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A Novel Analytic Method with Integral Transform for Solving Classes of Second and Third Order Ordinary Linear Differential Equations with Variable Coefficients

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Abstract: Analytical solutions of second- and third-order non-homogeneous Ordinary Linear Differential Equations (OLDEs) with variable coefficients have been investigated using an established mathematical tool, the integral transform, together with a new analytic method developed in this study. This study aims to utilize the integral transform alongside the new analytical method. The new method was derived from the concept of exactness in higher-order ODEs. Specifically, second- and third-order ODEs with variable coefficients are exact if there exist first- and second-order linear ODEs whose derivatives correspond to the given equations, respectively. In this new analytic method, an integrating factor function formula for second-order ODEs has been carefully formulated and derived, making every second-order ODE with variable coefficients reducible to its lower-order form, specifically first-order ODEs. To ensure the accuracy of the new method, two well-known classes of second-order linear ODEs, namely the Whittaker second-order linear ODE and the Modified Bessel equation, were applied. The results demonstrated that the new analytic method effectively solves these equations, producing exact analytical solutions. To validate the effectiveness and efficiency of the new analytic method, a comparative analysis was conducted using illustrative examples, followed by graphical representations of the solution results.

Keywords: Integral Transforms, Laplace Transform, New Analytic Method

1. Introduction

This paper presents a novel analytic method for obtaining solutions of linear nonhomogeneous second and third-order Ordinary Differential Equations (ODEs) using an integrating-factor approach. Their canonical forms of the ODEs are given:

$$y'' + p(x)y' + Q(x)y = R(x). \tag{1}$$

$$g''' + \alpha(x)g'' + \beta(x)g' + \gamma(x)g = M(x). \tag{2}$$

Where $y = y(x)$ and $g = g(x)$ are the associated analytical solutions for second and third-order ODEs respectively and the coefficients p(x), Q(x), $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are some real-valued functions of the independent variable x in I. The nature of the coefficients for the second and third-order linear ODEs considered in this study are polynomial functions of degree n where n lies in this range $(1 \leq n \leq 3)$. There have been many studies of second-order ODEs with variable coefficients see [1-3] for details. If one of the solutions of Equation (1) is known or given, then the other solution can be obtained using the method of reduction of order [4,5]. However, there is no universal analytical method to find even a single solution for ODEs of the form (1) except under certain restrictive conditions. Some well-known techniques, including reduction of order, change of independent variables, variation of parameter, and the method of undetermined coefficients, are applied to find the general solution or one specific solution for certain particular classes of ODEs of the form (1). Also, the third order in Equation (2) can only be solved in its general form when it is written in the form of an Euler-Cauchy equation. Hence, this area of differential equations especially second-order linear ODEs has received numerous applications in the theory of electric circuits and establishing a connection with vibrations in mechanics see [6-8] in details. These are of great importance, as their solution has important implications and builds the basic phenomena of electromagnetics, wave motion, heat conduction, fluid mechanics, stress analysis, and aerodynamics, among others [9,10] see in details. Meanwhile, for the third- and higher-order ODEs do not come much at an elementary level, but some of the applications can be seen here in much more detail [11]. When the applications of integral transforms for solving differential equations are discussed, many attempts have been made to transform Equation (1) into a simple form to make it solvable with the help of different transformations, i.e. Laplace transformation [12].

Wilmer III, A. and Costa, G. [13] developed a method in which an analytical solution is obtained for certain classes of second-order differential equations with coefficients of polynomial variables. A desired solution is obtained by the use of transformations and repeated iterated integration. This alternative method represents a different way to acquire a solution from classic power series techniques and other approaches. Ahmed, Z. and Kalim, M. [14] introduced a new analytical approach toward the general solution of Ordinary Linear Differential Equations (OLDEs) of order two. The method involves a transformation based on an integral function in an exponential form which leads to the general solution of a given differential equation. The authors discussed a special case of second-order OLDEs to develop the formulae and solution procedure and different problems have been solved to explain the solution method. They extended the idea to solve the general form of second-order OLDEs.

Recently, Beccar-Varela et al. [15] developed two analytic methods for solving higher-order ODEs without numerical solutions: the self-adjoint type formulation and the integrating-factor approach. They proposed a Riccati equation for a variable b to solve a second-order Ordinary Differential Equation (ODE), using its particular solution to find the integrating factor. This type of integrating factor technique was able to obtain particular solution for the second-order ODEs. However, they introduced the second-order integrating factor equation $u'' - u'P + u(Q - P') = 0$ without solving it. They emphasized that the integrating factor for second-order ODEs can still be obtained.

Pala, Y. and Kahya [16] in a study they conducted, they were able to present four different methods for solving second-order ODEs with variable coefficients analytically. The solution of the first method which is a special case is presented explicitly or integrally in the first method. The second approach involves solving similar two adjoined second-order homogeneous ordinary differential equations in order to solve the analytic solution of the general second-order ODEs. If the analytical solutions for both adjoined equations can be determined, the analytical solution for the original second-order ODE equation can always be obtained. In the third method, they utilized what is called a Riccati equation to transform the second-order ODE into a Riccati equation. Thus, the solution could be found using a recently developed method by Pala, Y. and Ertas [17]. In method four, they proposed a new transformation based on an integral function in an exponential form which leads to the Riccati equation and the solution once again relies on solving the Riccati equation. The results showed that the first method is limited to when $Q(x) = p'(x)$. The second method requires the solutions of the two adjoined equations. As long as the analytical solution of the two adjoined equations can be obtained, the analytical solution of the second-order ODEs can be found. The third method is a method of transforming second-order ODE into a Riccati equation and the fourth method is a newly modified method from the third method that involves a transformation based on an integral function in an exponential form and the solution of the fourth method again depends on the solution of the Riccati equation. However, if one of the two adjoined equations is presented here, it reads as follows; $N'' - PN' + (Q - P')N = 0$. So, the authors pointed out that the analytical solutions of two adjoined equations are still not known. Moreover, the adjoined equation appears identical to the one presented earlier, which is $u'' - u'P + u(Q - P') = 0$. Both equations are exactly the same despite being derived by two different independent researchers.

