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On Weakly $\mathcal{S}$-Prime Ideal Graph Of a Finite Commutative Ring

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Abstract: Let $\mathscr{H}$ be a finite commutative ring with unity. Let $\mathcal{I}$ be a proper ideal of $\mathscr{H}$ and $\mathcal{S}$ is the multiplicative closed subset of $\mathscr{H}$ which is disjoint with I . The weakly S-prime ideal graph denoted by $G_{\mathcal{I}}(\mathscr{H})$ is the undirected graph whose vertex set is the set of elements $\mathfrak{e}$ of $\mathscr{H}$ such that the non-zero product ef is in I and either se is in I or sf is in I for some f in H and the two distinct vertices $\mathfrak{e}$ and $\mathfrak{f}$ are connected by an edge if and only if either se is in I or sf is in I for some s in S. The purpose of this article is to investigate the graph theoretic properties of the weakly S-prime ideal graph associated with $\mathscr{H}$. This study focuses on rings of order $2\mathfrak{p}, 3\mathfrak{p}$ and $\mathfrak{pq}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are distinct primes. For these rings, the weakly S-prime ideal graph is a special type of graph and it is explained with examples. Furthermore, the graph theoretic concept of the weakly S-prime ideal graph $G_{\mathcal{I}}(\mathscr{H})$ namely its girth, diameter, radius and size are studied. The relation between the weakly S-prime ideal graph and annihilator ideal graph associated with a ring of order $2\mathfrak{p}$ is described and it is proved that these two graphs are isomorphic.

Keywords: Weakly S-prime Ideal, Degree, Diameter, Girth, Chromatic Number, Size

1. Introduction

Throughout this work, the ring $\mathscr{H}$ is assumed to be a finite commutative ring with identity. There is a lot of research relationship between graph theory and algebraic structure. The graphs are mainly used as mathematical systems involving groups, rings and modules etc., as well as in real-world applications. [9] A ring $\mathscr{H}$ is defined a non-empty set with two binary operations, addition and multiplication, where addition makes $\mathscr{H}$ an abelian group, multiplication is associative and the distributive laws holds.

The graph is a concept in discrete mathematics that is still being actively developed at this time and the study of graph from rings has become the most extensively utilized area of research as it advances both graph theory and algebra. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ denote a graph, where $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ are the set of vertices and edges respectively. The order of $\mathcal{G}$ is given by $|\mathcal{V}(\mathcal{G})|$ and the size of $\mathcal{G}$ is given by $|\mathcal{E}(\mathcal{G})|$. There are several individual graphs namely complete graphs, regular graphs, null graph, bipartite graphs etc.,

The study of zero-divisor graphs connect algebra and graph theory. The zero-divisor graph of $\mathscr{H}$ was first introduced by I. Beck [3] the graph of $\mathscr{H}$ is denoted by $\Gamma_0(\mathscr{H})$. Subsequently, Anderson and Livingston [2] gave a new definition of zero-divisor graph of a ring. The zero-divisor graph denoted by $\Gamma(\mathscr{H})$ is undirected graph whose vertices are the non-zero zero-divisors of $\mathscr{H}$ with two distinct vertices $\mathfrak{a}$ and $\mathfrak{b}$ joined by an edges if and only if $ab = 0$.

A graph can be represented using groups such as the generator graph, generating graph, commuting graph, cayley graph, power graph, automorphism groups in polyhedral graphs, solvable conjugacy class graph of groups etc., and a graph can be represented using rings such as the prime ideal graph, maximal ideal graph, prime intersection graph etc.,

In recent studies, Kalamani and Kiruthika [6] introduced the new concept of the vertex-order graph of the finite cyclic group. Further Kalamani and Ramya [7] introduced the product maximal graph of $\mathscr{H}$. The product maximal graph of $\mathscr{H}$, denoted by $\Gamma_{pm}(\mathscr{H})$ has a vertex set $\mathscr{H}$ and the two distinct vertices a and b are adjacent whenever $ab \in M$, where M is a maximal ideal and D. Kalamani and G. Ramya [8] introduced graph theoretical properties for $\Gamma_{pm}(\mathscr{H})$ and the resistance Distance Based indices. Subsequently, D. Kalamani and C.V. Mythily [10] introduced a new graph called S-prime ideal graph $\mathcal{G}_{S_d}$ has vertex set is elements of $\mathscr{H}$ and the two vertices $\mathfrak{a}$ and $\mathfrak{b}$ are joined by an edge if there exist $\mathfrak{s} \in S$ such that $\mathfrak{s}\mathfrak{a} \in S_d$ or $\mathfrak{s}\mathfrak{b} \in S_d$. This motivation led to the definition of a new graph.

Ahmed Hamed and Achraf Malek [15] introduced the S-prime ideal of a ring. Further, Fuad Ali Ahmahdi et.al [1] defined the notion of a weakly S-prime ideal of a $\mathscr{H}$. Let $\mathcal{S}$ be a multiplicative closed subset of $\mathscr{H}$ an ideal $\mathfrak{P}$ of $\mathscr{H}$ satisfying $\mathfrak{P} \cap \mathcal{S} = \emptyset$ is said to be weakly S- prime if there exist an element $\mathfrak{s} \in \mathcal{S}$ such that whenever $\mathfrak{a}, \mathfrak{b} \in \mathscr{H}$, $0 \neq \mathfrak{a}\mathfrak{b} \in \mathfrak{P}$ implies $\mathfrak{s}\mathfrak{a} \in \mathfrak{P}$ or $\mathfrak{s}\mathfrak{b} \in \mathfrak{P}$.

In this article, we define a new graph called weakly S-prime ideal graph of $\mathscr{H}$ for the ideal $\mathcal{I}$ and it is represented as $G_{\mathcal{I}}(\mathscr{H})$. The properties are studied from[4] and subsequently it is applied to $G_{\mathcal{I}}(\mathscr{H})$ and several new graph results are developed.

2. Preliminaries

This section provides the basic definitions related to the ring $\mathscr{H}$ and the graph definitions which are used in this paper.

Definition 2.1. A proper ideal $\mathscr{A}$ of a ring $\mathscr{H}$ is said to be weakly prime ideal of $\mathscr{H}$ if $\mathfrak{a}, \mathfrak{b} \in \mathscr{H}$ such that $0 \neq \mathfrak{a}\mathfrak{b} \in \mathscr{A}$ implies $\mathfrak{a} \in \mathscr{A}$ or $\mathfrak{b} \in \mathscr{A}$.

Definition 2.2. A subset $\mathcal{U} \subseteq \mathscr{H}$ is multiplicative closed if $1 \in \mathcal{U}$ and $\forall u, u' \in \mathcal{U}$, then their product $uu' \in \mathcal{U}$.

Definition 2.3. A graph is said to be complete if every vertex is connected to every other vertex. The notion $\mathcal{K}_n$ is used for a complete graph with n vertices.

Definition 2.4. The number of edges incident with a vertex is called degree of that vertex.

