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Positive and Negative Solutions of a Class of Fractional Schrödinger Equation

Jianing Wang

School of Mathematics Qilu Normal University, Jinan, China

Email address:
2545516792@qq.com

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Abstract: In this paper, we study the existence of positive and negative solutions for a class of fractional Schrodinger equations. Firstly, we give the definition of fractional Laplace operator and the conditions satisfied by nonlinear terms. This paper introduces the previous progress in this field, and gives the definitions of space and energy functional and the positive and negative parts of function. Then we introduce the main results of this paper. Next,we give the embedding relationship between workspace and L^p space and give the definition of inner product and norm of space. In order to obtain the existence of positive and negative solutions of the equation, we give the definitions of functions $u^+, u^-$and functional weak solutions. This paper mainly uses mountain pass lemma to prove. Firstly, according to the embedding relationship of workspace and the condition of nonlinear term f, it is proved that functional I satisfies mountain road structure. Secondly, we need to prove that functional I satisfies the (C_c) condition, we first prove that the sequence u_n is bounded, then prove that UN has convergent subsequence by the definition of inner product and holder inequality. Therefore, we prove that functional I satisfies the (C_c) condition. Then,we define functional $I^{\pm}$ and its inner product form to verify that functional $I^{\pm}$ also has mountain path structure and satisfies (C_c) condition. Finally, taking u^+ and u^- as experimental functions respectively, it is verified that they are the solutions of functional I. It is obtained that both u^+ and u^- are the solutions of functional I. Therefore,we get the conclusion.

Keywords: Fractional Schrödinger Equation, Mountain Pass Theorem, Positive and Negative Solutions

1. Introduction

This paper is concerned with the following fractional Schrödinger equation:

$$(-\Delta)^\alpha u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 2$, $\alpha \in (0,1)$, $(-\Delta)^\alpha$ stands for the fractional Laplacian of order α, V is a positive continuous potential, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Here the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ of a function $u \in \mathcal{S}$ is defined by

$$\mathcal{F}((-\Delta)^\alpha \phi)(\xi) = |\xi|^{2\alpha}\mathcal{F}(\phi)(\xi), \quad \forall\, \alpha \in (0,1),$$

where $\mathcal{S}$ denotes the Schwartz space of rapadly decreasing C^∞ functions in $\mathbb{R}^N$, $\mathcal{F}$ is the Fourier transform, i.e., $\mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{-2ui\xi\cdot x}u(x)dx$. If u is smooth enough, it can also be computed by the following singular integral:

$$(-\Delta)^\alpha u(x) = C_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}dy,$$

where $P.V.$ is the principal value and $C_{N,\alpha}$ is a normalization constant.

In recent years, the study of nonlinear problems has received much attention. Fractional Schrödinger equation has become an important research object for mathematicians and physicists, and has far-reaching influence on nonlinear analysis, differential geometry and mathematical physics, etc.

Chang had applied the variational methods to obtain the existence of ground state solutions for (1) when $f(x,u)$ is asymptotically linear with respect to u at infinity, [1]. i.e.,

$$\limsup_{t\to 0^+}\frac{f(x,t)}{t}<\mu^*<\liminf_{t\to+\infty}\frac{f(x,t)}{t}\le\limsup_{t\to+\infty}\frac{f(x,t)}{t}<+\infty,$$

Uniformly in $x\in\mathbb{R}^N$.

When f has subcritical growth, Simone Secchi had obtained that (1) has at least a non-trival solution by using the Mountain Pass Theorem [2].

When $f(x,u)=\lambda|u|^p u$ with $\lambda>0, p\in(0,\frac{2N}{N-2s}), N\ge 2$. Feng had obtained the existence of ground state solutions for (1) by using concentration compactness principle [3].

Khoutir had utilized the variational methods to obtain the existence of nontrivial solution for (1) when $V(x)$ which is allowed to be sign-changing and a sublinear nonlinearity $f(x,u)$ [4].

