity.

Theorem 3.7. Let $\mathfrak{B}$ be a ring of order n. If n = $p_1^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r}$ where $\alpha_i \geq 1$ and p_i's are distinct primes, $i = 1, 2, ..., r$ then number of radical ideals is 2^r.

Proof. Let I be the ideal of $\mathfrak{B}$.

Let n = $p_1^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r}$ where $p_1, p_2..., p_r$ are distinct primes and α_i, $i = 1, 2, ..., r$ are integers greater than or equal to 1. Every ideal is of the form $I = \left\langle p_1^{\beta_1}.p_2^{\beta_2}...p_r^{\beta_r}\right\rangle$ where $0 \leq \beta_i \leq \alpha_i$, $i = 1, 2, ..., r$ If $\beta_i \leq 1$ then the ideals are $I = \mathfrak{B}, \langle p_1\rangle, \langle p_2\rangle, ..., \langle p_r\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle ... \langle p_1p_r\rangle, \langle p_1p_2p_3\rangle, \langle p_1p_2p_4\rangle, ..., \langle p_1p_2...p_r\rangle$ Assume $I = \mathfrak{B}$ Obviously I is a radical ideal. Consider $I = \langle p\rangle$, where p is prime. Let $x \in rad(I)$ so $x^t \in \langle p\rangle$ (i.e) $p \mid x^t$. Since p is prime, $p \mid x$ so $x \in \langle p\rangle$.Thus, $rad(I) \subseteq I$ Therefore, I is a radical ideal and hence every prime ideal is a radical ideal.

Assume $I = \langle p_1.p_2...p_k\rangle, 2 \leq k \leq r$ Let $x \in rad(I)$ so $x^t \in \langle I\rangle = \langle p_1.p_2...p_k\rangle$ for all t.

$p_1.p_2...p_k \mid x^t$ so, $p_1.p_2...p_k \mid x$. Thus, $x \in I$ so $rad(I) \subseteq I$. Hence $I = rad(I)$ Therefore I is a radical ideal. Let $\beta_1 \neq 1$, and $\beta_i = 1$ for all i, $2 \leq i \leq r$ Let $x \in rad(I)$ so $x^t \in \langle I\rangle = \left\langle p_1^{\beta_1}.p_2...p_k\right\rangle$ $p_1^{\beta_1}.p_2...p_k \mid x^t$ there exist $x = p_1.p_2...p_k, t = \beta_1$ $1 \leq k \leq r$ $p_1^{\beta_1}.p_2...p_k \nmid x$. Hence $x \notin I$. Therefore $I \neq rad(I)$ Therefore, I is not a radical ideal. Since there are r-primes in $\mathfrak{B}$, the number of radical ideals is 2^r. Therefore, number of radical ideals is 2^r.

In view of theorem 3.8, we have the following theorem and corollary.

Theorem 3.8. If the ideal $I = \left\langle \frac{n}{d}\right\rangle$ of $\mathfrak{B}$ is a proper non-prime k-prime then k takes $(d - 1)$ values.

Proof. Let I be the proper non-prime k-prime ideal where $I = \left\langle \frac{n}{d}\right\rangle$ By theorem 3.2, every nonzero ideal of $\mathfrak{B}$ is k_M-prime. Hence, max $k = k_M = n - \phi(n) - 1$ By theorem 3.5, If I is non-prime then $k_m = n - \phi(n) - d + 1$ Hence, $k_m = n - \phi(n) - d + 1$ All proper non-prime ideals are k-prime where $k_m \leq k \leq k_M$ Difference $= k_M - k_m + 1 = n - \phi(n) - 1 - n + \phi(n) + d - 1 + 1 = d - 1$ Therefore, k takes $d - 1$ values.

Theorem 3.9. If the ideal I of $\mathfrak{B}$ is prime or $I = \mathfrak{B}$ then k takes $n - \phi(n) - 3$ values.

Proof. Assume I is prime. By theorem 3.1, every prime ideal of $\mathfrak{B}$ is k-prime where $3 \leq k \leq k_M$ Assume $I = \mathfrak{B}$ By theorem 3.5, $I = \mathfrak{B}$ is k-prime where $3 \leq k \leq k_z$ In both the cases, $k_m = 3$ max $k = k_M = n - \phi(n) - 1$ Difference $= k_M - k_m + 1 = n - \phi(n) - 3$ Therefore, k takes $n - \phi(n) - 3$ values.

