



Russell's Paradox, Our Solution, and the Other Solutions

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Abstract: The aim of this paper is proving that our solution is better than the solution presented by the own Russell and what is today the most accepted solution to the Russell's Paradox, which is the solution of Zermelo and Frankael. We presented our solution a few years ago, and that is a solution that we believe should be considered to be the actual solution. We decided to do what Dr. Hyde asked us to do in 2000 in terms of The Sorites Paradox and our solution to it, which is studying all objections to all solutions that have been previously presented and then saying why our solution does not suffer from those problems. We then analyze the solution presented by the own Russell back then on top of the most accepted solution, that of Zermelo and Frankael. We do all from the most unbiased perspective as possible. We seem to be able to prove, with solid argumentation, that our solution to the Russell's Paradox is the best solution so far. The methods we use are: analytical and synthetic studies, application of the results of the synthetic studies, and tests of soundness in reasoning based on the foundations of Logic. Each time something is found to be unsound we direct ourselves to our own solution and assess it in the same way to see if the study of ours also leads to unsoundness. Our results are that our solution is sounder than the solutions presented by Russell, Zermelo, and Frankael. The conclusion is that it is very likely to be the case that we have a final solution.

Keywords: Logic, Russell, Paradox, Language, Classical Logic

1. Introduction

The set $S = \{A/A \text{ is a set and } A \notin A\}$. If we assume S is not in the set S , then, by definition, it must belong to that set. If we assume S is in the set S , then it contradicts the definition of S . Here we have the paradox that Bertrand Russell (1872-1970) presented to Gottlob Frege (1848-1925) just as Frege's lifetime work on the logical foundations of arithmetic went to be published. In Gottlob's own words, *A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished* (Burton, 612).

Many ways have been found to give an example of Russell's Paradox for clearer understanding. Here we present a new example of this paradox. Imagine a middle school teacher who every week passes out a list of all the materials she will pass out that week that she expects each student to have in their binders. She plans to have the list, called the *weekly list*, be a table of contents for each week so the students can organize their binders. However, towards the end of the semester, the teacher realizes she sometimes forgot to include the weekly list itself as one of

the items she wants to be included in the binders. The teacher figures she can just create a supplementary list that will contain only the weekly lists that failed to include themselves as something needed to be in the binder. However, she finds now that she is faced with a problem. If she includes the supplementary list as one of the supplementary list's items, the list will no longer be the list of lists that do not contain themselves as an item. If she does not include the supplementary list as one of its items, then it would be considered one of the lists that failed to include itself and should be included! Thus, the teacher is faced with Russell's Paradox.

Bertrand Russell devised what he called the theory of types to prevent the paradox. In this theory, a set would be defined as being of a distinct type, like type 1. The elements of type 1 sets can then only be included in a set of type 2 because sets of type 2 are defined as containing only sets of type 1. Thus, we do not need to worry about whether or not a set of type 2 can contain itself because it's defined as only containing sets of type 1. This theory creates a sort of hierarchy of sets. In the example of the teacher and her lists, she would define the additional list as

containing only those lists that she had handed out weekly. Now she does not need to worry about whether or not she should include this additional list because it is not one of the weekly lists. While Russell's solution does succeed in avoiding the contradictions, mathematicians decided that the solution should be more intuitive for the foundations of mathematics.

(Early, [1], 1999)

According to Aatu [2], the own Russell addressed the issue well. See:

Russell's type theory, which is found in its mature form in the momentous *Principia Mathematica* avoids the paradoxes by two devices. First, Frege's fifth axiom is abandoned entirely: The extensions of predicates do not appear among the objects. Secondly, the predicates themselves are ordered into a ramified hierarchy so that the predicates at the lowest level can be defined by speaking of objects only, the predicates at the next level by speaking of objects and of predicates at the previous level and so forth.

The first of these principles has drastic implications to Mathematics. For example, the predicate *has the same cardinality* seemingly can't be defined at all. For predicates apply only to objects, and not to other predicates. In Frege's system, this is easy to overcome: The equicardinality predicate is defined for extensions of predicates, which are objects. In order to overcome this, Russell introduced the notion of types (which are today known as degrees). Predicates of degree 1 apply only to objects, predicates of degree 2 apply to predicates of degree 1, and so forth.

In this way, solving this paradox, for Russell, would be something like: On this level, we are dealing with sets from the lower level, so that the set we form with the rule we have just established cannot be included in it.

So, it does look like the own Russell had solved his paradox. We, mathematicians, could think that is counter-intuitive in the same sense that counting grains of sand was counter-intuitive in a soritical situation (Pinheiro, [3]): That is not the situation we had before, when the problem appeared, so that this is modifying something that we like as it is.

