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Research Article

An Effective Matrix Technique for the Numerical Solution of Second Order Differential Equations

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Abstract

In this paper, an effective technique for solving differential equations with initial conditions is presented. The method is based on the use of the Legendre matrix of derivatives defined on the close interval [-1,1]. Properties of the polynomial are outlined and further used to obtain the matrix of derivative which was used in transforming the differential equation into systems of linear and nonlinear algebraic equations. The systems of these algebraic equations were then solved using Gaussian elimination method to determine the unknown parameters required for approximating the solution of the differential equation. The advantage of this technique over other methods is that, it has less computational manipulations and complexities and also its availability for application on both linear and nonlinear second-order initial value problems is impressive. Other advantage of the algorithm is that high accurate approximate solutions are achieved by using a greater number of terms of the Legendre polynomial and once the operational matrix is obtained, it can be used to solve differential equations of higher order by introducing just a little manipulation on the operational matrix. Some existing sample problems from literature were solved and the results were compared to show the validity, simplicity and applicability of the proposed method. The results obtained validate the simplicity and applicability of the method and it also reveals that the method perform better than most existing methods.

Keywords

Legendre Polynomials, Matrix Calculus, Differential Equations

1. Introduction

The primary use of differential equations in general is to model motion, which is commonly called growth in economics. Specifically, a differential equation expresses the rate of change of the current state as a function of the current state.

In economics, differential calculus is used to compute marginal cost, marginal revenue, maxima and minima elasticity, partial elasticity and also enabling economists to predict maximum profit (or) minimum loss in a specific condition; one can also think of a change in general price level with respect to time as inflation. Second-order derivative with respect to time shows the rate of change of inflation, how inflation changes over time. Similarly, differentiating

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capital with respect to time shows investment.

Most ordinary differential equations arising in real-life applications cannot be solved exactly. These ordinary differential equations can be analyzed qualitatively. However, qualitative analysis may not be able to give accurate answers. A numerical method can be used to get an accurate approximate solution to a differential equation.

Motivated by these advantages, we will use Legendre operational matrix of derivatives through collocation method to approximate the solution of general second order differential equations with initial conditions.

The general second order differential equation is given as follows:

$$\frac{d^2y}{dx^2} = a(x)\frac{dy}{dx} + b(x)y + f(x) \tag{1}$$

where $a(x), b(x)$and $f(x)$ are functions of x. Conventionally, (1) can be solved using different methods such as the method of (educated) guess, the method of variation of parameters and it can also be solved by reducing it to a system of first order differential equations, and then any method of solving first order differential equations can then be applied to solve it. The setbacks of this technique were reported in ([2, 5, 12]). The method of collocation and interpolation of the power series and other polynomial basis functions were used to generate approximate solution and these techniques were reported by many scholars among them are ([1, 4, 6, 9, 11, 14]) to mention a few. Their approaches and techniques generated implicit continuous linear multistep methods which require separate predictors for implementation; this method is called the predictor-corrector method. There are major setbacks of these methods, numerical techniques such the block linear multistep methods lately introduced by researchers such as ([2-5, 7, 8, 10 13]) have shown allot of advantages over the predictor-corrector method. However, the advantages are compensated by tedious computational work and the use of more advance software to enable it handle the work.

In this paper, a collocation technique based on the Legendre operational matrix of derivatives for second order differential equations is proposed. The advantages of this technique over other methods is that it has less computational manipulations and complexities because it only involves operational matrix of derivatives and its transpose and thus, reduces the time involve in the derivation of the schemes, analysis and implementation as is in the case with linear multistep methods.

1.1. Legendre Polynomials

The Legendre polynomials exhibit simple and convenient form for calculation, compared with other orthogonal polynomials (Chebyshev polynomials, shifted Legendre polynomials...). They are well known family of orthogonal polynomials on the interval $[-1,1]$. They are solutions to the popular Legendre differential equation given as follows;

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{2}$$

They are widely used because of their smooth properties in the approximation of functions [15]. Equation (2) can be solved by series solution method (See [14]). The first few Legendre polynomials using the Rodriquez formula are:

$$l_0(x) = 1, l_1(x) = x, l_2(x) = \frac{1}{2}(3x^2-1), l_3(x) = \frac{1}{2}(5x^3-3x), l_4(x) = \frac{1}{8}(35x^4-30x^2+3)\ l_5(x) = \frac{1}{8}(63x^5-70x^3+15x), l_6(x) = \frac{1}{16}(231x^6-315x^4+105x^2-5),\ldots$$