In this paper, we assume

(V_1) $V\in C(\mathbb{R}^N,\mathbb{R}), V_0=\inf_{x\in\mathbb{R}^N}V(x)>0$.

(V_2) There exists $r_0>0$ such that, for any $K>0$,

$$meas(\{x\in B_{r_0}(y):V(x)\le K\})\to 0 \text{ as } |y|\to\infty.$$

(f_1) $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with a subcritical growth.

$$|f(x,t)|\le c(1+|t|^{q-1}),\ t\in\mathbb{R},\ x\in\mathbb{R}^N,$$

where $q\in(2,2^*_\alpha), 2^*_\alpha=\frac{2N}{N-2\alpha}$.

(f_2) $f(x,t)\ge 0$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$ and $f(x,t)=o(|t|)$ as $|t|\to 0$.

(f_3) $\lim_{|t|\to\infty}\frac{F(x,t)}{t^2}=+\infty$ uniformly for $x\in\mathbb{R}^N$, where $F(x,t)=\int_0^t f(x,s)ds$.

(f_4) There exist $\theta\ge 1, s\in[0,1]$ $s.t.$

$$\theta\mathcal{F}(x,t)\ge\mathcal{F}(x,st),(x,t)\in\mathbb{R}^N\times\mathbb{R},$$

where $\mathcal{F}(x,t)=f(x,t)t-2F(x,t)\ge 0$.

The main result is as follows.

Theorem 1.1 Assume that $(V_1),(V_2)$ and $(f_1)-(f_4)$ hold. Then the problem (1) admits a positive solution and a negative solution.

2. Preliminaries

Consider the Sobolev space:

$$H^\alpha(\mathbb{R}^N)=\{u\in L^2(\mathbb{R}^N):\int_{\mathbb{R}^N}(|\xi|^{2\alpha}\hat{u}^2+\hat{u}^2)d\xi<\infty\}.$$

where $\hat{u}=\mathcal{F}(u)$.The norm is defined by

$$\|u\|_{H^\alpha(\mathbb{R}^N)}=(\int_{\mathbb{R}^N}(|\xi|^{2\alpha}\hat{u}^2+\hat{u}^2)d\xi)^{\frac{1}{2}}.$$

In view of the presence of potential $V(x)$, we consider its subspace:

$$E=\{u\in H^\alpha(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2dx<\infty\}.$$

We define the norm in E by

$$\|u\|_E=(\int_{\mathbb{R}^N}(|\xi|^{2\alpha}\hat{u}^2+\hat{u}^2)d\xi+\int_{\mathbb{R}^N}V(x)u^2dx)^{\frac{1}{2}},$$

where $\hat{u}=\mathcal{F}(u)$. Moreover, by [2], E is a Hilbert space with the inner product

$$(u,v)_E=\int_{\mathbb{R}^N}(|\xi|^{2\alpha}\hat{u}(\xi)\hat{v}(\xi)+\hat{u}(\xi)\hat{v}(\xi))d\xi+\int_{\mathbb{R}^N}V(x)u(x)v(x)dx,\ \forall\, u,v\in E.$$

Note that, by Plancherel's theorem we have $\|u\|_2=\|\hat{u}\|_2$ and

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2dx=\int_{\mathbb{R}^N}((\widehat{-\Delta)^{\frac{\alpha}{2}}u}(\xi))^2d\xi=\int_{\mathbb{R}^N}(|\xi|^\alpha\hat{u}(\xi))^2d\xi=\int_{\mathbb{R}^N}|\xi|^{2\alpha}\hat{u}^2d\xi<\infty,\ \ \forall u\in H^\alpha(\mathbb{R}^N).$$

Together with (V_1), it follows that the norm $\|\cdot\|_E$ is equivalent to the norm

$$\|u\|=(\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{\alpha}{2}}u|^2+V(x)u^2)dx)^{\frac{1}{2}}.$$

The corresponding inner product is

$$(u,v)=\int_{\mathbb{R}^N}((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v+V(x)uv)dx.$$

Throughout out this paper, we will use the norm $\|\cdot\|$ in E.