The most accepted solution today is that of Zermelo and Fraenkel. Zermelo's axiom of specification is, to every set A and every definite property $P(x)$ there corresponds a set whose elements are exactly those elements x in A for which the property $P(x)$ holds (Burton, 616). What this axiom does is require a preexisting set A and some property $P(x)$ to make a new set. Previously, only the property $P(x)$ was required. This changes the set S to $S = \{x \in A/x \text{ is a set and } x \notin x\}$. Now $S \in S$ is impossible because it would have to satisfy the two conditions that $S \in A$ and $S \notin S$, which it clearly cannot. That is clear because we just state that, in order for $S \in S$, it must satisfy the property that $S \notin S$. If we consider the other possibility that $S \notin S$ we see that it does satisfy the property $P(x)$, but cannot meet our second requirement

that $S \in A$. This is because if $S \in A$ then it follows, by our definition, that $S \in S$, which we already reasoned is not true. Therefore we conclude that, by the law of excluded middle, which says that every proposition is either true or false (Burton, 612), $S \notin A$. Therefore, $S \notin S$ failed the second of the two requirements to be in the set S so we conclude that $S \notin S$ and have avoided the paradox.

In the example of the teacher and her lists, she has a list we will call list x that contains lists known to exist. However, in order for an item to get onto the supplementary list, the teacher makes the requirement that it must now be on list x and not contain itself. If we consider the possibility that the supplementary list is listed on the supplementary list, we find it must be on list x and must not contain itself on a list. The latter requirement clearly fails because we just said it contains itself. Now consider the other possibility that the supplementary list is not listed on itself. We can see that it does meet the first requirement of being a list that does not include itself, but it can't meet the second requirement that it's included on list x . This follows because if it were included on list x , it would imply that the supplementary list should be included on the supplementary list, but we already showed this is not allowed. Therefore, the teacher can safely avoid the paradox and not include the supplementary list as an item of itself.

(Early, [1], 1999)

2. Development

Set is a rustic, primitive, item in Mathematics, one of the most basic: We put a few things together, pass a graphical or imaginary line around those, and call that a set.

Imposing new rules, so that prohibitions are added to what we can use to form a set sounds like something we should reject.

Besides, appealing to the fact that, in Classical Logic, we could not have both belong and do not belong with the same couple of items, in the same order, could just imply that our current paradigms are inconsistent, as for Mathematics.

Our solution (Pinheiro, [9]) is way simpler and preserves things as they are: We have an enthymeme each, and every, time we form a set, something that is part of our Inner Reality (Pinheiro, [4]), but that we don't explicitly say accompanies the creation of the set. We all create a set at a certain time and on a certain date, so that date and time are things that come attached to the creation of sets, but, so far, we do not make this fact appear in writing together with our set.

When we are judging what will be part of a set or not, the set in formation is in the situation it is: In formation. As such, it can never be considered as a possible inclusion, for only what is already a set, is already formed, could.

We dealt with the issue time in *Infinis* (Pinheiro, [10]) as well. That is when we suggested that Cartesian plots should have time and date somewhere around them. Time DOES exist in Mathematics and it is more than time to tell it in an

explicit way.

Perhaps the source explains Zermelo's solution in a confused way, but it seems that, in the way they explain it, it actually agrees with ours, since it would have to do with time of creation of the set somehow, like first we have to have the set, then the set of sets, basically.

See how it appears on (Baldwin, [5]):

Zermelo's solution to Russell's paradox was to replace the axiom for every formula $A(x)$ there is a set $y = \{x: A(x)\}$ with the axiom for every formula $A(x)$ and every set b there is a set $y = \{x: x \text{ is in } b \text{ and } A(x)\}$.

According to the just-mentioned source, before the replacement, we have:

But Russell (and independently, Ernst Zermelo) noticed that $x = \{a: a \text{ is not in } a\}$ leads to a contradiction in the same way as the description of the collection of barbers. Is x itself in the set x ? Either answer leads to a contradiction.

How replacing what we see in the above paragraph with what we see in the extract that comes before this one on this post is a bit of a puzzle, like we can't really see how that would be a solution.

Another extract from the same source (Baldwin, [5]):

What became of the effort to develop a logical foundation for all of Mathematics? Mathematicians now recognize that the field can be formalized using the so-called Zermelo-Fraenkel set theory. The formal language contains symbols such as \in to express *is a member of*, $=$ for equality and \emptyset to denote the set with no elements. So one can write formulas such as $B(x)$: if $y \in x$ then y is empty. In set-builder notation, we could write this as $y = \{x: x = \emptyset\}$ or more simply as $y = \{\}$. Russell's paradox becomes: Let $y = \{x: x \text{ is not in } x\}$. Is y in y ?