The recurrence relation for Legendre polynomial is given by

$$l_0(x) = 1, l_1(x) = x$$

$$l_{k+1}(x) = \frac{(2k+1)}{(k+1)}x l_k(x) - \frac{k}{(k+1)}l_{k-1}(x), k = 1,2,3,\ldots$$

The Rodrigues formula for the Legendre polynomial is

$$l_k(x) = \frac{1}{2^k k!} \times \frac{d^k}{dx^k}(x^2-1)^k, k = ,0,1,2,\ldots$$

Properties of Legendre polynomials

The following properties hold for Legendre polynomials are

$$l_k(-x) = (-1)^n l_k(x)$$

i. $l_k(1) = 1$
ii. $l_k(-1) = (-1)^n$
iii. $l_k(0) = 0, k$ odd
iv. $l'_k(0) = 0, k$ even

Thus, the condition for orthogonality is:

$$\int_0^1 l_k(x) l_i(x)\, dx = \begin{cases} \frac{1}{2k+1}, if\ k = i \\ 0, if\ k \neq i \end{cases}$$

This implies that any function $y(x) \in [-1,1]$ can be approximated by Legendre polynomials as follows:

$$y(x) \cong \sum_{k=0}^{\infty} c_k l_k(x) \tag{3}$$

where

$$c_k =< y(x), l_k(x) >= (2k+1)\int_0^1 y(x), l_k(x)dx$$

1.2. Preliminaries

We introduce the Legendre vector $L(x)$ in the form $L(x) = [l_0(x), l_1(x), \ldots, l_n(x)]$, then the derivative of the vector $L(x)$, can be expressed in matrix form by

$$(L(x)')^T = \mathcal{M}\left(L(x)\right)^T = L(x)(\mathcal{M})^T \quad (4)$$

$$\text{Where } (L(x)')^T = \begin{pmatrix} l_0{}'(x) \\ l_1{}'(x) \\ l_2{}'(x) \\ \vdots \\ l'(x) \end{pmatrix}, \mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 6 & 0 & \cdots & 0 & (2n-1) & 0 \end{pmatrix} \text{ and } \left(L(x)\right)^T = \begin{pmatrix} l_0(x) \\ l_1(x) \\ l_2(x) \\ \vdots \\ l_n(x) \end{pmatrix}$$

Where the matrix $\mathcal{M}$ is an $(N+1) \times (N+1)$ matrix calculus, similarly, the k^{th} derivative of $L(x)$ can be obtained from the following relation;

$$\left.\begin{aligned} (L(x)')^T &= \mathcal{M}\left(L(x)\right)^T \rightarrow L(x)^1 = L(x)\mathcal{M}^T \\ (L(x)^2)^T &= L(x)^1\mathcal{M}^T = L(x)\mathcal{M}^T\mathcal{M}^T = L(x)(\mathcal{M}^T)^2 \\ (L(x)^3)^T &= L(x)^2\mathcal{M}^T = L(x)(\mathcal{M}^T)^2\mathcal{M}^T = L(x)(\mathcal{M}^T)^3 \\ &\vdots \\ (L(x)^k)^T &= L(x)(\mathcal{M}^T)^k \end{aligned}\right\} \quad (5)$$

In this paper, we shall use the collocation method based on Legendre matrix calculus to solve numerically the general second order differential equation.

2. Derivation of the Method

We now derive the algorithm for solving (1.1), that is

$$y'' = f(x, y, y')$$

Let us suppose the solution of (1) is to be approximated by the first $(N+1)$ terms of the Legendre polynomial; thus, we can write (3) as

$$y_N(x) \cong \sum_{j=0}^{N} c_j(x) l_j(x) = L(x)C^T \quad (6)$$

where the Legendre coefficients C vector and the Legendre vector $L(x)$ are given by

$$\left.\begin{aligned} C &= [c_0, c_1, c_2, \ldots, c_N] \\ L(x) &= [l_0(x), l_1(x), l_2(x), \ldots, l_N(x)] \end{aligned}\right\} \quad (7)$$