Definition 2.5. The two graphs are said to be isomorphic if

1. The number of vertices are equal,
2. The number of edges are equal,
3. The number of sequence are equal.

Definition 2.6. The distance $d(u,v)$ between the two vertices u and v is the length of the shortest path between u and v. The eccentricity, diameter and radius of the vertex in G are defined as follows: $e(v) = Max\{d(u,v) : u, v \in V(\mathcal{G})\}$, $diam = Max\{e(v) : v \in \mathcal{V}(\mathcal{G})\}$ and $rad = Min\{e(v) : v \in \mathcal{V}(\mathcal{G})\}$.

Definition 2.7. The girth of a graph $\mathcal{G}$ denoted by $gr(\mathcal{G})$is the length of the shortest cycle in $\mathcal{G}$. If $\mathcal{G}$ has no cycle, then the girth of $\mathcal{G}$ is infinite or $gr(\mathcal{G}) = \infty$.

Definition 2.8. A graph $\mathcal{G}$ is called bipartite if its vertex set V is partitioned into two non-empty subsets X and Y such that each edge is incident with one vertex from X and the other from Y.

Definition 2.9. The least number of colors required for coloring the graph $\mathcal{G}$ is called chromatic number.

Definition 2.10. The triangular book graph with n-pages is defined as n copies of cycle C_3 sharing a common edge. The common edge is called the spine or base of the book. This graph is denoted by $B(3,n)$. In other words it is the complete tripartite graph $K_{1,1,n}$.

Definition 2.11. Let $\mathscr{H}$ be a ring. The nilpotent graph of $\mathscr{H}$ denoted by $\mathcal{G}(\mathscr{H})$ is defined as follows: the vertex set is $\mathcal{G}(\mathscr{H}) \setminus Nil\mathscr{G}(\mathscr{H})$, where $Nil\mathscr{G}(\mathscr{H})$ denotes the set of all nilpotent elements of $\mathscr{H}$ and the two distinct vertices u and v are adjacent if $u + v$ is nilpotent.

3. Main Results

In this part, we introduced a new definition of the weakly S-prime ideal graph of the ring of order n.

Definition 3.1. Let $\mathscr{H}$ be a ring . Let $\mathcal{I}$ be a proper ideal of $\mathscr{H}$ and $\mathcal{S}$ is the multiplicative closed subset of $\mathscr{H}$ which is disjoint with $\mathcal{I}$. The weakly S-prime ideal graph denoted by $G_{\mathcal{I}}(\mathscr{H})$ is the undirected graph whose vertex set is the set of elements $\mathfrak{e}$ of $\mathscr{H}$ such that $0 \neq \mathfrak{e}\mathfrak{f} \in \mathcal{I}$ and either $\mathfrak{s}\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{s}\mathfrak{f} \in \mathcal{I}$ for some $\mathfrak{f} \in \mathscr{H}$ and the two distinct vertices $\mathfrak{e}$ and $\mathfrak{f}$ are connected by an edge if and only if either $\mathfrak{s}\mathfrak{e} \in \mathcal{I}$ or $\mathfrak{s}\mathfrak{f} \in \mathcal{I}$ for some $\mathfrak{s} \in \mathcal{S}$.

In this present work, Let $\mathscr{H}$ be a ring of order n. The weakly S-prime ideal graph and the multiplicative closed subset of $\mathscr{H}$ denoted as $G_{\mathcal{I}}(\mathscr{H})$ and $\mathcal{S}$ respectively. If d is a divisor of n, then d denote the ideal generated by d as $\mathcal{I}_d$ and its complement as $\mathcal{I}'_d$. The corresponding weakly S-prime ideal graph $G_{\mathcal{I}_d}(\mathscr{H})$ can be represented by $G_d(\mathscr{H})$.

4. Weakly S-prime Ideal Graph of Ring of Order 2p

There are two graphs namely $G_2(\mathscr{H})$ and $G_{\mathfrak{p}}(\mathscr{H})$ for the ring of order $2\mathfrak{p}$. The graph $G_{\mathfrak{p}}(\mathscr{H})$ forms a special type of graph and is discussed with suitable examples in this section. The vertices of the graphs $G_2(\mathscr{H})$ and $G_{\mathfrak{p}}(\mathscr{H})$ are $\mathcal{I}'_{\mathfrak{p}}$ and $\mathcal{I}'_2$ respectively.

Theorem 4.1. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then $G_{\mathfrak{p}}(\mathscr{H})$ is a star graph.

Proof: The only vertex $\mathfrak{p}$ of $G_{\mathfrak{p}}(\mathscr{H})$ is the ideal element of $\mathcal{I}_{\mathfrak{p}}$ and other vertices are non-ideal elements of $\mathscr{H}$.

Assume $\mathfrak{e}(or\ \mathfrak{f}) \in \mathcal{I}_{\mathfrak{p}}$, $\mathfrak{f}(or\ \mathfrak{e}) \in \mathscr{H} \setminus \{\mathcal{I}_{\mathfrak{p}}\}$ in $G_{\mathfrak{p}}(\mathscr{H})$. The product $\mathfrak{e}\mathfrak{f}$ will be in $\mathcal{I}_{\mathfrak{p}}$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}\mathfrak{e} \in \mathcal{I}_{\mathfrak{p}}\ or\ \mathfrak{s}\mathfrak{f} \in \mathcal{I}_{\mathfrak{p}}$. Thus, $\mathfrak{e}(or\ \mathfrak{f})$ adjacent to $\mathfrak{f}(or\ \mathfrak{e})$. Let $\mathfrak{e}$,$\mathfrak{f} \notin \mathscr{H} \setminus \{\mathcal{I}_{\mathfrak{p}}\}$ in $G_{\mathfrak{p}}(\mathscr{H})$.

Consider $\mathfrak{e}$ and $\mathfrak{f}$ are the elements of multiplicative closed subset $\mathcal{S}$. The product of $\mathfrak{e}\mathfrak{f}$ belongs to $\mathcal{S}$ but it does not belongs to $\mathcal{I}_{\mathfrak{p}}$.

Claim: $\mathfrak{s}\mathfrak{e} \notin \mathcal{I}_{\mathfrak{p}}$ and $\mathfrak{s}\mathfrak{f} \notin \mathcal{I}_{\mathfrak{p}}$ for all $\mathfrak{s}$ in $\mathcal{S}$.

Assume $\mathfrak{s}\mathfrak{e} \in \mathcal{I}_{\mathfrak{p}}$ for some $\mathfrak{s} \in \mathcal{S}$. This implies that

$$\mathfrak{s}\mathfrak{e}\mathfrak{f} \in \mathcal{I}_{\mathfrak{p}} \tag{1}$$

Since $\mathfrak{e} \notin \mathcal{I}_{\mathfrak{p}}$, $\mathfrak{f} \notin \mathcal{I}_{\mathfrak{p}}$, $\mathfrak{e}$ and $\mathfrak{f}$ are elements of $\mathcal{S}$. The product of $\mathfrak{e}\mathfrak{f}$ is an element of $\mathcal{S}$. Thus, $\mathfrak{e}\mathfrak{f} \in \mathcal{S}$ and $\mathfrak{s} \in \mathcal{S}$. The product of $\mathfrak{s}\mathfrak{e}\mathfrak{f} \in \mathcal{S}$. This is contradiction to (1). Hence, $\mathfrak{s}\mathfrak{e} \notin \mathcal{I}_\mathfrak{p}$ and $\mathfrak{s}\mathfrak{f} \notin \mathcal{I}_\mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$.