Therefore, in light of these two prior studies [15, 16], what they have in common is that unsolved second-order integrating factor function equation which is very important for finding the analytic solution for second-order Ordinary Linear Differential Equation with variable coefficients. Thus, the present study is motivated to solve this second-order integrating factor function equation using a new analytic method constructed from the concept of exactness of higher-order ODEs.

The present work motivation entails the following novel aspects: first, constructing a new analytic method from the concept of higher-order ODEs exactness. Secondly, deriving a formula for writing the general solution of second-order linear non-homogeneous ODEs of the form Equation (1) without knowing or finding one of the associated solutions. Thirdly, deriving the general integrating factor function formula for second-order ODEs. Fourthly, the same new analytic method is applied to a general third-order linear ODE to derive its integrating factor function formula. Finally, a comparison analysis between the new analytic method and the integral transformations was done via examples to validate the findings of the new analytic method.

2. Constructing the New Analytical Method

2.1. Using the Concept of Exactness of Differential Equations

Higher-order ODEs involving functions of one variable $y(x)$ are said to be exact if they stem from the process of differentiating a lower-order equation. This means that for example, for a second-order linear ODE with variable coefficients to be exact, there must be a first-order linear

ODE whose derivative is the given second-order linear ODE. Likewise, for a third-order linear ODE to be exact, there must be a second-order linear ODE whose derivative is the given third-order linear ODE. And so on the concept extends to higher-order ODE and beyond. Mathematically this definition of higher-order linear ODEs exactness can be constructed as the following way. Given by a second-order linear ODE;

$$y'' + p(x)y' + Q(x)y = 0. \tag{3}$$

Multiplying Equation (3) by an integrating factor function gives

$$\mu(x)y'' + \mu(x)p(x)y' + \mu(x)Q(x)y = 0. \tag{4}$$

If Equation (4) is exact, then it can be written as a first-order linear ODE with an integrable form: $(\mu(x)y' + k(x)y)' = 0$. If the derivative of this first-order linear ODE gives Equation (4), then Equation (4) is exact. Provided that the integrating-factor function $\mu(x)$ and k(x) are hopefully to be determined. The same way can be done to third-order linear ODE

$$y''' + P(x)y'' + Q(x)y' + R(x)y = 0. \tag{5}$$

To make this Equation (5) exact, multiply an integrating-factor function gives

$$\mu(x)y''' + \mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0. \tag{6}$$

If this Equation (6) is exact, then it can be written as a second-order linear ODE with an integrable form: $(\mu(x)y'' + W(x)y' + V(x)y)' = 0$. If the derivative of this second-order linear ODE gives Equation (6), then Equation (6) is exact. Provided that the integrating-factor function $\mu(x)$, W(x) and v(x) are hopefully determined. This concludes the construction new analytic method; the method just re-writes any linear ODEs into its lower order equation with integrable form. In a similar way as has been demonstrated above.

Theorem 2.1. Consider the second-order linear non-homogeneous ODEs of the form Equation (1) in an interval I. provided that $p(x)$ and $Q(x)$ are non-zero differentiable functions in I. Then, the general solution of the ODEs of the form Equation (1) in I is given as

$$y(x) = e^{-\int \frac{k(x)}{\mu(x)}dx} \cdot \int \left\{ e^{\int \frac{k(x)}{\mu(x)}dx} \left[\frac{1}{\mu(x)} \int \mu(x)R(x)dx + \frac{c_1}{\mu(x)} \right] \right\} dx + c_2 e^{-\int \frac{k(x)}{\mu(x)}dx} \tag{7}$$

where $k(x) = \mu(x)p(x) - \mu'(x)$. And the $\mu(x)$ is the integrating-factor function and c_1 and c_2 are arbitrary real constants.

Proof of Theorem 2.1 first, let us give the general form of a second-order linear nonhomogeneous ODE with variable coefficients.

$$y'' + p(x)y' + Q(x)y = R(x). \tag{8}$$

To make this equation exact, multiply it by an integrating-factor function gives

$$\mu(x)y'' + \mu(x)p(x)y' + \mu(x)Q(x)y = \mu(x)R(x). \tag{9}$$

If Equation (9) is exact, then it can be written as a first-order linear ODE with an integrable form as

$$(\mu(x)y' + k(x)y)' = \mu(x)R(x). \tag{10}$$

Integrating both sides gives

$$\mu(x)y' + k(x)y = \int \mu(x)R(x)dx + c_1, \tag{11}$$

$$y' + \frac{k(x)}{\mu(x)}y = \frac{1}{\mu(x)} \int \mu(x)R(x)dx + \frac{c_1}{\mu(x)}. \tag{12}$$

This is a first-order linear ODE with this integrating-factor function $IF = e^{\int \frac{k(x)}{\mu(x)}dx}$. Then,

$$\left(e^{\int \frac{k(x)}{\mu(x)}dx} y\right)' = e^{\int \frac{k(x)}{\mu(x)}dx} \left[\frac{1}{\mu(x)} \int \mu(x)R(x)dx + \frac{c_1}{\mu(x)} \right]. \tag{13}$$

Integrating both sides gives

$$e^{\int \frac{k(x)}{\mu(x)}dx} y = \int \left[e^{\int \frac{k(x)}{\mu(x)}dx} \left[\frac{1}{\mu(x)} \int \mu(x)R(x)dx + \frac{c_1}{\mu(x)} \right] \right] dx + c_2, \tag{14}$$

$$y(x) = e^{-\int \frac{k(x)}{\mu(x)}dx} \cdot \int \left\{ \left[e^{\int \frac{k(x)}{\mu(x)}dx} \left[\frac{1}{\mu(x)} \int \mu(x)R(x)dx + \frac{c_1}{\mu(x)} \right] \right] \right\} dx + c_2 e^{-\int \frac{k(x)}{\mu(x)}dx}. \tag{15}$$

Where $k(x) = \mu(x)p(x) - \mu'(x)$. How $k(x)$ equals this value can be observed in the next section where equation (19) is equated with equation (17). Additionally, the general integrating factor function formula for second-order linear OLDES will be derived. This concludes the proof!

