Applied and Computational Mathematics
2026, Vol. 15, No. 3, pp. 75–88
https://doi.org/10.11648/j.acm.20261503.12

Research Article

Solution of Nonlinear Equations Using the Modified Steffensen's Type Method

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Abstract

The root-finding problem is one of the most important problems in Numerical Analyis. It arises in a wide variety of practical applications in physics, chemistry, biosciences, engineering, etc. As a matter of fact, determination of any unknown appearing implicitly led to the evolution of root-finding problem. Therefore, this paper focuses on the modification and analysis of a high order derivative-free iteration methods for finding roots of nonlinear algebraic equations of the form $f(x) = 0$. The methods require only one initial approximation. The proposed method is seen as an extension of the second-order Steffensen's scheme, which is an Iterative method for approximating roots of non-linear equations which often breaks down when the derivative of the function value is zero or near zero at the point of iteration. This work therefore, seeks to introduce a method that overcomes such breakdown. The method herein is a combination of forward difference formula with Simpson's quadrature in spirit of Steffensen. The idea is to modify the Steffensen's method, which were recently developed to obtain derivative-free methods. The modified methods are shown to converge. We also describe how to obtain derivative-free methods to find solutions to multiple roots. Several numerical examples are provided to validate the theoretical order of convergence for nonlinear functions with simple roots and results obtained show the comparative advantage the proposed method has over well-known methods.

Keywords

Nonlinear Equations, Steffensen, Convergence

1. Introduction

One of the most significant problems in Numerical Analysis is root- finding problems. These problems It arise in several fields of science such a as physics, chemistry, biosciences, engineering, etc.

Severa methods are different forms of the standard Newton's scheme or Newton-like methods, which often require additional functional evaluations and computational resources. Despite these advancements, there is still need for iterative methods that balance high-order convergence with computational efficiency, particularly for solving nonlinear problems. In many instances, it is impossible to determine an exact solution to a nonlinear problem [2, 10, 11, 13-17, 26, 27, 36, 51]. To address this drawback, numerical schemes have been developed to approximate the solutions of nonlinear equations. Newton's method is one such classical method that requires evaluation of the function and its derivative to estimate a linear approximation to the root. [39, 41] developed the Chebyshev-Halley methods with sixth-order convergence for solving non-linear equations. It essentially improved on the Jarratt method.

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Received: 7 April 2026; **Accepted:** 22 April 2026; **Published:** 16 May 2026

The fourth order was developed by [20] which required the same number of function and derivative evaluations. Recently, sixth and higher order convergence was constructed [1, 28, 40-43, 57]. The Secant method is an improvement over the Newton Ralphson's method [2-8, 35]. The Method requires two initial guesses unlike the Newton Raphson's scheme. A new class of Secant like methods were recently developed which uses more than one point of the iteration. The methods include the methods developed by [18-20, 29, 37, 38, 47-49, 52, 53]. [45] carried out the Comparative Study on some Methods of Handling Nonlinear Equations while [46, 47] carried out a survey on the new root-finding algorithm using exponential series.

Challenges of convergence of the traditional Newton and Secant methods have been handled through using a hybrid of methods. [50] used a method that involves a multi-point Secant method in which an n-degree polynomial is fitted using the previous points of iteration and Newton method is applied in which the first derivative of the fitted polynomial replaces the derivative of the actual function in the Newton formula. Dekker [22] developed a method which combines Bisection method and the Newton Ralphson's method. In this method the function evaluations of bisection and Secant approaches are compared and the new point resulting in estimate of function value that is closer to the root is chosen for the next trial and error procedure. However, [9-12, 30-33] suggested a procedure using root bracketing and inverse quadratic extrapolation to the root. It is an improvement over [22] method in terms of improving the rate of convergence.

Inspired and motivated by ongoing research in this field, we propose and analyze a novel iterative method for solving nonlinear equations. The primary objective of this paper is to introduce a family of modified Steffesen's method with a high order of convergence. Specifically, the proposed method achieves a sixteenth-order convergence rate, which is significantly higher than many existing methods.

2. Materials and Methods

A nonlinear equation is an equation in which the unknown variable appears in a nonlinear form. This means that the unknown variables could appear in powers greater than 1, products of variables, transcendental functions (logarithmic, exponential, trigonometric) or combinations thereof.

Generally, a nonlinear equation can be expressed as

$$f(x) = 0 \tag{1}$$

where $f:\mathbb{R}\rightarrow\mathbb{R}$ is a nonlinear function of the variable.

The root-finding problem arises in a wide variety of practical applications in physics and engineering and many others. This problem also has a direct application in the multiple shooting method for two-point boundary-value problems. The basic concept to all root finding numerical methods is iteration or successive approximation. The main idea of an iterative method is to first choose a suitable guess of the root, and then repeatedly improve upon this guess, using some well-defined operations, until we obtain an approximate root that is sufficiently close to actual root [21-25].