There is no edge between $\mathfrak{e}$ and $\mathfrak{f}$. The vertex $\mathfrak{p}$ is connected to every other vertices of $G_\mathfrak{p}(\mathscr{H})$ and hence there is no adjacency between the vertices other than $\mathfrak{p}$.

Hence, it is a star graph whose central vertex is $\mathfrak{p}$.

Example 4.1. Let $R = \mathbb{Z}_{10}$.

The weakly S-prime ideals are $\mathcal{I}_2$ and $\mathcal{I}_5$. The associated graphs are illustrated in Figure 1.

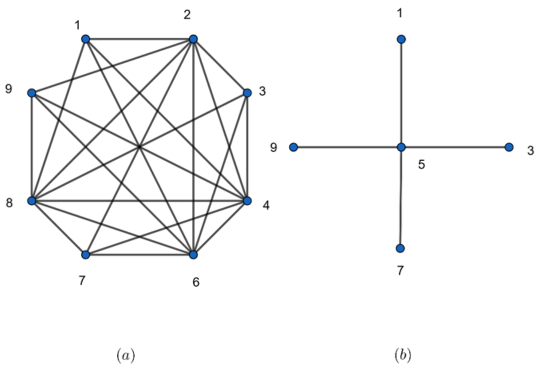


Figure 1. $(\mathfrak{a})G_2(\mathscr{H})$ *and* $(\mathfrak{b})G_5(\mathscr{H})$.

The $G_\mathcal{I}(\mathscr{H})$ of $G_2(\mathscr{H})$ is a connected graph and $G_5(\mathscr{H})$ is a star graph. The vertices of $G_2(\mathscr{H})$ is $\mathcal{I}_5'$ and $G_5(\mathscr{H})$ is $\mathcal{I}_2'$ where $\mathcal{I}_2'$ and $\mathcal{I}_5'$ are the complements of $\mathcal{I}_2$ and $\mathcal{I}_5$ respectively.

Theorem 4.2. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then the graphs $G_\mathfrak{p}(\mathscr{H})$ and $\Gamma(\mathscr{H})$ are isomorphic.

Proof: The graphs $G_\mathfrak{p}(\mathscr{H})$ and $\Gamma(\mathscr{H})$ are star graph for the ring of given order.

By theorem 4.1, the $G_\mathfrak{p}(\mathscr{H})$ has $\mathfrak{p}$ vertices and [11] the $\Gamma(\mathscr{H})$ has $2\mathfrak{p}$ - $[\phi(2\mathfrak{p})$ + 1] vertices which is equivalent to $\mathfrak{p}$. Therefore, the number of vertices in $G_\mathfrak{p}(\mathscr{H})$ and $\Gamma(\mathscr{H})$ are the same.

$$\text{Hence, } G_\mathfrak{p}(\mathscr{H}) \cong \Gamma(\mathscr{H})$$

Remark: The central vertex of both $\Gamma(\mathscr{H})$ and $G_\mathfrak{p}(\mathscr{H})$ is $\mathfrak{p}$.

Theorem 4.3. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then the diameter of $G_2(\mathscr{H})$ is 2 and radius 1.

Proof: Let v be any vertex of $G_2(\mathscr{H})$ and choose another vertex v_1 in $G_2(\mathscr{H})$.

If v is an element of $\mathcal{I}_2$, then the product vv_1 lies in $\mathcal{I}_2$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v \in \mathcal{I}_2$ for every $\mathfrak{s} \in \mathcal{S}$. Hence all other vertices of $G_2(\mathscr{H})$ are adjacent to the vertex v.

$$d(v, v_1) = 1 \text{ for all } v_1 \in G_2(\mathscr{H})$$

If v does not belong to $\mathcal{I}_2$ and v_1 belongs to $\mathcal{I}_2$, then the product $vv_1 \in \mathcal{I}_2$. Since $\mathfrak{s}v_1 \in \mathcal{I}_2$ for every $\mathfrak{s} \in \mathcal{S}$. Hence, an edge exists between the vertices v and v_1

$$d(v, v_1) = 1$$

Both v and v_1 are not elements of $\mathcal{I}_2$, then their product vv_1 does not belong to $\mathcal{I}_2$. Thus, the vertices v and v_1 are not adjacent.

$$d(v, v_1) = 2.$$

$$d(v, v_1) = \begin{cases} 1 & if \ \ v_1 \in \mathcal{I}_2 \\ 2 & if \ \ v_1 \notin \mathcal{I}_2 \end{cases}$$

Hence, the eccentricity of vertex v is

$$e(v) = \begin{cases} 1 & if \ \ v \in \mathcal{I}_2 \\ 2 & otherwise \end{cases}$$

Thus, the diameter of $G_2(\mathscr{H})$ is 2 and radius 1.

Theorem 4.4. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then the girth of $G_2(\mathscr{H})$ is

$$gr[G_2(\mathscr{H})] = \begin{cases} 3 & if \ \ v \in \mathcal{I}_2 \\ \infty & otherwise \end{cases}$$

Proof: Let $\mathfrak{e}, \mathfrak{f} \in G_2(\mathscr{H})$, where $\mathfrak{e} \neq \mathfrak{f}$.

If $\mathfrak{e} \in \mathcal{I}_2$ and $\mathfrak{f} \in \mathcal{I}_2$, then their product $\mathfrak{e}\mathfrak{f}$ also lies in $\mathcal{I}_2$. Moreover, since $\mathcal{S} \subseteq \mathscr{H}$, both $\mathfrak{s}\mathfrak{e}$ and $\mathfrak{s}\mathfrak{f}$ are the elements of the ideal $\mathcal{I}_2$ for all $\mathfrak{s} \in \mathcal{S}$. Thus, the vertices $\mathfrak{e}$ and $\mathfrak{f}$ are connected by an edge. Consequently, the ideal elements form a complete subgraph $\mathcal{K}_{(\mathfrak{p}-1)}$ with $(\mathfrak{p}-1)$ vertices.

If $\mathfrak{p} \geq 5$, then the graph $G_2(\mathscr{H})$ form a cycle. Therefore,

$$\text{gr}[G_2(\mathscr{H})] = 3$$

If $\mathfrak{p} < 5$, then the graph $G_2(\mathscr{H})$ contains no cycle. Thus,

$$\text{gr}[G_2(\mathscr{H})] = \infty$$

If $\mathfrak{e}, \mathfrak{f} \notin \mathcal{I}_2$, then the product $\mathfrak{e}\mathfrak{f}$ is not in $\mathcal{I}_2$. Thus, there is no edge between $\mathfrak{e}$ and $\mathfrak{f}$. Hence the non-ideal elements of $G_2(\mathscr{H})$ form an independent graph with $(\mathfrak{p}-1)$ vertices. Therefore, $G_2(\mathscr{H})$ has no cycle. Hence,

$$\text{gr}[G_2(\mathscr{H})] = \infty$$

If $\mathfrak{e} \in \mathcal{I}_2$, $\mathfrak{f} \notin \mathcal{I}_2$, then the product $\mathfrak{e}\mathfrak{f}$ is an element of $\mathcal{I}_2$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}\mathfrak{e} \in \mathcal{I}_2$ for all $\mathfrak{s} \in \mathcal{S}$. Thus, there is an edge between e and f. Therefore, each non-ideal element of $\mathscr{H}$ is adjacent to all the ideal elements of $\mathcal{I}_2$ in $G_2(\mathscr{H})$ and hence it gives a $\mathcal{K}_{(\mathfrak{p}-1),(\mathfrak{p}-1)}$ with $2(\mathfrak{p}-1)$ vertices.
Therefore, $G_2(\mathscr{H})$ has no cycle. Thus,

$$\text{gr}[G_2(\mathscr{H})] = \infty$$

$$gr[G_2(\mathscr{H})] = \begin{cases} 3, & \text{if } v \in \mathcal{I}_2 \\ \infty, & \text{otherwise} \end{cases}$$

The following theorem determines the chromatic number of $G_2(\mathcal{R})$.