The second derivative of (6) can be expressed as follows

$$y_N^{(2)}(x) = \sum_{j=0}^{N} c_j(x) l_j^{(2)}(x) = L^{(2)}(x)C^T = L(x)(\mathcal{M}^T)^{(2)}C^T \quad (8)$$

where $\mathcal{M}$ is the matrix calculus defined in (4) above. Now substituting (6) and (8) into (1), we have

$$L(x)(\mathcal{M}^T)^{(2)}C^T = a(x)L(x)(\mathcal{M}^T)^{(1)}C^T + b(x)L(x)C^T + f(x) \quad (9)$$

Finally, to find the approximate solution, we collocate the transformed equation (9) at different collocation points $x_j = \left(\frac{j}{N-k}\mathrm{h}_j\right), j = 0,1,2,\ldots,N-k$, to obtain $N-2$ nonlinear algebraic equations using

$$L(x_j)(\mathcal{M}^T)^{(2)}C^T = a(x_j)L(x_j)(\mathcal{M}^T)^{(1)}C^T + b(x_j)L(x_j)C^T + f(x_j), j = 0,1,.,N-2 \quad (10)$$

These equations together with the initial conditions give $(N+1) \times (N+1)$ nonlinear systems of algebraic equations which can be solved using Newton's iterative method for the unknown constants. Finally, $y_N(x)$ given in (6) can be calculated.

3. Numerical Illustrations

The following numerical experiments are performed with the aid of MAPLE 18 and Scientific Workplace software packages in order to further affirm the applicability, simplicity and accuracy of the proposed method.

Example 1

Let us first consider the second order pantograph equation solved by [9] given by

$$y''(x) = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2, x \in [0,1], h = 0.01$$

Subject to the initial conditions $y(0) = y'(0) = 0$, the exact solution to this problem is known to be $y(x) = x^2$.

Applying our technique with $N = 3$, we have the following expression;

$$L(x)(\mathcal{M}^T)^{(2)}C^T = \frac{3}{4}L(x)C^T + L\left(\frac{x}{2}\right)C^T - x^2 + 2\ (\vartheta_1)$$

Using the initial conditions, we have respectively,

$$L(0)C^T = 0, L(0)(\mathcal{M}^T)^{(1)}C^T = 0\ (\vartheta_2)$$

Collocating (ϑ_1) at $x = 0, \frac{1}{2}$ and evaluating (ϑ_2) at $x = 0$, we have the following algebraic systems of equations

$$\left(\begin{array}{c} 3c_2 = \frac{7}{4}c_0 - \frac{7}{8}c_2 + 2 \\ 3c_2 + \frac{3}{40}c_3 = \frac{7}{4}c_0 + \frac{1}{160}c_1 - \frac{69997}{80000}c_2 - \frac{239993}{25600000}c_3 + \frac{79999}{40000} \\ c_1 - \frac{3}{2}c_3 = 0 \\ c_0 - \frac{1}{2}c_2 = 0 \end{array}\right)$$

Solving for the unknown coefficients $[c_0, c_1, c_2, c_3]$, we have

$$\left[c_0 = \frac{1}{3}, c_1 = 0, c_2 = \frac{2}{3}, c_3 = 0\right]$$

Substituting these approximate values into (6), we get the approximate solution to the problem as

$$y_N(x) = x^2$$

The approximate solution is the same as the exact solution showing the accuracy of the method.

Example 2

Consider the second order differential equation solved by [5] given by

$$y''(x) = y(x) + xe^{3x}, x \in [0,1], h = 0.0025$$

Subject to the initial conditions $y(0) = -\frac{3}{32}, y'(0) = -\frac{5}{32}$, the exact solution to this problem is known to be $y(x) = \frac{4x-3}{32e^{-3x}}$.

Applying our technique with $N = 5$, we have the following expression:

$$L(x)(\mathcal{M}^T)^{(2)}C^T = L(x)C^T + xe^{3x}\ (\vartheta_1)$$

Using the initial conditions, we have respectively,

$$L(0)C^T = -\frac{3}{32}, L(0)(\mathcal{M}^T)^{(1)}C^T = -\frac{5}{32}\ (\vartheta_2)$$

Collocating (ϑ_1) at $x = 0, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$ and evaluating (ϑ_2) at $x = 0$, we have the following Values of the unknown coefficients

$$[c_0 = -6.015660797\,02 \times 10^{-2}, c_1 = 2.794908665\,50 \times 10^{-2}, c_2 = 0.109373977\,228, c_3 = 0.159771274\,744, c_4 = 5.624959089\,12 \times 10^{-2}, c_5 = 2.957750691\,22 \times 10^{-2}]$$