The new modified method for root finding is an iterative technique that is based on applying Modified Steffensen's type method (MSTM). The Modified Steffensen's Type Method (MSTM) is presented. The scheme is based on the combination of forward difference formula with Simpson quadrature formulae, as presented by [53, 54]. [34] introduced the convergence analysis of a family of Steffensen-type methods for generalized equations. Using the idea in the development of Steffensen's method, [55-57] consider the nonlinear equation (1), with starting value x_0 . The derivative $f'(x_n)$ at the point x_0 is given by the forward difference operator as:

$$f'(x_0)\approx\frac{f\left(x_0+f(x_0)\right)-f(x_0)}{f(x_0)} \tag{2}$$

$$x_1 = x_0 + f(x_0)$$

also,

$$h = x_1 - x_0 = f(x_0) \tag{3}$$

where h is the step length.

For every x_n, $n = 0,1,2,\ldots$, the derivative of in Newton Raphson's method is replaced by the forward approximation.

$$f'(x_n)\approx\frac{f\left(x_n+f(x_n)\right)-f(x_n)}{f(x_n)} \tag{4}$$

The method becomes

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f\left(x_n+f(x_n)\right)-f(x_n)} \tag{5}$$

The Steffensen's methods are derivative-free iterative techniques for solving nonlinear equations, improving upon basic fixed-point iteration by incorporating Aitken's -process for quadratic convergence, with higher-order variants achieving super-quadratic (e.g., fourth-order) or even super-cubic convergence (e.g., order ~3.383) using more function evaluations per step, often via interpolating polynomials or parameter estimation, all while maintaining efficiency by avoiding costly derivative calculations, making them popular for complex problems.

In recent years, researchers discussed Newton'smethod and made some modifications. In order to avoid f ′(x) = 0, Similarly, many authors investigate the Steffensen method [7-12].

Considering the third-order Newton-Steffensen method.

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (6)$$

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)[f(x_n) - f(z_n)]} \quad (7)$$

Steffensen's method (7) achieves quadratic convergence. It can be deduced in Newton's method by approximating.

$f'(x_n)$ with $f'(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$. For each step, the method requires two evaluations of the function and single evaluation of the derivative.

Approximating the derivative in (7) by the central-difference.

$$f'(x_n) \approx \frac{1}{2}\frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{f(x_n)} \quad (8)$$

A new method free from derivatives can now be achieved. If we take the reciprocal of (8), we have

$$z_n = x_n - \frac{2f^2(x_n)}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \quad (9)$$

$$x_{n+1} = x_n - \frac{2f(x_n)^3}{f(x_n + f(x_n)) - f(x_n - f(x_n))[f(x_n) - f(z_n)]} \quad (10)$$

Equation (10) can be simplified to obtain

$$x_{n+1} = x_n - \frac{9f(x_n)^3}{3[g(x_n) - f(x_n - f(x_n))M(x)]} \quad (11)$$

$$g(x_n) \approx f(x_n + f(x_n)) \quad (12)$$

$$M(x_n) \approx f(x_n - f(z_n)) \quad (13)$$

Equation (11) is the proposed method, hereafter referred to as Modified Steffensen's type method (MSTM) for solving a nonlinear equation.

The approach consists of approximating all derivatives appearing in the derivative-based methods. These modified derivative-free iterative methods are shown to achieve the same order of convergence as the derivative-based methods.

3. Convergence Analysis

Convergence of Bisection Method

Suppose that an algorithm produces iterates that converge as $lim_{n\to\infty} x_n = \delta$, if there is exists a sequence y_n that converges to zero and a positive constant K, such that

$$\epsilon_n = |x_n - \delta| \le k\,|y_n|$$

It implies that x_n is said to converge with rate y_n. Therefore, in the case of Bisection method,

$$|x_n - \delta| \le |b - a|\frac{1}{2^n}$$

So, the Bisection method has a convergence rate of $\frac{1}{2^n}$ with $|b - a|$ as the asymptotic convergence constant, that $K = |b - a|$

Theorem 1

Let $t^* \in I$ be a simple zero of sufficiently differentiable function $f: I \to R$ for an open interval I. If t_0 is sufficiently close to t^*, then the modified Steffensen method free from derivative has order of convergence three and satisfies the error equation.

Proof

Applying the Taylor series expansion theorem and taking account $f(t^*) = 0$. We can write $f(t_n) = e_n u_1 + e_n^2 u_2 + e_n^3 u_3 + e_n^4 u_4 + e_n^5 u_5 + e_n^6 u_6 + e_n^7 u_7 + e_n^8 u_8 + e_n^9 u_9 + o(e_n^3 u_{10})$

With $u_k = \frac{1}{k!} f^k(t^*)$, k = 1, 2,... and e_n be the error in t_n. after n iterations i.e. $e_n = x_n - t^*$.

$$f(t_n + f(t_n)) = (u_1^2 + u_1)e_n + (u_2 u_1^2 + 3u_2 u_1 + u_2)e_n^2$$
$$+(u_3 u_1^3 + 3u_3 u_1^2 + 2u_1 u_2^2 + 4u_3 u_1 + 2u_2^2 + u_3)e_n^3 +$$
$$+(u_4 + u_2(u_2^2 + 2u_1 u_3) + 5u_1 u_4 + 5u_2 u_3 + 6u_1^2 u_4 + 4u_1^3 u_4 + u_1^4 u_4 + 6u_1 u_2 u_3 + 3u_1^2 u_2 u_3)e_n^4 + 0(e_n^5) \quad (14)$$