4. Analysis of Minimal Primes over an Ideal I and Ideal Lengths Associated with k-prime Ideal

This section provides a generalization of the length of ideals and investigates some ring theoretic properties like minimal primes over an ideal I and length of each ideal related to k-prime are discussed with suitable examples.

Theorem 4.1. Let $\mathfrak{B}$ be a ring. If the ideal $I = \left\langle p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ...p_r^{\alpha_r}\right\rangle$ where $\alpha_i \geq 0$ and p_i's are distinct primes, $1 \leq i \leq r$. Then the length of the ideal I is $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_r$.

Proof. Let $I = \left\langle p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r} \right\rangle$ where $\alpha_i \geq 0$ and p_i's are distinct primes, $1 \leq i \leq r$. If I contains only one prime p_1 then its ascending chain of the ideal I is

$I = \langle p_1^{\alpha_1} \rangle \subset \left\langle p_1^{\alpha_1 - 1} \right\rangle \subset \left\langle p_1^{\alpha_1 - 2} \right\rangle \subset ... \subset \langle p_1 \rangle \subset \langle 1 \rangle = \mathfrak{B}$

Therefore, the length of the ideal I is α_1. If there are more than one distinct primes, say $r > 1$ in I, the ascending chain of the ideal I is

$I = \left\langle p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r} \right\rangle \subset \left\langle p_1^{\alpha_1 - 1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r} \right\rangle \subset \left\langle p_1^{\alpha_1 - 2} p_2^{\alpha_2} ... p_r^{\alpha_r} \right\rangle \subset$...(after α_1 steps)...$\subset \left\langle p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r} \right\rangle \subset \left\langle p_2^{\alpha_2 - 1} p_3^{\alpha_3} ... p_r^{\alpha_r} \right\rangle \subset$ ...(after α_2 steps)...$\subset \left\langle p_3^{\alpha_3} p_4^{\alpha_4} ... p_r^{\alpha_r} \right\rangle \subset$

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$\subset \left\langle p_{r-1}^{\alpha_{r-1} - 1} p_r^{\alpha_r} \right\rangle \subset$...(after α_{r-1} steps)... $\subset \left\langle p_r^{\alpha_r} \right\rangle \subset \left\langle p_{r-1}^{\alpha_{r-1}} \right\rangle \subset$...(after α_r steps)... $\subset \langle 1 \rangle = \mathfrak{B}$. Therefore, the length of the ideal I is $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_r$. Hence the result is proved for any values of r.

Example 4.1. Let $\mathfrak{B} = Z_{60}$ and ideals of $\mathfrak{B}$ are $< 1 >, < 2 >, < 3 >, < 4 >, < 5 >, < 6 >, < 10 >, < 12 >, < 15 >, < 20 >, < 30 >$ and $< 0 >$. The following hasse diagram represents the inclusion relation for these ideals and the chain they form, from which the length of each ideal can be determined.

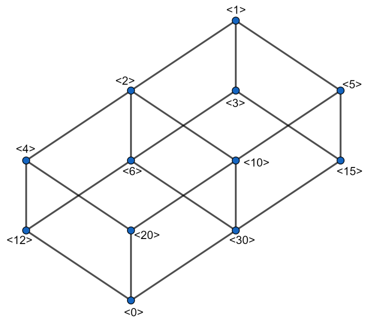

Figure 1. *Hasse diagram of ring of order 60.*

Table 1. *Length of ideals in* $\mathbb{Z}_{60}$.