Theorem 4.5. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then the chromatic number of $G_2(\mathscr{H})$ is $\chi[G_2(\mathscr{H})]$= $\mathfrak{p}$.

Proof: Assume $\mathfrak{e}$ and $\mathfrak{f}$ be two distinct vertices of the graph $G_2(\mathscr{H})$.

By the previous theorem, the ideal elements form a complete subgraph $\mathcal{K}_{(\mathfrak{p}-1)}$ with $(\mathfrak{p}-1)$ vertices of the $G_2(\mathscr{H})$ and the vertex colored with $\mathfrak{p}-1$ distinct colors.

The non-ideal elements of $\mathscr{H}$ forms an independent graph of $G_2(\mathscr{H})$ with a single color at each vertex.

Each non-ideal element in $G_2(\mathscr{H})$ is adjacent to all ideal elements of $\mathcal{I}_2$ and form a $\mathcal{K}_{(\mathfrak{p}-1),(\mathfrak{p}-1)}$ with $2(\mathfrak{p}-1)$ vertices. Thus, the vertices are colored by different color for the ideal element and non-ideal element of $G_2(\mathscr{H})$.

$$\chi[G_2(\mathscr{H})] = \mathfrak{p} - 1 + 1 = \mathfrak{p}$$

Hence, the chromatic number of $G_2(\mathscr{H})$ is $\chi[G_2(\mathscr{H})] = \mathfrak{p}$.

Theorem 4.6. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then the degree of $G_2(\mathscr{H})$ is given by

$$\deg(v) = \begin{cases} 2\mathfrak{p} - 3, & \text{if } v \in \mathcal{I}_2 \\ \mathfrak{p} - 1, & \text{if } v \notin \mathcal{I}_2 \end{cases}$$

Proof: Let v be any vertex of $G_2(\mathscr{H})$ and choose another vertex in $G_2(\mathscr{H})$.

If v is an element of $\mathcal{I}_2$, then the product vv_1 will be in the weakly S- prime ideal $\mathcal{I}_2$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v \in \mathcal{I}_2$ for all $\mathfrak{s} \in \mathcal{S}$. Thus, there are edges from all the other vertices of $G_2(\mathscr{H})$ to the vertex v. The degree of the vertex v is $|G_2(\mathscr{H})|$-1.

$$\deg(v) = 2\mathfrak{p} - 3 \quad \text{if } v \in \mathcal{I}_2$$

If v is not an element of $\mathcal{I}_2$, then the following cases occur:

Case(i): If $v_1 \in \mathcal{I}_2$, then the product vv_1 will be in the element of $\mathcal{I}_2$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v_1 \in \mathcal{I}_2$ for all $\mathfrak{s} \in \mathcal{S}$. Thus, there is an edge connecting v and v_1.

Case(ii): If v_1 is not an element of $\mathcal{I}_2$, then the product vv_1 is not an element of $\mathcal{I}_2$. Thus, there is no adjacency between v and v_1. The degree of the vertex v is $|\mathcal{I}_2|$ -1 in $G_2(\mathscr{H})$.

$$\deg(v) = \mathfrak{p} - 1 \quad \text{if } v \notin \mathcal{I}_2$$

$$\text{Hence, } \deg(v) = \begin{cases} 2\mathfrak{p} - 3, & \text{if } v \in \mathcal{I}_2 \\ \mathfrak{p} - 1, & \text{if } v \notin \mathcal{I}_2 \end{cases}$$

Theorem 4.7. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then the size of $G_2(\mathscr{H})$ is $\frac{1}{2}(\mathfrak{p}-1)(3\mathfrak{p}-4)$.

Proof: The $G_2(\mathscr{H})$ is a connected graph with $2(\mathfrak{p}-1)$ vertices.

Consider $\mathfrak{e}$ and $\mathfrak{f}$ be the distinct vertices of the graph $G_2(\mathscr{H})$.

Since the $\mathcal{I}_2$ elements of $G_2(\mathscr{H})$ form a $\mathcal{K}_{(\mathfrak{p}-1)}$ with $(\mathfrak{p}-1)$ vertices, the number of edges of $G_2(\mathscr{H})$ is $\frac{(\mathfrak{p}-1)(\mathfrak{p}-2)}{2}$.

The non ideal elements form an independent graph with $(\mathfrak{p}-1)$ vertices.

Each non-ideal element of $\mathcal{I}_2$ is adjacent to all the ideal elements of $G_2(\mathscr{H})$ and hence it forms a $\mathcal{K}_{(\mathfrak{p}-1),(\mathfrak{p}-1)}$ with $2(\mathfrak{p}-1)$ vertices, the number of edges of $G_2(\mathscr{H})$ is $(\mathfrak{p}-1)^2$.

The size of $G_2(\mathscr{H})$ is

$$\frac{1}{2}(\mathfrak{p}-1)(\mathfrak{p}-2) + (\mathfrak{p}-1)^2$$

$$|E(G_2(\mathscr{H}))| = \frac{1}{2}(\mathfrak{p}-1)(3\mathfrak{p}-4)$$

Theorem 4.8. Let $\mathscr{H}$ be a ring of order $2\mathfrak{p}$. Then the graphs $G_{\mathfrak{p}}(\mathscr{H})$ and annihilator ideal graph $AG(\mathscr{H})$ are isomorphic.

Proof: First to prove $AG(\mathscr{H})$ is a star graph.

The vertex set of $AG(\mathscr{H})$ is $Z(\mathscr{H})^*$ and hence $|Z(\mathscr{H})^*|$ contains $\mathfrak{p}$ vertices .

In a ring of order $2\mathfrak{p}$, the annihilator ideals are $\mathfrak{ann}(2k) = \{0, \mathfrak{p}\}$ where $1 \leq k \leq \mathfrak{p}-1$, $\mathfrak{ann}(\mathfrak{p}) = \{0, 2, 4, 6, ..., 2\mathfrak{p}-2\}$ and $\mathfrak{ann}(u) = \{0\}$ for every unit u.