Substituting these approximate values into (6), we get the approximate solution to the problem as

$$0.232922866\,934x^5 + 0.246091960\,149x^4 + 0.140625001\,378x^3 - 0.046875x^2 - 0.156250000\,001x - 0.09375$$

***Table 1**. Showing the numerical comparison of example 2 for $N = 5$.*

x	Exact	Approximate solution by LOM	Absolute error by LOM	Absolute error by [5]
0.0025	−0.094140915761800	−0.0941409157619	2.5000×10^{-15}	7.020×10^{-14}
0.0050	−0.094532404142338	−0.09 45324041423	5.6388×10^{-15}	1.217×10^{-13}
0.0075	−0.094924451608388	−0.0949244516084	1.9695×10^{-14}	3.396×10^{-12}
0.0100	−0.095317044390700	−0.0953170443908	9.6425×10^{-14}	8.122×10^{-12}
0.0125	−0.095710168480980	−0.0957101684814	3.8902×10^{-13}	1.453×10^{-11}
0.0150	−0.096103809629100	−0.0961038096304	1.2407×10^{-12}	2.233×10^{-11}
0.0175	−0.096497953340300	−0.0964979533436	3.3021×10^{-12}	3.156×10^{-11}
0.0200	−0.096892584872264	−0.0968925848799	7.6781×10^{-12}	4.220×10^{-11}
0.0225	−0.097289689232184	−0.0972876892483	1.6104×10^{-11}	5.421×10^{-11}
0.0250	−0.097683251173919	−0.0976832512051	3.1149×10^{-11}	6.754×10^{-11}

xample 3

Consider the following nonlinear second order boundary value problem solved in [16] given as:

$$y''(x) = \left(y'(x)\right)^2 - y(x) - 16x^6 + 2, x \in [-1,1], h = 0.0025$$

subject to the initial conditions $y(-1.0) = 0, y'(1.0) = -2$, the exact solution to this problem is known to be $y(x) = x^2 - x^4$.

Applying our technique with $N = 4$ we have the following expression;

$$L(x)(\mathcal{M}^T)^{(2)}C^T = \left(L(x)(\mathcal{M}^T)^{(1)}C^T\right)^2 - L(x)C^T - 16x^6 + 2 \ (\vartheta_3)$$

Using the initial conditions, we have respectively,

$$L(0)C^T = 0, L(0)(\mathcal{M}^T)^{(1)}C^T = 0 \ (\vartheta_4)$$

Collocating (ϑ_3) at $x = 0, \frac{1}{2}, \frac{1}{4}$ and evaluating (ϑ_4) at $x = 0$, we have the following Values of the unknown coefficients

$$[c_0 = 0.133333333\,333, c_1 = 0.0, c_2 = 9.523809523\,81 \times 10^{-2}, c_3 = 0.0, c_4 = -0.228571428\,571]$$

Substituting these approximate values into (6), we get the approximate solution to the problem as

$$-1.000000000\,00x^4 + 1.000000000\,00x^2 - 1.750000000\,00 \times 10^{-13}$$

Table 2. Showing the numerical comparison of example 3 for $N = 4$.

x	Exact	Approximate solution by LOM	Absolute error LOM $\lvert y_N(x) - y(x)\rvert$
0.0025	$6.24996093750 \times 10^{-6}$	$6.24996076250 \times 10^{-6}$	1.75×10^{-13}
0.0050	$2.4999\,3750000 \times 10^{-5}$	$2.49993748250 \times 10^{-5}$	1.75×10^{-13}
0.0075	$5.6246\,8359375 \times 10^{-5}$	$5.62468357625 \times 10^{-5}$	1.75×10^{-13}
0.0100	$9.999000000000 \times 10^{-5}$	$9.99899998250 \times 10^{-5}$	1.75×10^{-13}
0.0125	$1.562255859375 \times 10^{-4}$	$1.56225585763 \times 10^{-4}$	1.75×10^{-13}
0.0150	$2.249493750000 \times 10^{-4}$	$2.24949374825 \times 10^{-4}$	1.75×10^{-13}
0.0175	$3.061562109375 \times 10^{-4}$	$3.06156210763 \times 10^{-4}$	1.75×10^{-13}
0.0200	$3.998400000000 \times 10^{-4}$	$3.99839999825 \times 10^{-4}$	1.75×10^{-13}
0.0225	$5.059937109375 \times 10^{-4}$	$5.05993710763 \times 10^{-4}$	1.75×10^{-13}
0.0250	$6.24609375000 \times 10^{-4}$	$6.24609374825 \times 10^{-4}$	1.75×10^{-13}

Example 4

Consider the second order differential equation solved by [5] given by

$$y''(x) = x\left(y'(x)\right)^2, x \in [-1,1], h = 0.0025$$

Subject to the initial conditions $y(0) = 1, y'(0) = \frac{1}{2}$, the exact solution to this problem is known to be $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$.