2.2. Derivation of an Integrating-factor Function for Non-homogeneous Second Order ODEs with Variable Coefficients Using the New Analytic Method

The whole idea of this new analytic method is to reduce the order of linear ODEs to its lowest order linear ODEs with variable coefficients by an integrating factor function $\mu(x)$ which enables us to make the inexact (for example) second-order linear ODEs to exact ODEs to a form of integrable first-order ODEs. The integrating factor function formula for second-order linear ODEs with variable coefficients has been derived using a new analytic method. Consider the general form of non-homogeneous second-order linear DE.

$$y'' + p(x)y' + Q(x)y = R(x). \tag{16}$$

The coefficients $p(x)$, $Q(x)$ are some real-valued functions of the independent variable x in I. Multiply Equation (16) by an integrating-factor $'\mu(x)'$ gives,

$$\mu(x)y'' + \mu(x)p(x)y' + \mu(x)Q(x)y = \mu(x)R(x). \tag{17}$$

If Equation (17) is exact, the order of the equation can be reduced to first-order ODE of an integrable form as follows:

$$(\mu(x)y' + k(x)y)' = \mu(x)R(x). \tag{18}$$

Where $\mu(x)$ and $k(x)$ are functions to be determined. Differentiating LHS of Equation (18) yields

$$\mu(x)y'' + \mu'(x)y' + k(x)y' + k'(x)y = \mu(x)R(x). \tag{19}$$

Combining coefficients of y' and then comparing coefficients with Equation (17) gives the following equation in terms of an integrating-factor function alone

$$\mu''(x) - p(x)\mu'(x) + (Q(x) - p'(x))\,\mu(x) = 0. \tag{20}$$

This is the same equation that was left unsolved in these two researches [15,16]. The level of difficulty of finding the integrating factor in Equation (20) which is a second-order ODE, is the same as that of Equation (16). Now, to solve the integrating factor $'\mu(x)'$ in Equation (20), let's divide $\mu(x)$ on both sides of the equation provided that the $\mu(x) \neq 0$.

$$\frac{\mu''(x)}{\mu(x)} - p(x)\frac{\mu'(x)}{\mu(x)} = p'(x) - Q(x). \tag{21}$$

Let

$$W(x) = \frac{\mu'(x)}{\mu(x)}. \tag{22}$$

So that $\mu'(x) = W(x)\mu(x)$. Differentiating both sides gives

$$\mu''(x) = W(x)\mu'(x) + W'(x)\mu(x). \tag{23}$$

Putting Equation (23) in Equation (21) by substituting $\mu''(x)$ to its equivalent quantity gives

$$\begin{aligned}\left(W(x) - \frac{p(x)}{2}\right)^2 &= p'(x) - Q(x) - W'(x) \\ &\quad + \left(\frac{p(x)}{2}\right)^2.\end{aligned} \tag{24}$$

Let

$$\tau(x) = W(x) - \frac{p(x)}{2}. \tag{25}$$

So that,

$$W(x) = \tau(x) + \frac{p(x)}{2}. \tag{26}$$

Differentiating both sides gives

$$W'(x) = \tau'(x) + \frac{p'(x)}{2}. \tag{27}$$

Putting equations (27) and (25) in Equation (80) gives

$$\tau^2(x) + \tau'(x) = \underbrace{\frac{1}{2}p'(x) + \frac{1}{4}p^2(x) - Q(x),}_{f(x)}$$

$$\tau^2(x) + \tau'(x) = f(x). \tag{28}$$

If a solution to Equation (28) is known, then the solution of the integrating factor in Equation (22) is obtained. Now, from Equation (22) an explicit formula of the integrating-factor function $\mu(x)$ can be obtained in terms of $\tau(x)$ and $p(x)$. From Equation (22) gives

$$\frac{\mu'(x)}{\mu(x)} = W(x). \tag{29}$$

And also from Equation (26) states that

$$W(x) = \tau(x) + \frac{p(x)}{2}. \tag{30}$$

So, by re-writing the Equation (29) while changing $W(x)$ to its equivalent form in Equation (30) gives

$$\mu(x) = e^{\int\left(\tau(x) + \frac{p(x)}{2}\right)dx}. \tag{31}$$

In this Equation (31), a very important function called an integrating-factor function $''\mu(x)''$ for second-order ODEs with variable coefficients has been derived. As it can be seen the function $\mu(x)$ depends on another function called $\tau(x)$. the solution of $\tau(x)$ can be found from this equation $\tau^2(x) + \tau'(x) = \underbrace{\frac{1}{2}p'(x) + \frac{1}{4}p^2(x) - Q(x)}_{f(x)}$. which of course tells us that the $\tau(x)$ also depends on the outcome of the nature of $f(x)$. Those possible outcomes of f(x) found during this study are presented as the following cases:

Case 1: if $f(x) = \pm\beta$, where β is a $\mathbb{R} - \{0\}$. Then the expected solution for $\tau(x)$ and $\mu(x)$ will be obtained respectively in the following way:

For $f(x) = +\beta$

$$\tau^2(x) + \tau'(x) = \beta,$$
$$\tau'(x) = \beta - \tau^2(x).$$

Dividing both sides by $(\beta - \tau^2(x))$ and then integrating gives;

$$\tau(x) = \frac{\sqrt{\beta}\left(e^{2\sqrt{\beta}x} - 1\right)}{\left(e^{2\sqrt{\beta}x} + 1\right)}. \tag{32}$$

The solution for the integrating factor function $\mu(x)$ will be;

$$\mu(x) = \frac{1 + e^{2\sqrt{\beta}x}}{e^{2\sqrt{\beta}x}} e^{\int \frac{p(x)}{2} dx} \tag{33}$$

For $f(x) = -\beta$

$$\tau^2(x) + \tau'(x) = -\beta,$$
$$\tau'(x) = -\left(\beta + \tau^2(x)\right).$$

Dividing both sides by $(\beta + \tau^2(x))$ and then integrating gives;

$$\tau(x) = -\sqrt{\beta}\tan(\sqrt{\beta}x). \tag{34}$$

The solution for the integrating-factor function $\mu(x)$ will therefore be

$$\mu(x) = \cos(\sqrt{\beta}x)e^{\int \frac{p(x)}{2} dx}. \tag{35}$$

Case 2: if $f(x) = 0$, the expected solution for $\tau(x)$ and $\mu(x)$ will be obtained respectively in the following way

$$\tau^2(x) + \tau'(x) = 0,$$

Dividing both sides by $\tau^2(x)$ and then integrating both sides gives;

$$\tau(x) = \frac{1}{x}. \tag{36}$$

The solution for the integrating-factor function $\mu(x)$ will be

$$\mu(x) = xe^{\int \frac{p(x)}{2}) dx}. \tag{37}$$

The advantage of these two cases (Case 1 and Case 2) is that they can also be applied to second-order linear ODEs with constant coefficients to determine their integrating factor function without using the standard way of solving them which is generating their characteristic equation by letting $y = e^{mx}$ where m is constant.