If $\mathfrak{e} = p$ and $\mathfrak{f} = 2k$ are the vertices of $AG(\mathscr{H})$, then their product $\mathfrak{ef} = 0$ and hence $\mathfrak{ann}(\mathfrak{ef}) = \mathfrak{ann}(0) = \mathscr{H}$. Since $\mathfrak{e}$ and $\mathfrak{f}$ are the non-zero, $\mathfrak{ann}(\mathfrak{e})$ and $\mathfrak{ann}(\mathfrak{f})$ are the proper ideals of $\mathscr{H}$ and the union of $\mathfrak{ann}(\mathfrak{e})$ and $\mathfrak{ann}(\mathfrak{f})$ is a proper ideal of $\mathscr{H}$. Hence $\mathfrak{ann}(\mathfrak{ef}) \neq \mathfrak{ann}(\mathfrak{e}) \bigcup \mathfrak{ann}(\mathfrak{f})$. Thus, the vertices $\mathfrak{e}$ and $\mathfrak{f}$ are adjacent in the annihilator ideal graph $AG(\mathscr{H})$.

If $\mathfrak{e}$ and $\mathfrak{f}$ are $2k$ vertices of the annihilator ideal graph $AG(\mathscr{H})$, then their product $2k_1.2k_2 = 2k$ and hence $\mathfrak{ann}(\mathfrak{ef}) = \mathfrak{ann}(\mathfrak{p})$. Since $\mathfrak{ann}(\mathfrak{e}) = \mathfrak{ann}(\mathfrak{f}) = \mathfrak{ann}(\mathfrak{p})$ which implies $\mathfrak{ann}(\mathfrak{ef}) = \mathfrak{ann}(\mathfrak{e}) \bigcup \mathfrak{ann}(\mathfrak{f})$. Thus, the vertices $\mathfrak{e}$ and $\mathfrak{f}$ are not adjacent in the graph $AG(\mathscr{H})$.

Hence the vertex $\mathfrak{p}$ is adjacent to every other vertices $2k$ in annihilator ideal graph $AG(\mathscr{H})$ and it is a star graph.

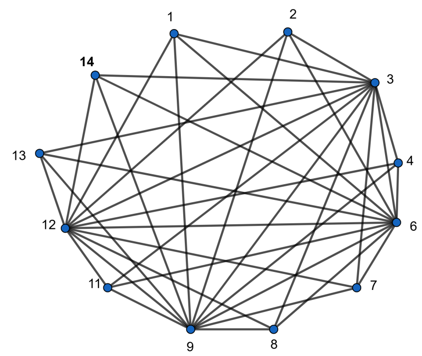
By theorem 4.1, the $G_{\mathfrak{p}}(\mathscr{H})$ is a star graph. Since the $|G_{\mathfrak{p}}(\mathscr{H})|$ is $\mathfrak{p}$, the graphs $G_{\mathfrak{p}}(\mathscr{H})$ and $AG(\mathscr{H})$ are isomorphic.

5. Weakly S-prime Ideal Graph of Ring of Order 3p

Let $\mathscr{H}$ be a ring of order $3\mathfrak{p}$. The weakly S-prime ideals of $\mathscr{H}$ are $\mathcal{I}_3$ and $\mathcal{I}_{\mathfrak{p}}$. This section investigates the properties of the graph of ring of order $3\mathfrak{p}$ through various theorems and examples. Let $G_3(\mathscr{H})$ and $G_{\mathfrak{p}}(\mathscr{H})$ be the ideals are $\mathcal{I}_3$ and $\mathcal{I}_{\mathfrak{p}}$ respectively. The vertex sets are $V(G_3(\mathscr{H})) = \mathcal{I}_{\mathfrak{p}}'$ and $V(G_p(\mathscr{H})) = \mathcal{I}_3'$.

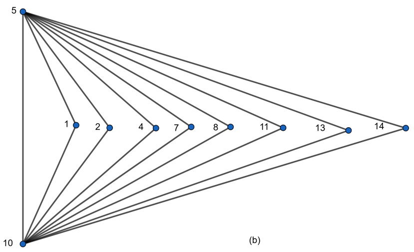
Example 5.1. Let $R = \mathbb{Z}_{15}$.

The weakly S-prime ideals are $\mathcal{I}_3$ and $\mathcal{I}_5$. The corresponding graphs are shown in Figure 2.



(a)

(a)



(b)

Figure 2. $(\mathfrak{a})G_3(\mathscr{H})$ *and* $(\mathfrak{b})G_5(\mathscr{H})$.

The $G_{\mathcal{I}}(\mathscr{H})$ of $G_3(\mathscr{H})$ is a connected graph and $G_5(\mathscr{H})$ is a triangular book graph. The vertices of $G_3(\mathscr{H})$ is $\mathcal{I}_5'$ and $G_5(\mathscr{H})$ is $\mathcal{I}_3'$ where $\mathcal{I}_3'$ and $\mathcal{I}_5'$ are the complements of $\mathcal{I}_3$ and $\mathcal{I}_5$ respectively.

Theorem 5.1. Let $\mathscr{H}$ be a ring of order $3\mathfrak{p}$. Then $G_{\mathfrak{p}}(\mathscr{H})$ is a triangular book graph.

Proof: Consider the two different vertices $\mathfrak{e}$ and $\mathfrak{f}$ in $G_{\mathfrak{p}}(\mathscr{H})$.

If both $\mathfrak{e}$ and $\mathfrak{f}$ belongs to $\mathcal{I}_{\mathfrak{p}}$ in $G_{\mathfrak{p}}(\mathscr{H})$, then their product $\mathfrak{ef} \in \mathcal{I}_{\mathfrak{p}}$. Since $\mathcal{S} \subseteq \mathscr{H}$, then their product of $\mathfrak{se} \in \mathcal{I}_{\mathfrak{p}}$ and $\mathfrak{sf} \in \mathcal{I}_{\mathfrak{p}}$ for all $\mathfrak{s} \in \mathcal{S}$. Thus, $\mathfrak{e}$ and $\mathfrak{f}$ are joined by an edge.

If $\mathfrak{e}, \mathfrak{f} \notin \mathcal{I}_{\mathfrak{p}}$ in $G_{\mathfrak{p}}(\mathscr{H})$, then their product $\mathfrak{ef} \notin \mathcal{I}_{\mathfrak{p}}$. Hence, the non-adjacent vertices are $\mathfrak{e}$ and $\mathfrak{f}$.

If $\mathfrak{e} \in \mathcal{I}_{\mathfrak{p}}$ and $\mathfrak{f} \notin \mathcal{I}_{\mathfrak{p}}$, then the product $\mathfrak{ef} \in \mathcal{I}_{\mathfrak{p}}$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{se} \in \mathcal{I}_{\mathfrak{p}}$ for all $\mathfrak{s} \in S$. Thus, the vertices $\mathfrak{e}$ and $\mathfrak{f}$ are adjacent.

Hence, each non-ideal element is adjacent to elements of $\mathcal{I}_{\mathfrak{p}}$ in $G_{\mathfrak{p}}(\mathscr{H})$.

The graph contains only two ideal elements, each non-ideal element forms a triangle and having the common edge with the ideal elements.

Therefore, $G_{\mathfrak{p}}(\mathscr{H})$ is a triangular book graph.

The graph $G_{\mathfrak{p}}(\mathscr{H})$ has the ideal elements as base of the book and non-ideal elements as the $2(\mathfrak{p}-1)$ pages of the book.