Associated with problem (1.1), we consider the energy functional $I:E\to\mathbb{R}$ defined by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{\alpha}{2}}u|^2+V(x)u^2)dx-\int_{\mathbb{R}^N}F(x,u)dx,\ \forall u\in E. \tag{2}$$

We assume that

$$I^{\pm}(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{\alpha}{2}}u|^2dx+V(x)u^2)dx-\int_{\mathbb{R}^N}f(x,u^{\pm})\phi dx,\ \forall u,\phi\in E.$$

where $u^+=\max\{u,0\},\ u^-=\min\{u,0\},\ u=u^++u^-$.

Lemma 2.1 (see[3], Lemma 1) Let (V_1) and (f_1) hold. Then $I\in C^1(E,\mathbb{R})$ and its derivative

$$\langle I'(u),\phi\rangle=\int_{\mathbb{R}^N}((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}\phi+V(x)u\phi)dx-\int_{\mathbb{R}^N}f(x,u)\phi dx,\ \ \forall u,\phi\in E.$$

Obviously, $I^{\pm}\in C^1(E,\mathbb{R})$, and

$$\langle (I^{\pm})(u),\phi\rangle=\frac{1}{2}\int_{\mathbb{R}^N}((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}\phi+V(x)u\phi)dx-\int_{\mathbb{R}^N}f(x,u^{\pm})\phi dx,\ \forall u,\phi\in E.$$

It is easily seen that the critical points of I correspond to weak solutions of problem (1). Moreover, if u_1,u_2 are the critical of I^+ and I^-, then $u_1>0,u_2<0$ are the positive solution and negative solution of (1).

Lemma 2.2(see[6] and [7]) E is continuously embedded into $L^p(\mathbb{R}^N)$ for $p\in[2,2^*_\alpha]$ and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for $p\in[2,2^*_\alpha)$.

Lemma 2.3 (see [5]) E is compactly embedded into $L^p(\mathbb{R}^N)$ for $p\in[2,2^*_\alpha)$ with $2^*_\alpha=\frac{2N}{N-2\alpha}$.

Definition 2.4 Let $(E,\|\cdot\|)$ be a real Banach space, $I\in C^1(E,\mathbb{R})$. We say that I satisfies the (C_c) condition if any sequence $\{u_n\}\subset E$ such that $I(u_n)\to c$ and $\|I'(u_n)\|(1+\|u_n\|)\to 0$ as $n\to\infty$ has a convergent subsequence.

Lemma 2.5 For any $u\in E$, we have

$$(u^+,u^-)\geq 0.$$

Proof. Assume that $u\in E, u^+,u^-\in E$, for any $u\in E$, we obtain

$$\begin{aligned}
(u^+,u^-) &= \int_{\mathbb{R}^N}((-\Delta)^{\frac{\alpha}{2}}u^+(-\Delta)^{\frac{\alpha}{2}}u^-+V(x)u^+u^-)dx\\
&= \int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u^+(-\Delta)^{\frac{\alpha}{2}}u^-dx\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{N+2\alpha}}dxdy\\
&= \int_{\{u(x)\geq0\}\times\{u(y)\geq0\}}\frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{N+2\alpha}}dxdy\\
&+ \int_{\{u(x)\geq0\}\times\{u(y)\leq0\}}\frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{N+2\alpha}}dxdy\\
&+ \int_{\{u(x)\leq0\}\times\{u(y)\geq0\}}\frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{N+2\alpha}}dxdy\\
&+ \int_{\{u(x)\leq0\}\times\{u(y)\leq0\}}\frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{N+2\alpha}}dxdy\\
&= \int_{\{u(x)\geq0\}\times\{u(y)\leq0\}}\frac{-u(x)u(y)}{|x-y|^{N+2\alpha}}dxdy+\int_{\{u(x)\leq0\}\times\{u(y)\geq0\}}\frac{-u(x)u(y)}{|x-y|^{N+2\alpha}}dxdy\\
&\geq 0,
\end{aligned}$$

Thus $(u^+,u^-)\geq 0$.