Similarly,

$$f(t_n - f(t_n)) = (-u_1^2 + u_1)e_n + (u_2 u_1^2 - 3u_2 u_1 + u_2)e_n^2$$
$$+(-u_3 u_1^3 + 3u_3 u_1^2 + 2u_1 u_2^2 - 4u_3 u_1 - 2u_2^2 + u_3)e_n^3 +$$

$$+(u_4+u_2(u_2^2+2u_1u_3)-5u_1u_4+5u_2u_3+6u_1^2u_4-4u_1^3u_4$$

$$+u_1^4u_4+6u_1u_2u_3-3u_1^2u_2u_3)e_n^4+0(e_n^5) \quad (15)$$

It can be deduced that

$$e_n+(\frac{-u_2}{u_1})e_n^2+(\frac{2u_2^2}{u_1^2}-\frac{2u_3}{u_1}-u_1u_3)e_n^3+(u_2u_3-4u_1u_4-\frac{3u_4}{u_1}-\frac{4u_2^3}{u_1^3}-\frac{7u_2u_3}{u_1^2})e_n^4+0(e_n^5) \quad (16)$$

Considering this relation and expression of z_n in the equation (26), we obtain

$$z_n=t^*+(\frac{u_2}{u_1})e_n^2-(\frac{2u_2^2}{u_1^2}-\frac{2u_3}{u_1}-u_1u_3)e_n^3-(u_2u_3-4u_1u_4-\frac{3u_4}{u_1}-\frac{4u_2^3}{u_1^3}+\frac{7u_2u_3}{u_1^2})e_n^4+0(e_n^5) \quad (17)$$

Expand $f(z_n)$ around the root by taking into consideration (17). Accordingly, we have

$$f(z_n)=u_2e_n^2+u_1(u_2u_3+\frac{2u_3}{u_1}-\frac{2u_2^2}{u_1^2})e_n^3$$
$$u_1(u_1u_4-u_2u_3+\frac{3u_4}{u_1}+\frac{4u_2^3}{u_1^3}-\frac{7u_2u_3}{u_1^2}+\frac{4u_2^3}{u_1^2})e_n^4+0(e_n^5) \quad (18)$$

using (14), (15), (16) and (17) in the last expression of (18), we obtain

$$e_{n+1}=\frac{c_2^2}{c_1^2}e_n^3+0(e_n^4) \quad (19)$$

Theorem 2

Let $t^*\in I$ be a simple zero of sufficiently differentiable function $f:I\to R$ for an open interval I. If x_0 is sufficiently close to t^*, then the order of convergence m of the methods is of order m=2.

Proof

Let $u_k=\frac{1}{k!}f^k\left(t^*\right)$, and $e_n=x_n-t^*$.

Since $f\left(t^*\right)$ is sufficiently differentiable, first Taylor expand $f\left(t_n\right)$ about t^* to obtain

$$f(t_n)=e_nu_1+e_n^2u_2+e_n^3u_3+\cdots \quad (20)$$

similarly, Taylor's expansion of the function about t^* is

$$f(t_n+f\left(t_n\right))=e_n\left(u_1+u_1^2\right)+e_n^2\left(u_2+3u_1u_2\right)+e_n^3\left(u_3+4u_1u_3+2u_2^2\right)+\cdots, \quad (21)$$

Subtracting (20) from (21), and dividing the result gives

$$g(t_n)=e_nu_1^2+e_n^2 2u_1u_2+e_n^3\left(3u_1u_2+2u_2^2\right)+\cdots \quad (22)$$

Also, Taylor expands $g\left(t_n\right)$ about α, first expand $\frac{f^2\left(t_n\right)}{f\left(t_n+f\left(t_n\right)\right)-f\left(t_n\right)}$ about x_n to obtain

$$\frac{f^2\left(t_n\right)}{f\left(t_n+f\left(t_n\right)\right)-f\left(t_n\right)}=e_n^2u_1^2+e_n^3u_1u_2+\cdots \quad (23)$$

substituting gives

$$z_n-\alpha=e_n-\left(e_n^2u_1^2+e_n^3u_1u_2+\cdots\right) \quad (24)$$

Now, substituting (24) in the Taylor's series of $f\left(z_n\right)$

$$f\left(z_n\right)=e_nu_1+e_n^2\left(u_2-u_1^3\right)+e_n^3\left(u_3-3u_1^3u_2\right)+\cdots \quad (25)$$

Similarly,

$$f\left(z_n+f\left(z_n\right)\right)=e_nu_1\left(1+u_1\right)+e_n^2\left(u_2+u_1^4-u_1^3+u_1^2u_2+2u_1u_2\right)+$$
$$e_n^3\left(2u_1^5-6u_1^3+u_1^3u_3+2u_1^2u_2-4u_1^2-u_1u_3+2u_2\right)+\cdots \quad (26)$$