Theorem 5.2. Let $\mathscr{H}$ be a ring of order $3\mathfrak{p}$. Then the degree of $G_{\mathfrak{p}}(\mathscr{H})$ is given by

$$\deg(v) = \begin{cases} 2\mathfrak{p}-1, & \text{if } v \in \mathcal{I}_{\mathfrak{p}} \\ 2, & \text{if } v \notin \mathcal{I}_{\mathfrak{p}} \end{cases}$$

Proof: Let v be any vertex of $G_{\mathfrak{p}}(\mathscr{H})$ and choose another vertex v_1 in $G_{\mathfrak{p}}(\mathscr{H})$.

If v is an element of $\mathcal{I}_{\mathfrak{p}}$, then the product vv_1 is in $\mathcal{I}_{\mathfrak{p}}$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v \in \mathcal{I}_{\mathfrak{p}}$ for all $\mathfrak{s} \in \mathcal{S}$. There is an edge between v and v_1. The degree of the vertex v is $|G_{\mathfrak{p}}(\mathscr{H})| - 1$.

Thus,

$$\deg(v) = 2\mathfrak{p} - 1 \quad \text{if } v \in \mathcal{I}_{\mathfrak{p}}$$

If v does not belong to weakly S-prime ideal $\mathcal{I}_{\mathfrak{p}}$, then v_1 is an element of $\mathcal{I}_{\mathfrak{p}}$, the product vv_1 belongs to $\mathcal{I}_{\mathfrak{p}}$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v_1 \in \mathcal{I}_{\mathfrak{p}}$ for all $\mathfrak{s} \in \mathcal{S}$. There is an edge between v and v_1. However, when v_1 is not an element of $\mathcal{I}_{\mathfrak{p}}$, the product vv_1 does not belong to $\mathcal{I}_{\mathfrak{p}}$. Thus, there is no adjacency between the vertices v and v_1. The degree of the vertex v is $|\mathcal{I}_{\mathfrak{p}}| - 1$ in $G_{\mathfrak{p}}(\mathscr{H})$.

Thus,

$$\deg(v) = 2 \quad \text{if } v \notin \mathcal{I}_{\mathfrak{p}}$$

$$\text{Hence, } \deg(v) = \begin{cases} 2\mathfrak{p}-1, & \text{if } v \in \mathcal{I}_{\mathfrak{p}} \\ 2, & \text{if } v \notin \mathcal{I}_{\mathfrak{p}} \end{cases}$$

Theorem 5.3. Let $\mathscr{H}$ be a ring of order $3\mathfrak{p}$. Then the size of $G_3(\mathscr{H})$ is $\frac{1}{2}(\mathfrak{p}-1)(5\mathfrak{p}-6)$.

Proof: The $G_3(\mathscr{H})$ is a connected graph with $3(\mathfrak{p}-1)$ vertices. Assume two distinct vertices $\mathfrak{e}$ and $\mathfrak{f}$ in $G_3(\mathcal{R})$.

If $\mathfrak{e}$ and $\mathfrak{f}$ are the elements of the ideal $\mathcal{I}_3$, then their product $\mathfrak{ef}$ also belongs to $\mathcal{I}_3$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{se}$ and $\mathfrak{sf}$ are the elements of ideal $\mathcal{I}_3$ for every $\mathfrak{s} \in \mathcal{S}$. Hence, there is an edge connecting $\mathfrak{e}$ and $\mathfrak{f}$. Therefore, every pair of elements of $\mathcal{I}_3$ is adjacent and hence they form a subgraph of $\mathcal{K}_{(\mathfrak{p}-1)}$ with $(\mathfrak{p}-1)$ vertices. Thus, the number of edges of the complete graph is $\frac{(\mathfrak{p}-1)(\mathfrak{p}-2)}{2}$.

If $\mathfrak{e}$ and $\mathfrak{f}$ does not belong to $\mathcal{I}_3$, then the product $\mathfrak{ef}$ does not belong to $\mathcal{I}_3$. Hence, there exists no edge between $\mathfrak{e}$ and $\mathfrak{f}$.

If $\mathfrak{e} \in \mathcal{I}_3$ and $\mathfrak{f} \notin \mathcal{I}_3$, then the product $\mathfrak{ef}$ lies in $\mathcal{I}_3$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{se}$ belongs to $\mathcal{I}_3$ for all $\mathfrak{s} \in \mathcal{S}$. Hence, there is an edge between $\mathfrak{e}$ and $\mathfrak{f}$. Therefore, each ideal element of $\mathcal{I}_3$ is adjacent to all the non-ideal elements of $G_3(\mathscr{H})$ and the number of non-ideal elements of $G_3(\mathscr{H})$ is $2(\mathfrak{p}-1)$. Hence, it forms a $\mathcal{K}_{(\mathfrak{p}-1),2(\mathfrak{p}-1)}$. Thus, the number of edges is $2(\mathfrak{p}-1)^2$. Hence, the size of $G_3(\mathscr{H})$ is

$$\frac{1}{2}(\mathfrak{p}-1)(\mathfrak{p}-2) + (\mathfrak{p}-1) \cdot 2(\mathfrak{p}-1)$$

$$|E(G_3(\mathscr{H}))| = \frac{1}{2}(\mathfrak{p}-1)(5\mathfrak{p}-6)$$

Theorem 5.4. Let $\mathscr{H}$ be a ring of order $3\mathfrak{p}$. Then the degree of $G_3(\mathscr{H})$ is given by

$$\deg(v) = \begin{cases} 3\mathfrak{p}-4, & \text{if } v \in \mathcal{I}_3 \\ \mathfrak{p}-1, & \text{if } v \notin \mathcal{I}_3 \end{cases}$$

Proof: Let v be any vertex of $G_3(\mathscr{H})$ and choose another vertex v_1 other than v in $G_3(\mathscr{H})$.

For any $v \in \mathcal{I}_3$, then the product vv_1 will be in $\mathcal{I}_3$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v \in \mathcal{I}_3$ for all $\mathfrak{s} \in \mathcal{S}$. Thus,there exists an edge between v and v_1. The degree of the vertex v is $|G_3(\mathscr{H})|$- 1.

$$\deg(v) = 3\mathfrak{p} - 4 \quad \text{if } v \in \mathcal{I}_3$$

suppose that $v \notin \mathcal{I}_3$. If $v_1 \in \mathcal{I}_3$ then the product v_1v will be in $\mathcal{I}_3$. Since $\mathcal{S} \subseteq \mathscr{H}$, the product $\mathfrak{s}v_1 \in \mathcal{I}_3$ for all $\mathfrak{s} \in \mathcal{S}$ which implies v_1 is adjacent to every other vertices in $G_3(\mathscr{H})$. If $v_1 \notin \mathcal{I}_3$, then the product v_1v will not be in $\mathcal{I}_3$ and no adjacency occurs between v_1 and v.

The degree of the vertex v is $|\mathcal{I}_3|$-1 in $G_3(\mathscr{H})$

$$\deg(v) = \mathfrak{p} - 1 \quad \text{if } v \notin \mathcal{I}_3$$

$$\deg(v) = \begin{cases} 3\mathfrak{p} - 4, & \text{if } v \in \mathcal{I}_3 \\ \mathfrak{p} - 1, & \text{if } v \notin \mathcal{I}_3 \end{cases}$$

Theorem 5.5. Let $\mathscr{H}$ be a ring of order $3\mathfrak{p}$. Then the chromatic number of $G_3(\mathscr{H})$ is $\chi[G_3(\mathscr{H})] = \mathfrak{p}$

Proof: The subgraph of $G_3(\mathscr{H})$ formed by the ideal elements is a complete graph with $\mathfrak{p} - 1$ vertices.