Ideal	Chain	Length
$\langle 1 \rangle$	$- - -$	0
$\langle 2 \rangle$	$\langle 2 \rangle \subset \langle 1 \rangle$	1
$\langle 3 \rangle$	$\langle 3 \rangle \subset \langle 1 \rangle$	1
$\langle 5 \rangle$	$\langle 5 \rangle \subset \langle 1 \rangle$	1
$\langle 4 \rangle$	$\langle 4 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	2
$\langle 6 \rangle$	$\langle 6 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	2
	$\langle 6 \rangle \subset \langle 3 \rangle \subset \langle 1 \rangle$	2
$\langle 10 \rangle$	$\langle 10 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	2
	$\langle 10 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	2
$\langle 15 \rangle$	$\langle 15 \rangle \subset \langle 3 \rangle \subset \langle 1 \rangle$	2
	$\langle 15 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	2
$\langle 12 \rangle$	$\langle 12 \rangle \subset \langle 6 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	3
	$\langle 12 \rangle \subset \langle 6 \rangle \subset \langle 3 \rangle \subset \langle 1 \rangle$	3
	$\langle 12 \rangle \subset \langle 4 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	3
$\langle 20 \rangle$	$\langle 20 \rangle \subset \langle 10 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	3
	$\langle 20 \rangle \subset \langle 10 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	3
	$\langle 20 \rangle \subset \langle 4 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	3
$\langle 30 \rangle$	$\langle 30 \rangle \subset \langle 10 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	3
	$\langle 30 \rangle \subset \langle 10 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	3
	$\langle 30 \rangle \subset \langle 15 \rangle \subset \langle 3 \rangle \subset \langle 1 \rangle$	3
	$\langle 30 \rangle \subset \langle 15 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	3
$\langle 0 \rangle$	$\langle 0 \rangle \subset \langle 12 \rangle \subset \langle 4 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 12 \rangle \subset \langle 6 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 12 \rangle \subset \langle 6 \rangle \subset \langle 3 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 20 \rangle \subset \langle 4 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 20 \rangle \subset \langle 10 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 20 \rangle \subset \langle 10 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 30 \rangle \subset \langle 6 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 30 \rangle \subset \langle 6 \rangle \subset \langle 3 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 30 \rangle \subset \langle 10 \rangle \subset \langle 2 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 30 \rangle \subset \langle 10 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 30 \rangle \subset \langle 15 \rangle \subset \langle 3 \rangle \subset \langle 1 \rangle$	4
	$\langle 0 \rangle \subset \langle 30 \rangle \subset \langle 15 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$	4

Corollary 4.1. Let $\mathfrak{B}$ be a ring of order n. If n = $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_r^{\alpha_r}$ where $\alpha_i \geq 0$ and $p_1, p_2, ..., p_r$ are distinct primes then $\mathfrak{B}$ is n-noetherian where n = $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_r$.

Theorem 4.2. Let I be a k-prime ideal of $\mathfrak{B}$. If $S_1, S_2, S_3, ..., S_n$ are minimal prime ideals over I then $\prod_{i=1}^{n} S_i$ is k-prime ideal.

Proof. Since $I \subseteq S_i$ for all i, the product of all minimal prime ideal is k-prime ideal of $\mathfrak{B}$.

Corollary 4.2. If I is a k-prime ideal of $\mathfrak{B}$, then I is not an ideal of maximum length.

Proof. In finite commutative ring, the longest chain of ideals starts from $\langle 0 \rangle$. If I has maximum length then $I = \langle 0 \rangle$. Since, $\langle 0 \rangle$ is not k-prime, it is not an ideal of maximum length.

Theorem 4.3. Let I be the k-prime ideal of B then cardinality of minimal primes over $I \leq$ maximum length of the ideal I.

Example 4.2. Let $\mathfrak{B} = Z_{24}$ and ideals of B are $< 1 >, < 2 >, < 3 >, < 4 >, < 6 >, < 8 >, < 12 >$ and $< 0 >$. The following tabular column shows the length and cardinality of the minimal primes over the ideal I.

Table 2. *Length and cardinality of minimal primes over I.*

Ideal	Length	Cardinality of minimal primes over I
$\langle 2 \rangle$	1	1
$\langle 3 \rangle$	1	1
$\langle 4 \rangle$	2	1
$\langle 6 \rangle$	2	2
$\langle 8 \rangle$	3	1
$\langle 9 \rangle$	4	2

5. Conclusion

In this paper, some of the basic properties namely sum, product, union and intersection of the k-prime ideals of a ring $\mathfrak{B}$ with unity are discussed and the value of k is generalized. Moreover , it is proved that radical of I is k-prime ideal of $\mathfrak{B}$. Finally the number of radical ideals and length of each ideals of a commutative ring $\mathfrak{B}$ with unity are generalized. The future research will be focused on k-prime ideal in different classes of rings as well as their relationship with other generalization of prime ideals and extending these results to hypergraphs.