Subtracting (26) from (25), gives

$$f\left(z_n+f\left(z_n\right)\right)-f\left(z_n\right)=e_n u_1^2+e_n^2\left(u_1^4+u_1^2u_2+2u_1u_2\right)+ \\ e_n^3\left(2u_1^5-4u_1^4-4u_1^3+u_1^3u_3+5u_1^2u_2+u_1^2u_3+2u_2-u_3\right)+\cdots \tag{27}$$

Therefore, the Taylor expansion of $g\left(z_n\right)$ become

$$g(z_n)=e_n u_1^2+e_n^2\left(u_1^4+u_1^2u_2+u_1u_2+u_1^3\right) \\ +e_n^3\left(2u_1^6+u_1^4u_2-4u_1^3u_2-u_2^2-u_1u_3\right)+\cdots \tag{28}$$

Similarly, Taylor expansion follows: Taylor expanding $\frac{1}{2}\frac{f^2\left(t_n\right)}{f\left(t_n+f\left(t_n\right)\right)-f\left(t_n\right)}$ gives

$$\frac{1}{2}\frac{f^2\left(t_n\right)}{f\left(t_n+f\left(t_n\right)\right)-f\left(t_n\right)}=\frac{1}{2}e_n^2u_1^2+\frac{1}{2}e_n^3u_1u_2+\cdots \tag{29}$$

Substitution gives

$$z_n-\alpha=e_n-\frac{1}{2}\left(e_n^2u_1^2+e_n^3u_1u_2+\cdots\right) \tag{30}$$

So that,

$$f\left(z_n\right)=e_n u_1+e_n^2\left(\tfrac{1}{2}u_2-u_1^3\right) \\ +e_n^3\left(u_3-\tfrac{1}{2}u_1^2u_2-\tfrac{1}{2}u_1^2u_3\right)+\cdots \tag{31}$$

Therefore

$$f\left(z_n+f\left(z_n\right)\right)=e_n u_1\left(1+u_1\right)+e_n^2\left(u_2+\tfrac{1}{2}u_1^4+u_1^3+u_1^2u_2+u_1u_2\right)+ \\ e_n^3\left(\tfrac{1}{2}u_1^2u_2-\tfrac{1}{2}u_1^3u_2+\tfrac{1}{2}u_1^3u_3-3u_1^3-u_1^4+2u_1^2u_2-2u_1^2+2u_1u_3-2u_2+u_3\right)+\cdots \tag{32}$$

Subtracting (31) from (32), results in

$$f\left(z_n+f\left(z_n\right)\right)-f\left(z_n\right)=e_n c_1^2+e_n^2\left(c_2\left(c_1+1\right)^2+\tfrac{1}{2}c_1^4+c_1^3-c_1c_2-\tfrac{1}{2}c_1^3-c_2\right)+ \\ e_n^3\begin{pmatrix}\tfrac{1}{2}u_1^2u_2-\tfrac{1}{2}u_1^3u_2-\tfrac{1}{2}u_1^3u_3-u_1u_3 \\ -3u_1^3-u_1^4+2u_1u_2+2u_1^2+\left(u_1+1\right)^3u_3+\tfrac{1}{2}u_1^2u_2+\tfrac{1}{2}u_1^2u_3-u_3\end{pmatrix}+\cdots \tag{33}$$

Then, Taylor's expansion of $g\left(z_n\right)$ is given as

$$g\left(z_n\right)=e_n u_1^2+e_n^2\left(\tfrac{1}{2}u_1^3+u_1^2u_2\right)+e_n^3(\tfrac{1}{4}u_1^5+\tfrac{1}{2}u_1^4u_2+2u_1^4+6u_1^3-\tfrac{1}{2}u_1^3u_3+\tfrac{3}{2}u_1^3u_2 \\ -4u_1^2u_3+\tfrac{3}{2}u_1^2u_2+4u_1^2+u_1^2u_2+2u_2+4u_1u_2-u_3)+\cdots \tag{34}$$

The Taylor's expansion of the denominator is therefore

$$g\left(t_n\right)+4g\left(v_n\right)+g\left(z_n\right)=6u_1^2e_n-e_n^2\left(3u_1^3+4u_1^2u_2+4u_1u_2\right)+\cdots \tag{35}$$

Hence, the Taylor expansion of $\frac{6f\left(t_n\right)}{g\left(t_n\right)+4g\left(v_n\right)+g\left(z_n\right)}$ about x_n is

$$\frac{6f(t_n)}{g(t_n)+4g(v_n)+g(z_n)} = e_n + e_n^2\left(\frac{2u_2}{3u_1}+\frac{u_1}{2}+\frac{2u_2}{3}\right)+\cdots \tag{36}$$

Substituting to obtain the error equation

$$e_{n+1} = e_n^2\left(\frac{2u_2}{3u_1}+\frac{u_1}{2}+\frac{2u_2}{3}\right)+O\left(e_n^3\right) \tag{37}$$

Hence, the method is of order p Equation (37) shows that the method is of order m=2.

4. Results and Discussions

In this section, the efficiency of the new method modified method verified, by applying it to solving nonlinear equations in Table 1.