Thus, the ideal elements of $G_3(\mathscr{H})$ are colored by distinct $\mathfrak{p} - 1$ colors.

The subgraph of $G_3(\mathscr{H})$ formed by the non-ideal elements is an independent graph. Since the non-ideal elements are adjacent to the ideal elements, the ideal elements and non-ideal elements of $G_3(\mathscr{H})$ do not share the same color.

Thus, the non-ideal elements of $G_3(\mathscr{H})$ are colored by a single color.

$$\chi[G_3(\mathscr{H})] = \mathfrak{p} - 1 + 1 = \mathfrak{p}$$

Hence, the chromatic number of $G_3(\mathscr{H})$ is $\chi[G_3(\mathscr{H})]$= $\mathfrak{p}$.

Theorem 5.6. Let $\mathscr{H}$ be a ring of order $3\mathfrak{p}$. Then the chromatic number of $G_\mathfrak{p}(\mathscr{H})$ is $\chi[G_\mathfrak{p}(\mathscr{H})] = 3$.

Proof: $G_\mathfrak{p}(\mathscr{H})$ is a triangular book graph by theorem 5.1 In a graph, the ideal elements form the base of the book and non-ideal elements form the pages of the book.

Each non-ideal element is adjacent to the ideal elements of $G_\mathfrak{p}(\mathscr{H})$. The graph contains two ideal elements of $\mathcal{I}_\mathfrak{p}$ which form the base of the book and $2(\mathfrak{p} - 1)$ non-ideal elements which form the pages of the book.

The base of the book is colored with distinct colors and pages of the book are colored with single color. The base of the book and the pages of the book are colored with different colors.

$$\chi[G_\mathfrak{p}(\mathscr{H})] = 3$$

Hence, the chromatic number of $G_\mathfrak{p}(\mathscr{H})$ is $\chi[G_\mathfrak{p}(\mathscr{H})]$= 3.

6. Weakly S-prime Ideal Graph of Ring of Order pq

The results from ring of orders $2\mathfrak{p}$ and $3\mathfrak{p}$ are generalized to the ring of order $\mathfrak{pq}$. The size and degree of the graph are derived. Furthermore, the nilpotent graph is a subgraph of $G_\mathfrak{p}(\mathscr{H})$ and $G_\mathfrak{q}(\mathscr{H})$ and it is explained in this section.

Theorem 6.1. Let $\mathscr{H}$ be a ring of order $\mathfrak{pq}$. Then the size of

$$G_\mathfrak{p}(\mathscr{H}) \text{ is } \frac{1}{2}(\mathfrak{q} - 1)[2\mathfrak{p}(\mathfrak{q} - 1) - \mathfrak{q}]$$

Proof: The $G_\mathfrak{p}(\mathscr{H})$ is a connected graph and the vertices of $G_\mathfrak{p}(\mathscr{H})$ are the elements of $\mathcal{I}_\mathfrak{q}^{'}$.

Therefore, the total number of vertices is $\mathfrak{p}(\mathfrak{q} - 1)$.

Let $\mathfrak{e}, \mathfrak{f} \in G_\mathfrak{p}(\mathscr{H})$ such that $\mathfrak{e} \neq \mathfrak{f}$.

If both $\mathfrak{e}$ and $\mathfrak{f}$ belongs to $\mathcal{I}_\mathfrak{p}$, then their product $\mathfrak{ef} \in \mathcal{I}_\mathfrak{p}$. Since $\mathcal{S} \subseteq \mathscr{H}$, both $\mathfrak{se} \in \mathcal{I}_\mathfrak{p}$ and $\mathfrak{sf} \in \mathcal{I}_\mathfrak{p}$ for every $\mathfrak{s} \in \mathcal{S}$. Hence, there is an edge between $\mathfrak{e}$ and $\mathfrak{f}$. Therefore, every pair of ideal elements $\mathcal{I}_\mathfrak{p}$ are adjacent and hence these elements form a subgraph of $\mathcal{K}_{(\mathfrak{q}-1)}$ with $(\mathfrak{q} - 1)$ vertices.

Thus, the number of edges of the complete graph

$$G_\mathfrak{p}(\mathscr{H}) \text{ is } \frac{(\mathfrak{q} - 1)(\mathfrak{q} - 2)}{2}$$

If both $\mathfrak{e}$ and $\mathfrak{f}$ does not belong to $\mathcal{I}_\mathfrak{p}$, then their product $\mathfrak{ef} \notin \mathcal{I}_\mathfrak{p}$. Hence, there is no edge between $\mathfrak{e}$ and $\mathfrak{f}$.

If $\mathfrak{e} \in \mathcal{I}_\mathfrak{p}$ and $\mathfrak{f} \notin \mathcal{I}_\mathfrak{p}$, then their product $\mathfrak{ef}$ is an element of $\mathcal{I}_\mathfrak{p}$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{se}$ is in $\mathcal{I}_\mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. Hence, there is an edge between $\mathfrak{e}$ and $\mathfrak{f}$. Therefore, each ideal element of $\mathcal{I}_\mathfrak{p}$ is adjacent to all the non-ideal elements of $G_\mathfrak{p}(\mathscr{H})$ and the number of non-ideal elements of $G_\mathfrak{p}(\mathscr{H})$ is $(\mathfrak{p} - 1)(\mathfrak{q} - 1)$. Hence, it forms a $\mathcal{K}_{(\mathfrak{q}-1),(\mathfrak{p}-1)(\mathfrak{q}-1)}$. Thus, the number of edges of the complete bipartite graph

$$G_\mathfrak{p}(\mathscr{H}) \text{ is } (\mathfrak{p} - 1)(\mathfrak{q} - 1)^2$$

Hence, the size of $G_\mathfrak{p}(\mathscr{H})$ is

$$\frac{1}{2}(\mathfrak{q} - 1)(\mathfrak{q} - 2) + (\mathfrak{p} - 1)(\mathfrak{q} - 1)^2$$

$$|E(G_p(\mathscr{H}))| = \frac{1}{2}(\mathfrak{q} - 1)[2\mathfrak{p}(\mathfrak{q} - 1) - \mathfrak{q}]$$

Theorem 6.2. Let $\mathscr{H}$ be a ring of order $\mathfrak{pq}$. Then the degree of the vertex v in $G_\mathfrak{p}(\mathscr{H})$ defined as

$$\deg(v) = \begin{cases} \mathfrak{pq} - \mathfrak{p} - 1, & \text{if } v \in \mathcal{I}_\mathfrak{p} \\ \mathfrak{q} - 1, & \text{if } v \notin \mathcal{I}_\mathfrak{p} \end{cases}$$

Proof: The vertices of $G_\mathfrak{p}(\mathscr{H})$ are the elements of $\mathcal{I}_\mathfrak{q}'$. Let v be any vertex of the $G_\mathfrak{p}(\mathscr{H})$ and choose another vertex v_1 in $G_\mathfrak{p}(\mathscr{H})$.