Case3: If $f(x) = \frac{n}{x^2}$ or $f(x) = \frac{n}{(x \pm k)^2}$, where n and k are $\mathbb{R} - \{0\}$, the expected solution for $\tau(x)$ and $\mu(x)$ will be obtained respectively in the following ways. For the case $f(x) = \frac{n}{x^2}$,

$$\tau^2(x) + \tau'(x) = \frac{n}{x^2}. \tag{38}$$

By assuming that the solution of $\tau(x)$ in Equation (38) as $\frac{v}{x}$, where v is $\mathbb{R} - \{0\}$. Then one can verify that the general solution of the differential equation in Equation (38) of $\tau(x)$ in terms of n will be

$$\tau(x) = \frac{(1 \pm \sqrt{1 + 4n})}{2x}. \tag{39}$$

The solution for the integrating-factor function $\mu(x)$ will therefore be

$$\mu(x) = e^{\int \left(\frac{(1 \pm \sqrt{1+4n})}{2x} + \frac{p(x)}{2}\right) dx}. \tag{40}$$

For the case $f(x) = \frac{n}{(x \pm k)^2}$

$$\tau^2(x) + \tau'(x) = \frac{n}{(x \pm k)^2}. \tag{41}$$

By letting the solution $\tau(x)$ in Equation (41) as $\frac{m}{(x \pm k)}$, where again m is $\mathbb{R} - \{0\}$. Then one can verify that the general solution of $\tau(x)$ in terms of n will be

$$\tau(x) = \frac{(1 \pm \sqrt{1 + 4n})}{2(x \pm k)}. \tag{42}$$

The solution for the integration factor function $\mu(x)$ will therefore be

$$\mu(x) = e^{\int \left(\frac{(1 \pm \sqrt{1+4n})}{2(x \pm k)} + \frac{p(x)}{2}\right) dx}. \tag{43}$$

Case 4: If $f(x) = \frac{n}{x^2(x-v)^2}$ or $f(x) = \frac{n}{x^2(x+v)^2}$ where n and v are $\mathbb{R} - \{0\}$, the expected solution for $\tau(x)$ and $\mu(x)$ will be obtained in the following way. By writing as a compact form for $f(x)$, that if $f(x) = \frac{n}{x^2(x \pm v)^2}$ then,

$$\tau^2(x) + \tau'(x) = \frac{n}{x^2(x \pm v)^2}. \tag{44}$$

Proposing a solution for $\tau(x)$ in Equation (44) to be $\tau(x) = \frac{kx+l}{x(x \pm v)}$, where k and l are unknown constants and are $\mathbb{R} - \{0\}$. Then one can verify that the general solution of the differential equation in Equation (44) of $\tau(x)$ in terms of n and v will be

$$\tau(x) = \frac{x + \left(\frac{\pm v \pm \sqrt{v^2 + 4n}}{2}\right)}{x(x \pm v)}. \tag{45}$$

The solution for the integrating-factor function $\mu(x)$ will therefore be

$$\mu(x) = e^{\int \left(\frac{x + \left(\frac{\pm v \pm \sqrt{v^2+4n}}{2}\right)}{x(x \pm v)} + \frac{1}{2}p(x)\right) dx}. \tag{46}$$

Case 5: If $f(x) = \frac{n}{x^4}$ where n is $\mathbb{R} - \{0\}$, the expected solution for $\tau(x)$ and $\mu(x)$ will be obtained respectively in the following ways: For the case $f(x) = \frac{n}{x^4}$

$$\tau^2(x) + \tau'(x) = \frac{n}{x^4}. \tag{47}$$

By proposing a solution of $\tau(x)$ in Equation (47) which takes the following form as $\tau(x) = \frac{kx+l}{x^2}$, where k and l are unknown constants and they are $\mathbb{R} - \{0\}$. Then one can verify that the general solution of the differential Equation in Equation (47) of $\tau(x)$ in terms of n will be

$$\tau(x) = \frac{x \pm \sqrt{n}}{x^2}. \tag{48}$$

The solution for the integrating-factor function $\mu(x)$ will therefore be;

$$\mu(x) = xe^{\left(\frac{\pm\sqrt{n}}{x} + \int \frac{p(x)}{2} dx\right)}. \tag{49}$$

2.3. Deriving an Integrating-factor Function for Third-order ODEs with Variable Coefficients Using the New Analytic Method

The methodology of using the integrating factor approach, which involves reducing the order of an equation to a lower-order equation, has been applied to third-order ODEs as well. Specifically, for a third-order linear ODE to be exact, there must exist a second-order linear ODE whose derivative corresponds to the given third-order ODE. The general form of third-order linear non-homogeneous ODEs with variable coefficients is given as

$$y''' + p(x)y'' + Q(x)y' + R(x)y = M(x). \tag{50}$$

Let's multiply an integrating factor throughout the Equation (50).

$$\mu(x)y''' + \mu(x)p(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = \mu(x)M(x). \tag{51}$$

If Equation (51) is exact after multiplying the integrating-factor function, then the left-hand side of the equation can be written as the result of differentiating its lower-order equation, in this case, second-order ODE of an integrable form.

$$(\mu(x)y'' + W(x)y' + V(x)y)' = \mu(x)M(x). \tag{52}$$

Where $\mu(x)$, $W(x)$ and $V(x)$ are functions to be determined. Let's differentiate the left-hand side of Equation (52), gives

$$\mu(x)y''' + (\mu'(x) + W(x))\, y'' + (W'(x) + V(x))\, y' + V'(x)y = \mu(x)M(x). \tag{53}$$