Table 1. *Test Functions and their simple roots.*

s/n	Nonlinear Equations	Simple Root	Initial point
1	$Cos\ x - x = 0$	$\alpha = 0.739085$	$x_0 = -2$
2	$Sinx - 1 + x = 0$	$\alpha = 0.510973$	$x_0 = 1$
3	$e^x - 3x = 0$	$\alpha = 0.619061$	$x_0 = 0$
4	$x^3 - 6x + 4 = 0$	$\alpha = 0.732051$	$x_0 = 1$
5	$3x^2 - 0.6x - 7 = 0$	$\alpha = -1.4308$	$x_0 = 0.1$
6	$xTan\ x + 1 = 0$	$\alpha = 2.79839$	$x_0 = 2.5$
7	$2Sin\ x - x = 0$	$\alpha = 1.89549$	$x_0 = 2.9$

All computations were carried out using MATLAB. For methods considered, the iterations are in successions and converge to an approximate solution of the nonlinear equations. The test equations on Table 1, are used to compare the new method MSTM with other well-known methods: Newton Method (NM), Steffensen's method (SM) and the New Iterative Method (NIM). All the test equations are adapted from [3]. All methods are of order p = 2 except for NIM which is of p = 3. The new method MSTM is similar to the classical methods in operation.

Table 2. *Numerical results for the nonlinear equation (1) on Table 1.*

Iterates (i)	Root (x_i) (MSTM)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)
1	-0.613394	0.734536	7.91330	3.98861
2	0.376528	0.7390892	0.874395	-4.47739
3	0.713909	0.739085	0.736226	0.520357
4	0.738968	-	0.739084	0.728841
5	0.739085	-	0.739085	0.739058
6	-	-	-	0.739085

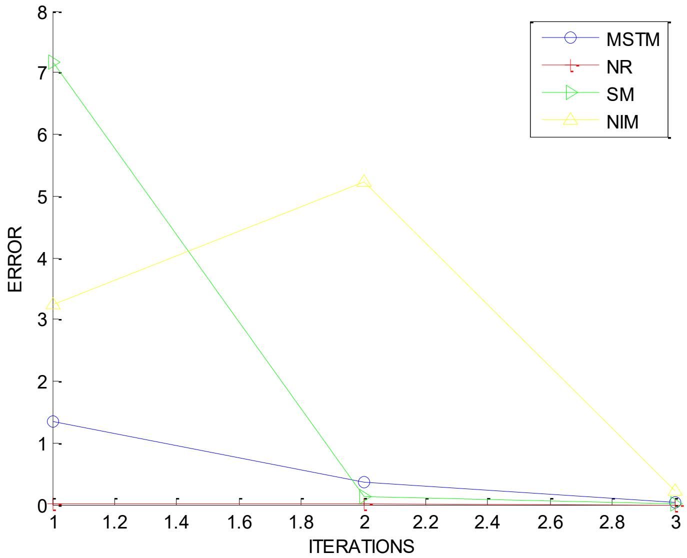

Figure 1. The graph of error versus iterations for problem 1.

Table 2 presents the results for solving problem 1 with each of the four methods. The results indicate clearly that the proposed methods did not fail to meet the convergence criteria.

The proposed Modified Steffensen Type Method (MSTM) is of order p=2 converges faster than the New Iterative Method of order p=3 as shown in Table 2. The oscillatory property of the equation is tracked by both MSTM and NIM.

Figure 1 shows the summary of the results of the iteration processes among the four methods for equations in which the iterations regularly convergence to the root.

Table 3. Numerical results for the nonlinear equation (2) on Table 1.

Iterates (i)	Root (x_i) (MSTM)	Root (x_i) NR)	Root (x_i) SM)	Root (x_i)(NIM)
1	0.498677	0.453698	0.265172	0.548969
2	0.510955	0.51058	0.50270	0.511279
3	0.510973	0.510973	0.510948	0.510973
4	-	-	0.510973	-

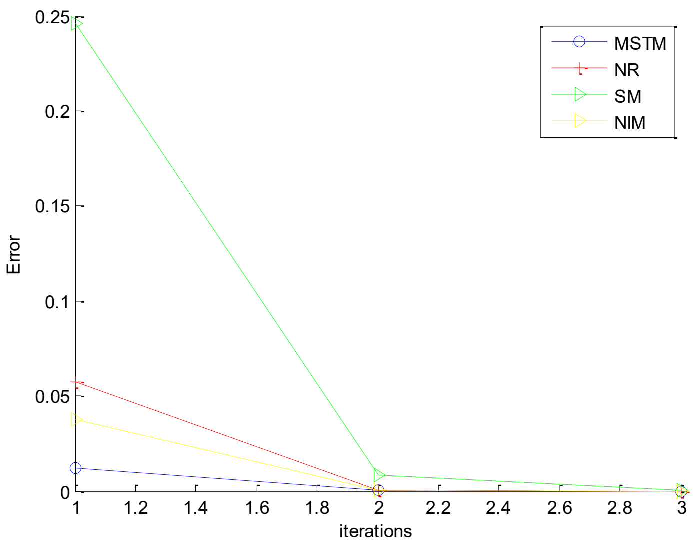

Figure 2. The graph of error versus iterations for problem 2.