If v is an element of $\mathcal{I}_\mathfrak{p}$, then the product vv_1 is an element of $\mathcal{I}_\mathfrak{p}$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v$ is an element of $\mathcal{I}_\mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. Thus, v is adjacent to v_1. Therefore, the degree of the vertex v is $|G_\mathfrak{p}(\mathscr{H})|$- 1.

$$\deg(v) = \mathfrak{pq} - \mathfrak{p} - 1 \quad \text{if } v \in \mathcal{I}_\mathfrak{p}$$

If $v \notin \mathcal{I}_\mathfrak{p}$, then v_1 is an element of $\mathcal{I}_\mathfrak{p}$, the product vv_1 will be in the weakly S-prime ideal $\mathcal{I}_p$. Since $\mathcal{S} \subseteq \mathscr{H}$, $\mathfrak{s}v_1$ belongs to $\mathcal{I}_\mathfrak{p}$. Hence v_1 is connected to all the vertices of $G_\mathfrak{p}(\mathscr{H})$. However, when v_1 is not an element of $\mathcal{I}_\mathfrak{p}$, the product vv_1 does not belong to $\mathcal{I}_\mathfrak{p}$ which implies there is no edge between v and v_1.Therefore, the degree of the vertex v is $|\mathcal{I}_\mathfrak{p}|$-1 in $G_\mathfrak{p}(\mathscr{H})$.

$$\deg(v) = \mathfrak{q} - 1 \quad \text{if } v \notin \mathcal{I}_\mathfrak{p}$$

$$\deg(v) = \begin{cases} \mathfrak{pq} - \mathfrak{p} - 1, & \text{if } v \in \mathcal{I}_p \\ \mathfrak{q} - 1, & \text{if } v \notin \mathcal{I}_p \end{cases}$$

Theorem 6.3. Let $\mathscr{H}$ be a ring of order $\mathfrak{pq}$. Then the chromatic number of $G_\mathfrak{p}(\mathscr{H})$ is $\chi[G_\mathfrak{p}(\mathscr{H})]$= $\mathfrak{q}$.

Proof: The graph $G_{\mathfrak{p}}(\mathscr{H})$ has vertex set $\mathcal{I}_{\mathfrak{q}}^{'}$.
The ideal elements of $\mathcal{I}_{\mathfrak{p}}$ in $G_{\mathfrak{p}}(\mathscr{H})$ forms a complete graph with $(\mathfrak{q}-1)$ vertices by theorem 6.1. Therefore, the ideal elements in $G_{\mathfrak{p}}(\mathscr{H})$ the vertices are colored by $\mathfrak{q}-1$ distinct colors.

Any pair of non-ideal elements are not adjacent in $G_{\mathfrak{p}}(\mathscr{H})$.

The non-ideal elements are adjacent to the ideal elements of $\mathcal{I}_{\mathfrak{p}}$. Since the ideal elements of $\mathcal{I}_{\mathfrak{p}}$ share the edges with all the non-ideal elements in $G_{\mathfrak{p}}(\mathscr{H})$.

Therefore, the ideal elements and non-ideal elements in $G_{\mathfrak{p}}(\mathscr{H})$ are colored by different colors. Thus, the non-ideal elements of $G_{\mathfrak{p}}(\mathscr{H})$ are colored by a single color.

$$\text{Hence, } \chi[G_{\mathfrak{p}}(\mathscr{H})] = \mathfrak{q}$$

Theorem 6.4. Let $\mathscr{H}$ be a ring of order $\mathfrak{pq}$. If the component C of the nilpotent graph $G(\mathscr{H})$ contains an element of $< p >$ or $< q >$, then it is a subgraph of the $G_{\mathfrak{p}}(\mathscr{H})$ or $G_{\mathfrak{q}}(\mathscr{H})$ respectively.

Proof: The vertices of $G(\mathscr{H})$ is $\mathscr{H} \setminus Nil(\mathscr{H})$.

The $G(\mathscr{H})$ is disconnected and has $C_1, C_2, C_3, ...C_k$ components [17].

If $\mathfrak{e} \in G(\mathscr{H})$, then either $\mathfrak{e}$ is an ideal element $\mathfrak{p}$ or $\mathfrak{q}$ or a unit.

If $\mathfrak{e} \in< p >$, then there exist $\mathfrak{f} \in< p >$ such that $\mathfrak{e}+\mathfrak{f}$ belongs to nilpotent element. Thus, the pair $(\mathfrak{e},\mathfrak{f})$ form a component of $G(\mathscr{H})$. Therefore, the element of $< p >$ form a $\frac{|\langle p\rangle|}{2}$ components in $G_{\mathfrak{p}}(\mathscr{H})$.

Similarly, if $\mathfrak{e} \in< q >$,there are $\frac{|\langle q\rangle|}{2}$ components in $G_{\mathfrak{p}}(\mathscr{H})$.

In $G_{\mathfrak{p}}(\mathscr{H})$, the ideal elements of $< p >$ form a complete subgraph of $G_{\mathfrak{p}}(\mathscr{H})$.

Each components C_i where $1 \leq C_i \leq \frac{|\langle p\rangle|}{2}$ is a subgraph of the complete graph and hence it is a subgraph of $G_{\mathfrak{p}}(\mathscr{H})$.

Similarly, Each components C_i where $1 \leq C_i \leq \frac{|\langle q\rangle|}{2}$ is a subgraph of the complete graph and hence it is a subgraph of $G_{\mathfrak{q}}(\mathscr{H})$.

7. Conclusion

In this paper, the weakly S-prime ideal graph for the rings of $\mathscr{H}$ of order $2\mathfrak{p}$,$3\mathfrak{p}$ and $\mathfrak{pq}$ are introduced and it is seen that they are connected graphs. In particular, $G_{\mathfrak{p}}(\mathscr{H})$ is a star graph which is isomorphic to both $\Gamma(\mathscr{H})$ and $AG(\mathscr{H})$ if $|\mathscr{H}| = 2\mathfrak{p}$. It is a triangular book graph if $|\mathscr{H}| = 3\mathfrak{p}$. In a ring of order $\mathfrak{pq}$, the nilpotent graph becomes the subgraph of $G_{\mathfrak{p}}(\mathscr{H})$.

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Abbreviations

$\mathscr{H}$	Finite Commutative Ring with Unity
S	Multiplicative Closed Subset of $\mathscr{H}$ Disjoint with $\mathcal{I}$
$G_{\mathcal{I}}(\mathscr{H})$	Weakly S-prime Ideal Graph
$\mathcal{I}$	Proper Ideal of $\mathscr{H}$
$\mathcal{I}_{\mathfrak{p}}, \mathcal{I}_{\mathfrak{q}}$	Prime Ideal of $\mathscr{H}$
$\mathcal{I}_{\mathfrak{p}}'$	Non-prime Ideal of $\mathscr{H}$
C_i	Components
$G(\mathscr{H})$	Nilpotent Graph
$\Gamma(\mathscr{H})$	Zero-divisor Graph
$AG(\mathscr{H})$	Annihilator Ideal Graph

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Vasuki Shanmugam: Conceptualization, Methodology, Investigation, Visualization, Writing - original draft

Conflicts of Interest

The authors declare no conflicts of interest.

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