Remark 2.6 Under the result of Lemma 2.4, for any $u\in E$, we have

(i) $(u,u^+)\geq(u^+,u^+)$,

(ii) $(u,u^-)\geq(u^-,u^-)$.

3. Proof of the Main Theorems

Lemma 3.1 Assume that $(V_1),(V_2)$ and $(f_1)-(f_4)$ satisfy. Then the functional I satisfying

(i) There exist $\delta,\rho>0$ such that $I(u)\geq\delta,\ \forall u\in E$ with $\|u\|=\rho$;

(ii) There exists $e\in E$ with $\|e\|>\rho$ such that $I(e)<0$.

Proof. (i) From the Lemma 2.2, E is continuously embedded into $L^p(\mathbb{R}^N)$,
which implies that

$$\|u\|_p^q\leq c_*\|u\|^q,\ \forall p\in[2,2^*_\alpha], \tag{3}$$

where c_* is a constant.

By $(f_1),(f_2)$, there exist $c(\varepsilon)>0$, such that

$$|F(x,u)| \leq \varepsilon|u|^2 + c(\varepsilon)|u|^q, \quad \forall \varepsilon \in (0, \frac{1}{4c_1}),\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{4}$$

Consequently, by (3),(4), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon\int_{\mathbb{R}^N}|u|^2dx - c(\varepsilon)\int_{\mathbb{R}^N}|u|^q dx \\ &\geq \frac{1}{2}\rho^2 - \varepsilon c_1\|u\|^2 - c_2\|u\|^q := \delta \quad \forall u \in E, \end{aligned}$$

where $c_1,\ c_2 \in R$.

It is not difficult to see that there exists $\rho > 0$ sufficiently small such that
$I(u) \geq \delta > 0,\ \forall u \in E$ with $\|u\| = \rho$.

(ii) By $(f_1), (f_3)$,there exist $\nu > 2$ such that

$$F(x,u) \geq B_1|u|^\nu - B_2,\ \ \forall x \in \mathbb{R}^N. \tag{5}$$

Therefore, for $t > 0, u \in E$ we have

$$\begin{aligned} I(t\vartheta) &= \frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{\alpha}{2}}t\vartheta|^2 + V(x)(t\vartheta)^2)dx - \int_{\mathbb{R}^N} F(x,t\vartheta)dx \\ &\leq \frac{t^2}{2}\|\vartheta\|^2 - \int_{\mathbb{R}^N}(B_1|t\vartheta|^\nu - B_2)dx \\ &\leq \frac{t^2}{2}\|\vartheta\|^2 - B_1t^\nu\|\vartheta\|_2^\nu + B_3, \end{aligned}$$

where $B_1, B_2, B_3 > 0$.

Combining the $\nu > 2$, take $e = t^*\vartheta$ with $t^* = t$. It is easily seen that $I(e) < 0$.

Lemma 3.2 Under the assumptions of Theorem 1.1, the functional I satisfies the (C_c) condition for any $c \in \mathbb{R}$.

Proof. We first proof the any (C_c) sequence is bounded, then proof the (C_c) sequence have convergent subsequence.

Let $\{u_n\} \in E$ such that

$$I(u_n) \to c,\ \ \|I'(u_n)\|(1 + \|u_n\|) \to 0, \quad n \to \infty, \tag{6}$$

This implie that

$$c = I(u_n) + o(1),\ \ I'(u_n)u_n = o(1), \quad n \to \infty. \tag{7}$$

We assume by contradiction that $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}, (1, 2, 3\ldots)$ Clearly, $\{v_n\} \subset E$ and $\|v_n\| = 1$, for any n. There exists $v \in E$ such that, passing to a subsequence if necessary, we have

$$\begin{cases} v_n \rightharpoonup v & \text{in } E, \\ v_n \to v & \text{in } L^p(\mathbb{R}^N),\ \ p \in [2, 2^*_\alpha), \\ v_n(x) \to v(x) & \text{a.e. } x \in \mathbb{R}^N. \end{cases} \tag{8}$$