Applying our technique with $N = 4$, we have the following expression;

$$L(x)(\mathcal{M}^T)^{(2)}C^T = x\left(L(x)(\mathcal{M}^T)^{(1)}C^T\right)^2 \ (\vartheta_5)$$

Using the initial conditions, we have respectively,

$$L(0)C^T = 1, L(0)(\mathcal{M}^T)^{(1)}C^T = \frac{1}{2} \ (\vartheta_6)$$

Collocating (ϑ_5) at $x = 0, \frac{1}{2}, \frac{1}{4}$ and evaluating (ϑ_6) at $x = 0$, we have the following values of the unknown coefficients

$$[c_0 = 1.000003906\,25, c_1 = 0.524999990\,234, c_2 = 1.116072282\,34 \times 10^{-5}, c_3 = 1.666666015\,62 \times 10^{-2}, c_4 = 4.464289129\,35 \times 10^{-6}]$$

Substituting these approximate values into (6), we get the approximate solution to the problem as

$$1.953126494\,09 \times 10^{-5}x^4 + 4.166665039\,05 \times 10^{-2}x^3 + 3.75 \times 10^{-17}x^2 + 0.500000000\,000x + 1.000000000\,00$$

Table 3. Numerical comparison of example 4 for $N = 4$.

x	Exact	Approximate solution by LOM	Absolute error LOM $\lvert y_N(x) - y(x) \rvert$	Absolute error by [5] $\lvert y_N(x) - y(x) \rvert$
0.0025	1.001250000 65	1.001250000 60	1.017×10^{-16}	1.339×10^{-14}
0.0050	1.002500005 21	1.002500004 82	9.359×10^{-15}	7.321×10^{-14}
0.0075	1.003750017 58	1.003750016 29	9.339×10^{-14}	1.000×10^{-13}
0.0100	1.005000041 67	1.005000038 63	4.460×10^{-13}	1.250×10^{-12}
0.0125	1.006250081 38	1.006250075 49	1.462×10^{-12}	1.372×10^{-12}
0.0150	1.007500140 63	1.007500130 52	3.812×10^{-12}	4.824×10^{-12}
0.0175	1.008750223 32	1.008750207 37	8.514×10^{-12}	6.314×10^{-12}
0.0200	1.010000333 35	1.010000309 73	1.701×10^{-11}	2.801×10^{-12}
0.0225	1.011250474 65	1.011250441 25	3.122×10^{-11}	4.322×10^{-12}
0.0250	1.012500651 10	1.012500605 63	5.367×10^{-11}	6.757×10^{-11}

4. Conclusion

In this work, a collocation technique based on the Legendre matrix calculus for solving general second order linear and nonlinear differential equations was presented. The derivation of this algorithm was essentially based on choosing a set of Legendre polynomials. The advantage of this technique over other methods is that it has less computational manipulations and complexities and also its availability for application on both linear and nonlinear second-order initial value problems. Other importance of the algorithm is that high accurate approximate solutions are achieved by using a few numbers of terms of the Legendre polynomial which result to simple matrix calculus and its transpose and thus, reduces the time involve in the derivation of the schemes to be used for implementation as compared to the case of linear multistep methods, more importantly the same matrix can be used to solve higher order differential equations by introducing just a little manipulation io it and also reduces the computational run time. The comparison of the results shows that the method is a very simple and efficient mathematical tool for solving initial value problems of differential equations.

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Author Contributions

Nathaniel Mahwash Kamoh: Conceptualization, Methodology, Project administration, Software, Writing – original draft, Writing – review & editing

Bwebum Cleofas Dang: Conceptualization, Methodology, Writing – original draft

Comfort Soomiyol Mrumun: Methodology, Project administration, Writing – original draft, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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