And by comparing coefficients of Equation (53) with Equation (51), gives

$$\mu'(x) + W(x) = \mu(x)p(x), \tag{54}$$

$$W'(x) + V(x) = \mu(x)Q(x), \tag{55}$$

$$V'(x) = \mu(x)R(x). \tag{56}$$

Finding $W'(x)$ from Equation (54) gives the following.

$$W'(x) = \mu(x)p'(x) + \mu'(x)p(x) - \mu''(x). \tag{57}$$

Also from Equation (55), making $W'(x)$ as a subject of formula gives

$$W'(x) = \mu(x)Q(x) - V(x). \tag{58}$$

Equate equations (57) and (58) and put $V(x) = \int \mu(x)R(x)dx$.

$$\mu(x)p'(x) + \mu'(x)p(x) - \mu''(x) = \mu(x)Q(x) - \int \mu(x)R(x)dx, \tag{59}$$

$$\int \mu(x)R(x)dx = \mu''(x) - \mu'(x)p(x) + (Q(x) - p'(x))\, \mu(x). \tag{60}$$

Now, let's differentiate both sides of the Equation (60) to eliminate the integral

$$\left(\int \mu(x)R(x)dx\right)' = (\mu''(x) - \mu'(x)p(x) + (Q(x) - p'(x))\, \mu(x))', \tag{61}$$

$$\mu(x)R(x) = \mu(x)''' - \mu'(x)p'(x) - \mu''(x)p(x) + Q(x)\mu'(x) + Q'(x)\mu(x) - p'(x)\mu'(x) - p''(x)\mu(x), \tag{62}$$

$$\mu(x)''' - \mu''(x)p(x) + (Q(x) - 2p'(x))\, \mu'(x) + (Q'(x) - p''(x) - R(x))\, \mu(x) = 0. \tag{63}$$

If a solution to the associated integrating factor Equation (63) is known, a particular solution to Equation (50) can be found. As observed, solving the associated integrating factor equation is quite challenging. However, imposing a condition on the last term can simplify the problem, allowing us to resolve the remaining three terms and solve the integrating factor equation. The condition is given by $R(x) = Q'(x) - p''(x)$. This condition allows the general form of the third-order ODE in Equation (50) to be expressed as follows: $y''' + p(x)y''(x) + Q(x)y'(x) + (Q'(x) - p''(x))\,y(x) = M(x)$. By continuing our derivation and applying this condition, the following equation for the integrating factor was obtained.

$$\mu'''(x) - p(x)\mu''(x) + (Q(x) - 2p'(x))\,\mu'(x) = 0. \tag{64}$$

Let

$$T(x) = \mu'(x). \tag{65}$$

then,

$$T''(x) - p(x)T'(x) + (Q(x) - 2p'(x))\,T(x) = 0. \tag{66}$$

Dividing both sides by $T(x)$, provided that $T(x) \neq 0$ yields

$$\frac{T''(x)}{T(x)} - p(x)\frac{T'(x)}{T(x)} = 2p'(x) - Q(x). \tag{67}$$

Let

$$Z(x) = \frac{T'(x)}{T(x)}. \tag{68}$$

By cross multiplication gives us, $T'(x) = Z(x)T(x)$. Differentiating both sides gives

$$T''(x) = Z(x)T'(x) + Z'(x)T(x). \tag{69}$$

Putting this Equation (69) in Equation (67) gives

$$\begin{aligned}\left(Z(x) - \frac{p(x)}{2}\right)^2 &= 2p'(x) - Q(x) - Z'(x)\\ &\quad + \left(\frac{p(x)}{2}\right)^2.\end{aligned} \tag{70}$$

Let

$$\tau(x) = Z(x) - \frac{p(x)}{2}. \tag{71}$$

Differentiating both sides gives

$$\tau'(x) = Z'(x) - \frac{p'(x)}{2}. \tag{72}$$

Going back to the equation (70) gives

$$\tau^2(x) = 2p'(x) - Q(x) - \left(\tau'(x) + \frac{p'(x)}{2}\right) + \frac{p^2(x)}{4},$$

$$\tau^2(x) + \tau'(x) = \underbrace{\frac{3}{2}p'(x) + \frac{1}{4}p^2(x) - Q(x)}_{g(x)},$$

$$\tau^2(x) + \tau'(x) = g(x). \tag{73}$$

If the solution to Equation (73) is known, then the solution of the integrating factor in Equation (65) is obtained. Now, from Equation (65) an explicit formula of $\mu(x)$ can be obtained in terms of $\tau(x)$ and $p(x)$. It is known from Equation (65) that

$$T(x) = \mu'(x). \tag{74}$$

Then from Equation (68) becomes

$$Z(x) = \frac{T'(x)}{T(x)} = \frac{\mu''(x)}{\mu'(x)}. \tag{75}$$

But, from Equation (71), $Z(x) = \tau(x) + \frac{p(x)}{2}$. Using this value and placing it in Equation (75) gives

$$\frac{\mu''(x)}{\mu'(x)} = \tau(x) + \frac{p(x)}{2}. \tag{76}$$

Integrating twice gives;

$$\mu(x) = \int \left(e^{\int\left(\tau(x) + \frac{p(x)}{2}\right)dx}\right) dx. \tag{77}$$

This is an integrating factor function for the following special non-homogeneous third-order ODE of the form: $y''' + p(x)y''(x) + Q(x)y'(x) + (Q'(x) - p''(x))\,y(x) = M(x)$. The objective was to derive the integrating factor function for general third-order ODEs with variable coefficients. However, the derivation led to Equation (63), which is challenging to solve. Nonetheless, by applying a condition to Equation (63), Equation (77) was derived as the integrating factor function for the special type of third-order ODE mentioned earlier, resulting in a complete solution.

3. Comparative Analysis Between the New Analytical Method and the Integral Transform

This section presents a comparative study through two experiments, each solved using integral transforms, specifically, the Laplace transform, along with a new analytic method that employs the integrating factor function approach. The first experiment addresses the Whittaker second-order linear ODE, while the second experiment addresses the modified Bessel equation of order $1/2$. Laplace equations used in these two experiments are provided in [12,18].