The rate of convergence of the MSTM as shown in Table 3, is as fast as that of NR and NIM as it has the same number of iterations as the Newton and New Iterative Method but the MSTM converges to the exact root α of the function in each iteration. The new method shows superiority over the other methods.

Table 4. *Numerical results for the nonlinear equation (3) on Table 1.*

Iterates (i)	Root (x_i) (MSTM)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)
1	0.816451	0.5	0.553574	
2	0.638342	0.61006	0.619727	Diverges
3	0.619234	0.618997	0.619061	
4	0.619061	0.619061	-	

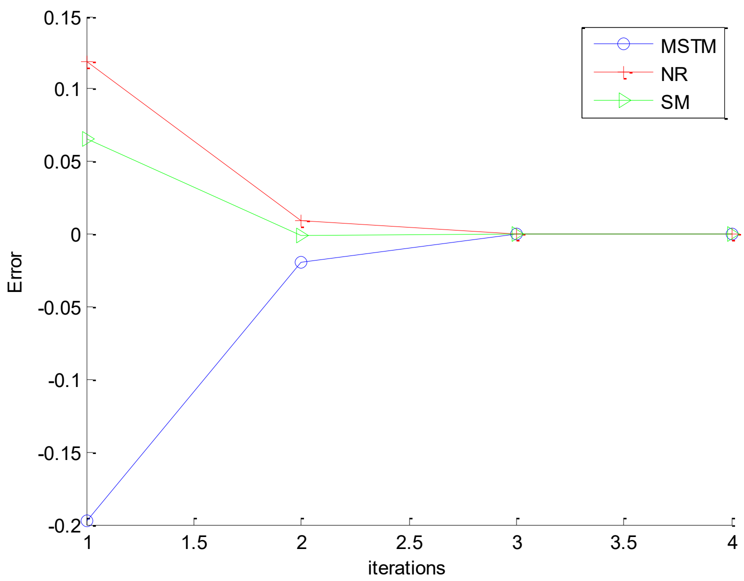

Figure 3. *The graph of error versus iterations for problem 3.*

In a similar pattern, for the equation:

$e^x - 3x = 0$ with the approximated root $\alpha = 0.619061$. The graph indicates the total number of iterations required is 4 for the MST method, for Newton method and 3 for Steffensen's method. The table reveals clearly that the new method diverges.

The table shows the number of iterations required for finding roots of the equations among the different methods where by the proposed Modified Steffensen's method shows convergence rate similar to Newton methods and better overall rate of convergence.

The Steffensen's converges to α after three iterations while proposed method required additional iteration like the Newton's method. The NIM of order p=3 diverges.

The new modified method has; therefore, better order of convergence compared to the regular Secant method, Newton Raphson's method which have order of convergence.

Table 5. *Numerical results for the nonlinear equation (4) on Table 1.*

Iterates (i)	Root (x_i) (MSTM)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)
1	0.796016	0.666667	0.8	0.790698
2	0.730373	0.730159	0.73857	0.743282
3	0.732071	0.732049	0.732122	0.732721
4	0.732051	0.732051	0.732051	0.732054
5	-	-	-	0.732051

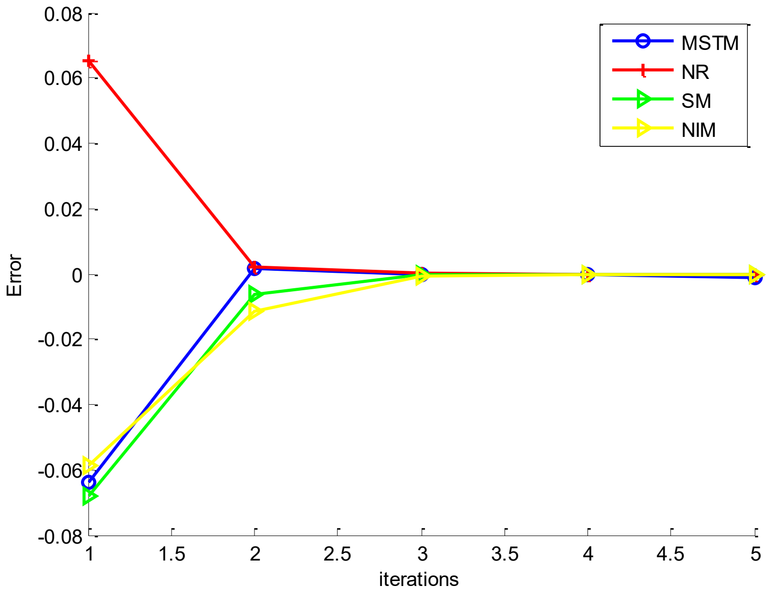

Figure 4. The graph of error versus iterations for problem 4.

The MSTM approximate the root of the third-degree polynomial (4) at same number of iterations with the Newton's method and Steffensen's method as indicated on Table 5.

In Table 6, the efficiency of the proposed method is demonstrated via its application on problem (5) in Table 1. The Newton method (NR) and NIM diverges while Steffensen's method (SM) and its variants MSTM approximated the root efficiently. Whereas, the Steffensen's Method required ten (10) iterations, the proposed MSTM required nine (9) iterations to converge.