Suppose that $v \neq 0$, in E. Dividing by $\|u_n\|^2$ in both sides of (2), noting that $I(u_n) \to c$, we obtain

$$\int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}dx = \frac{1}{2} + o(\|u_n\|^{-2}) < +\infty, \tag{9}$$

On the other hand, denote $\Omega_{\neq} := \{x \in \mathbb{R}^N : v(x) \neq 0\}$, by (f_3), for all $x \in \Omega_{\neq}$, and we have

$$\frac{F(x,u_n)}{\|u_n\|^2} = \frac{F(x,u_n)}{|u_n|^2} \cdot \frac{|u_n|^2}{\|u_n\|^2} = \frac{F(x,u_n)}{|u_n|^2} \cdot |v_n|^2 \to +\infty,$$

If $|\Omega_{\neq}| > 0$, using Fatou's lemma, we obtain

$$\int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}dx \to +\infty, \quad n \to \infty,$$

This contradicts (9). Therefore, $\Omega_{\neq}$ has zero measure, i.e., $v = 0$ a.e. in $\mathbb{R}^N$. Let $t_n \in [0,1]$ such that

$$I(t_n u_n) = \max_{t \in [0,1]} I(tu_n).$$

Then we claim $I(u_n t_n)$ is bounded. If $t_n = 0, I(0) = 0$; if $t_n = 1, I(t_n u_n) = I(u_n) \to c$. Hence, $I(t_n u_n)$ is bounded when $t_n = 0, 1$. if $0 < t_n < 1$, for n large enough

$$\begin{aligned} \langle I'(t_n u_n), t_n u_n \rangle &= \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{\alpha}{2}} t_n u_n|^2 + V(x)(t_n u_n)^2)dx - \int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n dx, \\ &= t_n \frac{d}{dt} |_{t=t_n} I(tu_n) = 0. \end{aligned}$$

Consequently, there exists $\theta \geq 1$, by (f_3) and (7), we have

$$\begin{aligned} I(tu_n) &\leq I(t_n u_n) - \frac{1}{2}\langle I'(t_n u_n), t_n u_n \rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n dx - \int_{\mathbb{R}^N} F(x, t_n u_n) dx \\ &\leq \int_{\mathbb{R}^N} \theta[\frac{1}{2} u_n f(x, u_n) - F(x, u_n)]dx \\ &= \theta[I(u_n) - \frac{1}{2}(I'(u_n), u_n)] \\ &\leq d_1, \end{aligned} \tag{10}$$

where d_1 is a positive constant. But fixing any $m > d^*(d^* \in R)$, we let $\overline{v}_n = \sqrt{2m} \cdot \frac{u_n}{\|u_n\|} = \sqrt{2m} v_n$. Note that from $(f_1), (f_2)$, we see that there exist $d_2 > 0, d_3 > 0$ such that

$$F(x,u) \leq d_2|u|^2 + d_3|u|^p, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Then by (8) we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, \overline{v}_n)dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} (d_2|\overline{v}_n|^2 + d_3|\overline{v}_n|^p)dx = 0.$$

Then for n large enough,

$$I(t_n u_n) \geq I(\sqrt{2m}\frac{u_n}{\|u_n\|}) = I\overline{v}_n) = m - \int_{\mathbb{R}^N} F(x, \overline{v}_n)dx \geq m,$$

This also contradicts (10).