3.1. The First Experiment of Comparing the Two Methods

Experiment 1: Solution by the New Analytical Method. The new analytic method was applied for solving the well-known Whittaker second-order differential equation of the form; $y'' + \left(\frac{-1}{4} + \frac{k}{x} + \frac{(1/4)-m^2}{x^2}\right)y = 0$, where k and m are constant parameters. The method is able to solve this equation by finding the integrating-factor function for all values of m and k held by this relation: $k = \frac{2m-1}{2}$. Where m is an integer. The general form of the Whittaker Equation is;

$$y'' + \left(\frac{-1}{4} + \frac{k}{x} + \frac{(1/4) - m^2}{x^2} \right) y = 0. \tag{78}$$

Solution

$$y'' + \left(\frac{-1}{4} + \frac{k}{x} + \frac{(1/4) - m^2}{x^2} \right) y = 0. \tag{79}$$

The general solution of this Whittaker second-order ODE for all values of m and k was derived such that when $k = \frac{2m-1}{2}$. This relation between k and m is not for our choice selection but it is based on derivation of the integrating factor of the above Whittaker equation. The general solution when considering the relation between m and k mentioned above is given as;

$$\begin{aligned} y(x) = c_1 x^{\left(\frac{1-2m}{2}\right)} e^{\frac{-x}{2}} \cdot \int x^{(2m-1)} e^{-x} dx \\ + c_2 x^{\left(\frac{1-2m}{2}\right)} e^{\frac{x}{2}} \end{aligned} \tag{80}$$

To find the integrating-factor function for the above Whittaker Equation (79), we must first find the nonlinear first-order equation in the integrating-factor formula which is $\tau(x)$. Knowing that $p(x) = 0$ and $Q(x) = \left(\frac{-1}{4} + \frac{k}{x} + \frac{(1/4)-m^2}{x^2} \right)$, then $\tau(x)$ has this relation equation: $\tau'(x) + \tau^2(x) = \frac{1}{2}p'(x) + \frac{1}{4}p^2(x) - Q(x)$. Then $\tau'(x) + \tau^2(x) = \frac{(1/4)x^2 - kx + \left(m^2 - 1/4\right)}{x^2}$. To find the solution of $\tau(x)$, an assumption was taken that such solution takes the form as $\tau(x) = \frac{sx+l}{x}$ for s and I are $\mathbb{R}$. Then, $\tau'(x) + \tau^2(x) = \frac{s^2x^2 + 2slx + l^2 - l}{x^2}$. Comparing coefficients gives; $\left(s^2 = 1/4\right), (2sl = -k), \left(l^2 - l = \left(m^2 - 1/4\right)\right)$. Taking the positive value of $s = 1/2$ and taking the value of l to be $l = \frac{1-2m}{2}$. To keep the consistency of the second equation, $2sl = -k$, the following relation between m and k such that $k = \frac{2m-1}{2}$ was derived. In this experiment 1, let us consider positive integer values for m, such that $m = 1, 2, 3, ...$, corresponding to k's values become: $k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ and so on. starting with the first value for $m = 1$ we have,

$$y'' + \left(\frac{-1}{4} + \frac{1/2}{x} - \frac{3/4}{x^2} \right) y = 0. \tag{81}$$

Integrating-factor function, $\mu(x)$ and $k(x)$ are given by;

$$\mu(x) = e^{\int \left(\tau(x) + \frac{1}{2}p(x)\right) dx} = x^{\frac{-1}{2}} e^{\frac{1}{2}x}, \tag{82}$$

$$k(x) = \mu(x)p(x) - \mu'(x) = \left(x^{\frac{-3}{2}} - x^{\frac{-1}{2}} \right) \frac{1}{2} e^{\frac{1}{2}x}. \tag{83}$$

Putting values of these equations (82), (83) to the general solution formula in Theorem 1 gives the solution of the Whittaker equation;

$$y(x) = c_1 x^{\frac{-1}{2}} e^{\frac{-1}{2}x} (x+1) + c_2 x^{\frac{-1}{2}} e^{\frac{1}{2}x}. \tag{84}$$

Or just simply put $m = 1$ in Equation (80) gives the same solution. This is the analytic solution of the Whittaker equation when $m = 1$ and $k = 1/2$. To prove the validity of this solution, the Laplace transform is used.

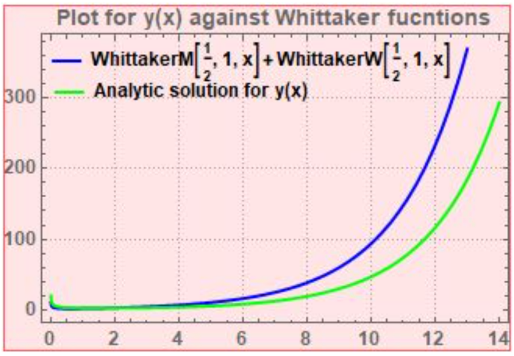

Figure 1. *Solution graph for $m = 1$, $k = 1/2$ for the above Whittaker second-order equation.*

In Figure 1, the graph of the solution $y(x) = c_1 x^{\frac{-1}{2}} e^{\frac{-1}{2}x}(x+1) + c_2 x^{\frac{-1}{2}} e^{\frac{1}{2}x}$ was drawn in comparing with the solution of Whittaker functions for the Whittaker equation when $m = 1$ and $k = \frac{1}{2}$. Figure 1 displays the Whittaker functions, plotted in terms of the real part of the solution while letting $c_1 = c_2 = 1$. Wolfram Mathematica Software was used to plot the graph.