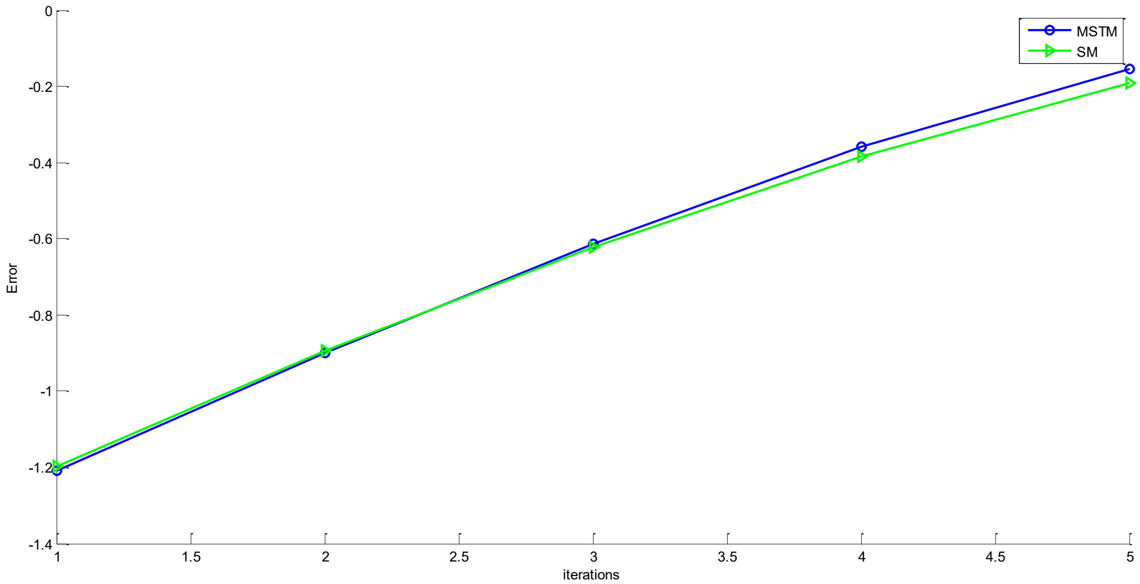

Figure 5. The graph of error versus iterations for problem 5.

Table 6. Numerical results for the nonlinear equation (5) on Table 1.

Iterates (i)	Root (x_i) (MSTM)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)
1	-0.223119	Diverges	-0.233333	Diverges
2	-0.53067		-0.536487	

Iterates (i)	Root (x_i) (MSTM)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)
3	-0.817694		-0.809952	
4	-1.07353		-1.04799	
5	-1.27705		-1.23887	
6	-1.39654		-1.36606	
7	-1.42901		-1.42127	
8	-1.43079		-1.43056	
9	-1.4308		-1.43079	
10	-		-1.4308	

This result shows that the proposed modified Steffensen's Methods has a relative advantage for this type of problem.

Table 7. Numerical results for the nonlinear equation (6) on Table 1.

Iterates (i)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)	Root (x_i) (MSTM)
1	2.77558	2.53062	2.64056	2.33474
2	2.79838	2.6102	2.77668	2.71079
3	2.79839	2.73879	2.79832	2.79672
4	-	2.79680	2.79839	2.79839
5	-	2.79839	-	-

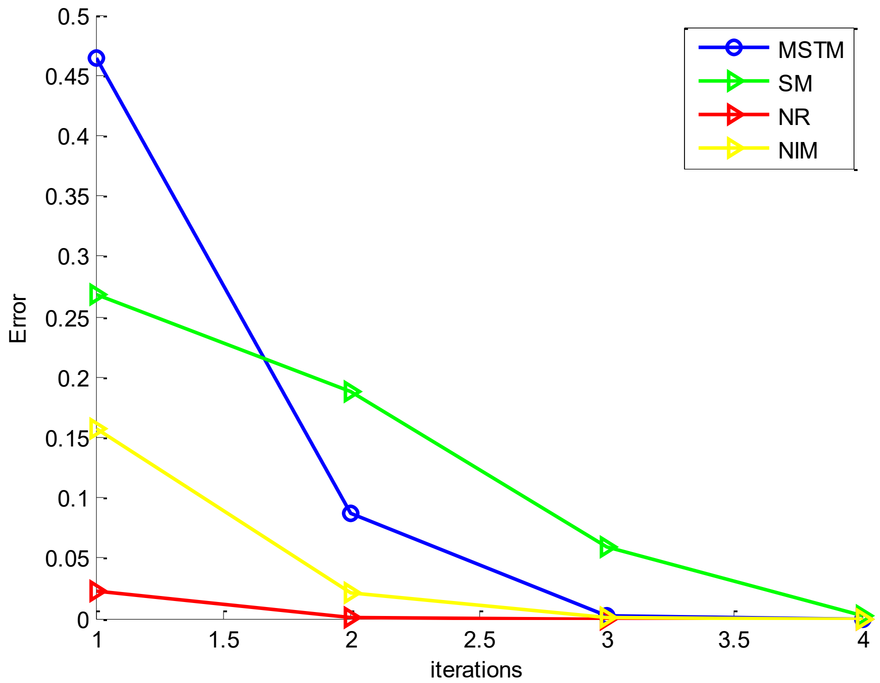

Figure 6. The graph of error versus iterations for problem 6.