Now, the sequence $\{u_n\}$ is bounded, as required. Next we confirm that $\{u_n\}$ has a convergent subsequence. Without loss of generality, we assume that

$$u_n \rightharpoonup u \quad \text{in } E,$$

$u_n \to u \quad \text{in } L^p(\mathbb{R}^N), \; p \in [2, 2_\alpha^*),$

Combining this with (f_1) and the Hölder inequality, we see

$$\begin{aligned} |\int_{\mathbb{R}^N} [f(x,u_n) - f(x,u)](u_n - u)dx| &\leq \int_{\mathbb{R}^N} [2c + c(|u_n|^{p-1} + |u|^{p-1})]|u_n - u|dx \\ &\leq 2c|u_n - u|_2 + c(\int_{\mathbb{R}^N} (|u_n| + |u|)^p)^{\frac{p-1}{p}} \cdot (\int_{\mathbb{R}^N} |u_n - u|^p)^{\frac{1}{p}} \\ &\leq 2c\|u_n - u\|_2 + c(\|u_n\|_p^{\frac{p-1}{p}} + \|u\|_p^{p-1}) \cdot \|u_n - u\|_p \to 0 \;\; as \; n \to \infty. \end{aligned}$$

Therefore,

$$\|u_n - u\|^2 = \langle I'(u_n) - I'(u), u_n - u\rangle + \int_{\mathbb{R}^N} [f(x,u_n) - f(x,u)](u_n - u)dx \to 0,$$

with the fact that $(I'(u_n) - I'(u), u_n - u) \to 0$, when $n \to \infty$, this implies that $u_n \to u \in E$. Hence, we prove that I satisfies the (C_c) condition for any $c > 0$.

Proof of Theorem 1.1.

Definition 3.3 Define

$$f(x,u^+) = \begin{cases} f(x,u), & u \geq 0, \\ 0, & u < 0, \end{cases}$$

$$f(x,u^-) = \begin{cases} 0, & u \geq 0, \\ f(x,u), & u < 0, \end{cases}$$

Let

$$F(x,t^\pm) = \int_0^t f(x,u^\pm)du.$$

We note that $f(x,u^\pm)$ satisfies the $(f_2),(f_3)$. Hence, $f(x,u^\pm)$ and $F(x,t^\pm)$ satisfies (f_4) when $t > 0$; when $t < 0$, $f(x,u^-)$ and $F(x,u^-)$ satisfies (f_4).

Let us consider the functional $I^\pm : E \to \mathbb{R}$

$$I^\pm(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{\alpha}{2}}u|^2 dx + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u^\pm)dx, \tag{11}$$

It is easy to verify $I^\pm \in C^1(E,\mathbb{R})$ and

$$\langle (I^\pm)(u), \phi\rangle = \frac{1}{2}\int_{\mathbb{R}^N} ((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}\phi + V(x)u\phi)dx - \int_{\mathbb{R}^N} f(x,u^\pm)\phi dx,\ \forall u,\phi \in E. \tag{12}$$

By lemma 3.1 and Lemma 3.2, we can also prove functional $I^\pm$ has the mountain pass geometry and satisfies (C_c) condition. Therefore, u_1 is a critical point of I^+; u_2 is a critical point of I^-.

Using u_1^- as the experimental function, By (12), (f_4) and remark 2.6, we have

$$\begin{aligned} 0 &= \langle (I^+)'(u_1), u_1^-\rangle \\ &= \int_{\mathbb{R}^N} [(-\Delta)^{\frac{\alpha}{2}}u_1(-\Delta)^{\frac{\alpha}{2}}u_1^- + V(x)u_1u_1^-]dx - \int_{\mathbb{R}^N} f(x,u_1^+)u_1^- dx \\ &= \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}}u_1(-\Delta)^{\frac{\alpha}{2}}u_1^- dx + \int_{\mathbb{R}^N} V(x)(u_1^-)^2 dx. \\ &\geq \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}}u_1^-(-\Delta)^{\frac{\alpha}{2}}u_1^- dx + \int_{\mathbb{R}^N} V(x)(u_1^-)^2 dx. \\ &= \|u_1^-\|^2, \end{aligned}$$

Which implies that $u_1^- = 0$, we also obtain $u_1 \geq 0$. On the other hand, we note that $u_1 \not\equiv 0$, by Maximum principle [9], we have $u_1 > 0$. In the same way, we can verify $u_2 < 0$. Thus, u_1 and u_2 are the positive solution and negative solution of problem (1), respectively.

Competing Interests

The author declares that they have no competing interests.

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