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Abbreviations

$\mathfrak{B}$	Finite Commutative Ring with Unity of Order n
n	Order of the Ring $\mathfrak{B}$
$Z(\mathfrak{B})^*$	Set of Non-zero Zero-divisors of $\mathfrak{B}$
G	Set of Non-zero, Non-unit Elements of $\mathfrak{B}$
I	Ideal of $\mathfrak{B}$
P	Prime Ideal of $\mathfrak{B}$
k_m	Minimum Value of k
k_M	Maximum Value of k
s	Cardinality of the Ideal I
$Rad(I)$	Radical Ideal of I

Author Contributions

Kalamani Duraisamy: Conceptualization, Methodology, Supervision, Validation, Writing – review & editing

Jayasree Anbuselvi Sharavanan: Conceptualization, Investigation, Methodology, Visualization, Writing – original draft

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Amor Haouaoui, Ali Benhissi, The k-zero-divisor hypergraph. Ricerche mat. (2012). 61, 83-101. https://doi.org/10.1007/s11587-011-0117-x

[2] Anderson, D.F, Badawi. A. On n-Absorbing Ideals of Commutative Rings. Com-munications in Algebra. (2011). 39, 1646-1672. https://doi.org/10.1080/00927871003738998

[3] Anderson, D.D, Bataineh. M. Generalizations of prime ideals. Communications in Algebra. (2008). 36, 686-696. https://doi.org/10.1080/00927870701724177

[4] Badawi, A. On 2-absorbing ideals of commutative rings. Bulletin of the Australian Mathematical Society. (2007). 75(03), 417-429. https://doi.org/10.1017/S0004972700039344

[5] Beck I. Coloring of commutative rings. Journal of Algebra. (1988). 116(1), 208-226. https://doi.org/10.1016/0021-8693(88)90202-5

[6] Ch. Eslahchi, A.M. Rahimi. The k-zero-divisor hypergraph of a commutative ring.International Journal of Mathematics and Mathematical Sciences. (2007). (3).

[7] David F.Anderson and Livingston, P.S. The Zero-Divisor Graph of a Commutative Ring. Journal of Algebra. (1999). 217, 434-447. https://doi.org/10.1007/978-1-4419-6990-3_2

[8] David S. Dummit and Richard M.Foote. Abstract Algebra (third edition), John Wiley and Sons Inc. (2004).

[9] Elham Mehdi-Nezhad and Amir M.Rahimi. A note on k-zero-divisor hypergraph of some commutative rings. Italian Journal of Pure and Applied Mathematics. (2023).50, 495-502.

[10] Gallian J. Contemporary Abstract Algebra (tenth edition), Chapman and Hall/CRC. (2021).

[11] Mythily C V, Kalamani D. Study on S-prime Ideal as Nilpotent Ideal. Journal of applied mathematics and informatics. (2024). 42(5), 1171 - 1182. http://doi.org/10.14317/jami.2024.1171

[12] Nalawade N.B,Bapat M.S., Jakkewad S.G., Dhanorkar G.A. and Bhosale D.J. Structural Properties of Zero-Divisor Hypergraph and Superhypergraph over Z_n: Girth and Helly Property. Panamerican Mathematical Journal. (2025). 35(4s), 485-495. https://doi.org/10.52783/pmj.v35.i4s.4977

[13] Selvakumar K, Ramanathan V. On the genus of the k-annihilating-ideal hyper-graph of commutative rings. Indian Journal of Pure and Applied Mathematics. (2019). 50(2), 461-475. https://doi.org/10.1007/s13226-019-0338-3

[14] V.C Amritha and K.Selvakumar. Ideal based hypergraph of a commutative ring. AKCE International Journal of Graphs and Combinatorics. (2021). 18-23. https://doi.org/10.1080/09728600.2021.2007039

[15] V.Ramanathan and K.Selvakumar. The k-annihilating-ideal hypergraph of commu-tative ring. AKCE International Journal of Graphs and Combinatorics. (2020). 241-252. https://doi.org/10.1016/j.akcej.2019.02.008

[16] Selvakumar Krishnan, Beautlin Jemi J. and Ali Reza Moniri Hamzekolaee. A top-ological property of a hypergraph assigned to commutative rings. Journal of algebra and its ap-plications. (2023).24(03). https://doi.org/10.1142/S0219498825500938

[17] Selvakumar Krishnan and Subajini ms. Ideal based k-zero-divisor hypergraph of commutative rings. Algebra Colloquium. (2021). 28(04):655-672. https://doi.org/10.1142/S1005386721000511