For the equation $xTan\,x + 1 = 0$ with the approximated root $\alpha = 2.79839$. For this equation, the total number of iterations required is 4 for the MSTM method with a sharper rate of convergence; for Newton method 3 iterations but with comparative low rate of convergence and 3 for Steffensen's method also with comparative low rate of convergence.

The table shows the number of iterations required for finding roots of the equations among the different methods where by the proposed Modified Steffensen's method shows convergence rate similar to Newton methods and better overall rate of convergence.

The Steffensen's converges to α after three iterations while proposed method required additional iteration like the Newton's method. The NIM of order p=3 diverges.

The modified method has; therefore, better order of convergence compared to the regular Steffensen's method, Newton Raphson's method which have order of convergence.

Table 8. Numerical results for the nonlinear equation (7) on Table 1.

Iterates (i)	Root (x_i) (NR)	Root (x_i) (SM)	Root (x_i) (NIM)	Root (x_i) (MSTM)
1	2.0769	0.852556	-2.68597	1.07413
2	1.91056	3.4634	1.78590	1.58535
3	1.89562	1.26769	1.65651	1.85322
4	1.89549	1.88755	1.87742	1.89466
5	-	1.89547	1.89529	1.89549
6	-	1.89549	1.89549	-

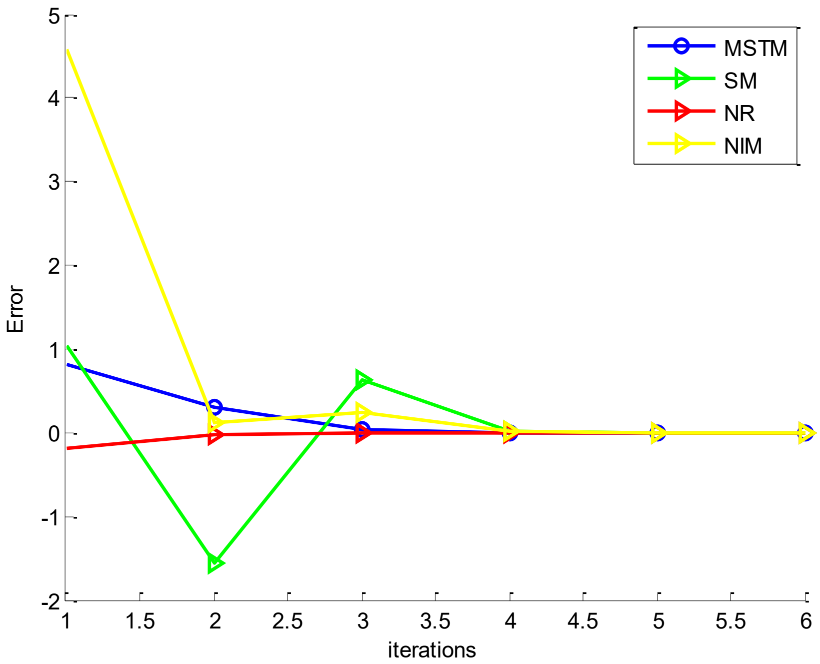

Figure 7. The graph of error versus iterations for problem 7.

It is known that when a set of vectors converges, the norm

$$\| x^{(k+1)} - x^{(k)} \| = 0$$

Thus, by our table above, the norm is equal to zero at the sixth iteration. This indicates that our method has converged to the solution, which will be denoted by $\bar{x}$.

MSTM is investigated for its convergence for this type of problem. Table 1 summarizes the results of the iteration for these pathological cases in which comparison is method between the proposed new method and the traditional NR, SM, NIM. The Tables show that, whereas either NRM, SM or the new method (and in some cases both) fail to converge to the root; the proposed method almost always converges to the root. This result shows that the proposed modified Steffensen's method has a relative advantage.

Figures 2 to 7 show the characteristics of convergence for the three different methods when applied to the equation:

In Tables 7 and 8, the approximation of the sinusoidal equation (7) in Table 1 reveals that the NIM yielded oscillatory approximation. The proposed MSTM generated stable approximated solution which converged after five iterations.

5. Conclusion

The computational efficiency of the new modified method makes it a promising tool for solving a wide range of nonlinear problems in both theoretical and applied contexts. This study provides a detailed analysis of the method's convergence properties, supported by extensive numerical experiments that validate its superior performance compared to existing methods. By addressing the challenges of traditional approaches and providing a more efficient alternative, this work contributes to the ongoing advancement of numerical methods for solving nonlinear equations.

Abbreviations

MSTM	Modified Steffensen's Type Method
SM	Secant Method
NIM	New Iterative Method
NR	Newton-Raphson's Method

Author Contributions

Clement Adaku Nnedinma: Conceptualization, Data curation, Methodology, Project administration, Writing – original draft

Bazuaye Frank Etin-Osa: Software, Supervision, Validation, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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