Experiment 1: Solution by the Laplace Transform.

$$y'' + \left(\frac{-1}{4} + \frac{1/2}{x} - \frac{3/4}{x^2} \right) y = 0. \tag{85}$$

Solution

$$\mathcal{L}\left(x^2 y''\right) - \frac{1}{4}\mathcal{L}\left(x^2 y\right) + \frac{1}{2}\mathcal{L}(xy) - \frac{3}{4}\mathcal{L}(y) = 0. \tag{86}$$

Applying the Laplace transform equations, the next equation is obtained;

$$Y''(s) + \frac{4s - \frac{1}{2}}{(s^2 - 1/4)} Y'(s) + \frac{5/4}{(s^2 - 1/4)} Y(s) = 0. \tag{87}$$

The solution of this second-order linear ODEs with variable coefficients in the Laplace domain can be solved analytically by applying the following general solution formula given in theorem 1.

$$Y(s) = e^{-\int \frac{k(s)}{\mu(s)} ds} \cdot \int \left\{ e^{\int \frac{k(s)}{\mu(s)} ds} \left[\frac{1}{\mu(s)} \int \mu(s) R(s) ds + \frac{c_1}{\mu(s)} \right] \right\} ds + c_2 e^{-\int \frac{k(s)}{\mu(s)} ds}. \tag{88}$$

Given that $R(s) = 0$ as the equation is homogeneous. Provided that $\mu(s)$ and $k(s)$ are given below respectively;

$$\mu(s) = e^{\int\left(\tau(s)+\frac{1}{2}p(s)\right)ds} = (1-2s)(1+2s)^{\frac{5}{2}}, \tag{89}$$

$$k(s) = \mu(s)p(s) - \mu'(s) = -(1+2s)^{\frac{5}{2}}. \tag{90}$$

Putting these two values into the general solution formula in Equation (88) gives the solution in the Laplace domain;

$$Y(s) = -c_1\frac{(s+1)}{3(1+2s)^{\frac{3}{2}}} + \frac{c_2}{(1-2s)^{\frac{1}{2}}}. \tag{91}$$

Taking Laplace inverse on both sides gives;

$$y(x) = c_1 x^{\frac{-1}{2}} e^{\frac{-x}{2}}(x+1) + c_2 x^{\frac{-1}{2}} e^{\frac{x}{2}}. \tag{92}$$

This is the same solution derived by the new analytic method.

3.2. The Second Experiment of Comparing the Two Methods

Experiment 2: Solution by the New Analytical Method.

This experiment 2 presents a modified Bessel equation of order $1/2$. The general form of the modified Bessel equation is:

$$x^2y'' + xy' - \left(x^2+v^2\right)y = 0. \tag{93}$$

Considering order $1/2$ for v in this experiment 2 , we get;

$$x^2y'' + xy' - \left(x^2+\frac{1}{4}\right)y = 0. \tag{94}$$

Solution

$$x^2y'' + xy' - \left(x^2+\frac{1}{4}\right)y = 0. \tag{95}$$

Writing in standard form gives;

$$y'' + \frac{1}{x}y' - \left(\frac{x^2+1/4}{x^2}\right)y = 0. \tag{96}$$

The solution of this second-order linear ODE can be solved analytically by applying the general solution formula given in theorem 1.

$$y(x) = e^{-\int\frac{k(x)}{\mu(x)}dx}\cdot\int\left\{e^{\int\frac{k(x)}{\mu(x)}dx}\left[\frac{1}{\mu(x)}\int\mu(x)R(x)dx + \frac{c_1}{\mu(x)}\right]\right\}dx + c_2e^{-\int\frac{k(x)}{\mu(x)}dx}. \tag{97}$$

Given that $R(x) = 0$ as the equation is homogeneous. Provided that $\mu(x)$ and $k(x)$ are given by;

$$\mu(x) = e^{\int\left(\tau(x)+\frac{1}{2}p(x)\right)dx} = x^{\frac{1}{2}}\left(e^x+e^{-x}\right), \tag{98}$$

$$k(x) = \mu(x)p(x) - \mu'(x) = \frac{1}{2}x^{\frac{-1}{2}}\left(e^x+e^{-x}\right) + x^{\frac{1}{2}}\left(e^{-x}-e^x\right). \tag{99}$$

Substituting these two values back to the general solution formula given in Equation (97) gives the solution of this modified Bessel equation;

$$y(x) = c_1x^{\frac{-1}{2}}e^{-x} + c_2x^{\frac{-1}{2}}e^x. \tag{100}$$

In figure 2, the graph of the solution $y(x) = c_1x^{\frac{-1}{2}}e^{-x} + c_2x^{\frac{-1}{2}}e^x$ was drawn in comparison with the solution of the Modified Bessel functions in which c_1 and c_2 were taken to be equal to 1.

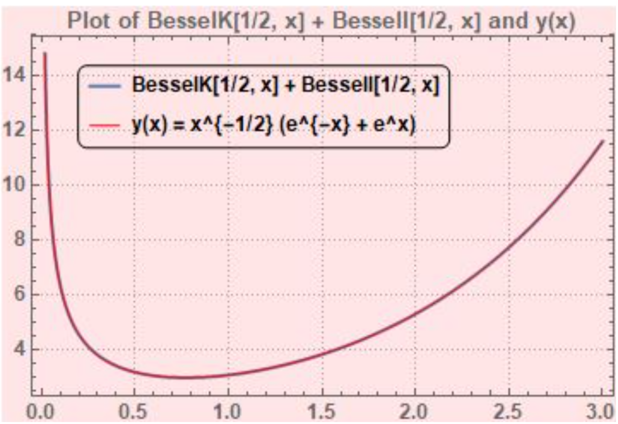

Figure 2. *Solution graph for Modified Bessel equation of order 1/2.*

Experiment 2: Solution by Laplace transform

$$x^2y'' + xy' - \left(x^2+\frac{1}{4}\right)y = 0. \tag{101}$$

Solution

$$x^2y'' + xy' - \left(x^2+\frac{1}{4}\right)y = 0. \tag{102}$$

Applying Laplace transform gives;

$$\mathcal{L}\left(x^2y''\right) + \mathcal{L}\left(xy'\right) - \mathcal{L}\left(x^2y\right) - \frac{1}{4}\mathcal{L}(y) = 0. \tag{103}$$

Applying Laplace equations gives the next equation;

$$Y''(s) + \frac{3s}{s^2-1}Y'(s) + \frac{3/4}{s^2-1}Y(s) = 0. \tag{104}$$

This transformed second-order linear ODE in the Laplace domain can be solved analytically by applying the general solution formula given in Theorem 1.

$$Y(s)=e^{-\int\frac{k(s)}{\mu(s)}ds}\cdot\int\left\{e^{\int\frac{k(s)}{\mu(s)}ds}\left[\frac{1}{\mu(s)}\int\mu(s)R(s)ds+\frac{c_1}{\mu(s)}\right]\right\}ds+c_2e^{-\int\frac{k(s)}{\mu(s)}ds}. \tag{105}$$

Given that $R(s)=0$ as the equation is homogeneous. Provided that $\mu(s)$ and $k(s)$ are given below respectively;

$$\mu(s)=e^{\int\left(\tau(s)+\frac{1}{2}p(s)\right)ds}=(1+s)(1-s)^{\frac{3}{2}}, \tag{106}$$

$$k(s)=\mu(s)p(s)-\mu'(s)=\frac{1}{2}(1-s)^{\frac{3}{2}}. \tag{107}$$

Putting these two values into the general solution formula in Equation (105) gives the solution in the Laplace domain;

$$Y(s)=\frac{c_1}{(1-s)^{\frac{1}{2}}}+\frac{c_2}{(1+s)^{\frac{1}{2}}}. \tag{108}$$

Taking Laplace inverse on both sides gives in terms of $y(x)$;

$$\mathcal{L}^{-1}[Y(s)]=c_1\mathcal{L}^{-1}\left(\frac{1}{\sqrt{1-s}}\right)+c_2\mathcal{L}^{-1}\left(\frac{1}{\sqrt{1+s}}\right), \tag{109}$$

$$y(x)=c_1x^{\frac{-1}{2}}e^{x}+c_2x^{\frac{-1}{2}}e^{-x}. \tag{110}$$

This is the exact same solution obtained by the new analytic method.

4. Conclusions

In this study, a novel analytical method for solving classes of second-order linear ODEs with variable coefficients was introduced. By utilizing the concept of higher-order ODE exactness, a new analytic method was constructed and derived the following key findings; firstly, an important integrating-factor function formula for second-order linear ODEs with variable coefficients was derived. Secondly, in Theorem 1, a general solution formula was provided for the second-order linear ODE expressed only in terms of the integrating factor function, $\mu(x)$, and another function called $k(x)$. cases were given for the non-linear first-order term function, $\tau(x)$, in the integrating factor function formula, as this term is governed by non-linear relation: $\tau'(x)+\tau^2(x)=\underbrace{\frac{1}{2}p'(x)+\frac{1}{4}p^2(x)-Q(x)}_{f(x)}$, five cases were constructed with their solutions for the possibilities that f(x) becomes constants or function of x so as to simplify the computation of the integrating-factor formula. A comparative analysis was performed to validate the new analytic method, demonstrated through two experiments. The first experiment showed that the well-known Whittaker second-order linear ODE can be solved for all values m and k such that m and k held by this relation: $k=\frac{2m-1}{2}$, when $m=[1,2,3,...]$. The second experiment solves the modified Bessel equation of order $1/2$. In both experiments 1 and 2, the results showed that the solutions of the new analytic method were consistent with those obtained from the Laplace transform. The following major conclusions can be drawn from the study:

1. Any second-order linear ODE with variable coefficients may be analytically solvable if the non-linear term $\tau(x)$ in the integrating factor function formula is determined. This function plays a critical role in the existence of an integrating factor function for the given second-order ODE.
2. For a general third-order linear ODE with variable coefficients to be reducible to a second-order linear ODE, a solution to the associated integrating factor Equation (63) must be known.

It would be of interest to Further pursue on generalizing the integrating factor function for solving nth higher order ODEs using this new analytic method. Also worthy of investigation is the non-linear term,$\tau(x)$, in the integerating-factor formula that possesses the nonlinearity property which adds further complexity in the integerating-factor function formula.

Remark

1. The following observations were noted about the plots of two experiments.
   (a) In experiment 1, the Whittaker functions, $c_1\text{WhittakerW}\left[\frac{1}{2},1,x\right]+c_2\text{WhittakerM}\left[\frac{1}{2},1,x\right]$ which represent the exact known solution to the Whittaker second-order ODE, were plotted alongside the analytical solution y(x) derived using the new analytical method. The analytical solution partially align with the Whittaker functions. This partial allignment arises because c_1 WhittakerW $\left[\frac{1}{2},1,x\right]=c_1e^{\frac{-x}{2}}\left[x^{\frac{-1}{2}}+x^{\frac{1}{2}}\right]$ and c_2 WhittakerM $\left[\frac{1}{2},1,x\right]=c_2e^{\frac{-x}{2}}\left[-2x^{\frac{-1}{2}}-2x^{\frac{1}{2}}\right]+c_2x^{\frac{-1}{2}}e^{\frac{x}{2}}$. While $y(x)=c_1x^{\frac{-1}{2}}e^{\frac{-x}{2}}(x+1)+c_2x^{\frac{-1}{2}}e^{\frac{x}{2}}$. Clearly, the first term of y(x) matches the structure of c_1WhittakerW $\left[\frac{1}{2},1,x\right]$ completely. However,

the second term of $y(x)$ only match with the second term of c_2 WhittakerM $\left[\frac{1}{2}, 1, x\right]$ and differs with the first term. This explains why the two curves don't coincide.

(b) In experiment 2, the Modified Bessel equation of order $1/2$ was solved, giving the solution as $y(x) = x^{\frac{-1}{2}}\left(e^x + e^{-x}\right)$, which is identical to the well-known expression for the modified Bessel functions of first and second kind, given as; BesselK $\left[\frac{1}{2}, x\right]$ + BesselI $\left[\frac{1}{2}, x\right]$ $= x^{\frac{-1}{2}}\left(e^x + e^{-x}\right)$. As a result, the graph of $y(x)$ naturally overlaps with the graph of BesselK $\left[\frac{1}{2}, x\right]$ + BesselI $\left[\frac{1}{2}, x\right]$ across the entire plotted domain. Both curves are identical and coincide because they represent the same analytical solution to the Modified Bessel equation of order $1/2$.

Symbols and Abbreviations

OLDEs	Ordinary Linear Differential Equations
ODEs	Ordinary Differential Equations
ODE	Ordinary Differential Equation
$\mathbb{R}$	The set of Real Numbers
$\mathcal{L}^{-1}$	Inverse Laplace transform
$\mu(x)$	Integrating factor function
$\int$	An intgeral
$\mathbb{R} - \{0\}$	The set of Real Numbers except zero
$\mathcal{L}\{f(t)\}$	Laplace transform of a given function f

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Duncan Gathungu Kioi: Supervision, Validation, Writing - review & editing

Kang'ethe Giterere: Supervision, Validation, Writing - review & editing

All authors have read and agreed to the publication of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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