We then understand that perhaps Zermelo and Fraenkel simply helped us write things more properly, like perhaps they did not really solve the paradox.

In fact, the paper about our solution brings the paradox in a shape that is very similar to the shape we have just seen in the extract.

From (Klement, [6]), we get:

Aussonderung: A quite different approach is taken in Zermelo-Fraenkel (ZF) set theory. Here too, a restriction is placed on what sets are supposed to exist. Rather than taking the *top-down* approach of Russell and Frege, who originally believed that for any concept, property or condition, one can suppose there to exist a class of all those things in existence with that property or satisfying that condition, in ZF set theory, one begins from the *bottom up*. One begins with individual entities, and the empty set, and puts such entities together to form sets. Thus, unlike the early systems of Russell and Frege, ZF is not committed to a universal set, a set including all entities or even all sets. ZF puts tight restrictions on what sets exist. Only those sets that are explicitly postulated to exist, or which can be put together from such sets by means of iterative processes, etc., can be concluded to exist. Then, rather than having a naive class abstraction principle that states that an entity is in a certain class if and only if it

meets its defining condition, ZF has a principle of separation, selection, or as in the original German, *Aussonderung*. Rather than supposing there to exist a set of all entities that meet some condition *simpliciter*, for each set already known to exist, *Aussonderung* tells us that there is a subset of that set of all those entities in the original set that satisfy the condition. The class abstraction principle then becomes: If set A exists, then for all entities x in A , x is in the subset of A that satisfies condition C if and only if x satisfies condition C . This approach solves Russell's paradox, because we cannot simply assume that there is a set of *all* sets that are not members of themselves. Given a set of sets, we can separate or divide it into those sets within it that are in themselves and those that are not, but since there is no universal set, we are not committed to the set of all such sets. Without the supposition of Russell's problematic class, the contradiction cannot be proven.

Now the key-element seems to be: If set A exists, then, for all entities x in A , x is in the subset of A that satisfies condition C if and only if x satisfies condition C . Translated into Russell's Paradox, we get that if the set of all sets that do not belong to themselves exists, then, for all entities x in this set, x is in the subset of this set that satisfies the condition of not belonging to itself if and only if x satisfies the condition of not belonging to itself.

Assume the set of sets that do not belong to themselves (set A from the previous paragraph) exists. Make the condition C be: Does not belong to itself. For all entities x in the set of sets that do not belong to themselves, x is in the subset of this set that satisfies condition C if and only if x satisfies condition C .

Now, if x does not belong to itself, it will satisfy the condition, so that x will be in the subset of the set if it is an entity in the set. In this way, we will say that x belongs to the subset of the set of sets that do not belong to themselves if it does not belong to itself and it is in the set, but we would like to call the set of the sets that do not belong to themselves x , so that we still get the paradox.

Our source says that we don't get the paradox anymore instead. The reason would be that Zermelo-Fraenkel solution would mean that sets are not universal sets anymore: They are sets of subsets. We would then have a subset of sets that belong to themselves and a subset of sets that do not belong to themselves inside of a set. If that were the case, x would have to belong to exclusively one of those subsets, and it would not be a member of itself for some reason, but if a set is the conjunction of all its subsets, then it should be inside of itself. A set is not equal to its subsets however and this interpretation seems incompatible with what we see written right before it.

Poincare seems to agree with our criticism to the claim that Zermelo solved Russell's Paradox. See (Hallett, [7]):

Mr. Zermelo does not allow himself to consider the set of all the objects which satisfy a certain condition because it seems to him that this set is never closed; that it will always be possible to introduce new objects. On the other

hand, he has no scruple in speaking of the set of objects which are part of a certain *Menge M* and which also satisfy a certain condition. It seems to him that one cannot possess a *Menge* without possessing at the same time all its elements. Among these elements, he will choose those which satisfy a given condition, and will be able to make this choice very calmly, without fear of being disturbed by the introduction of new and unforeseen elements, since he already has all these elements in his hands. By positing beforehand this *Menge M*, he has erected an enclosing wall which keeps out the intruders who could come from without. But he does not query whether there could be intruders from within whom he enclosed inside his wall. (Poincaré 1909: 477; p. 59 of the English translation)

There is one source (Hart, [8]) that seems to explain things better in what regards Zermelo-Fraenkel's solution to Russell's Paradox. See:

Thus we must conclude that

$$\{X : X \notin X\}$$

is not a set, and we must revise the intuitive notion of a set.

The safe way to eliminate paradoxes of this type is to abandon the Schema of Comprehension and keep its weak version, the *Schema of Separation*:

If P is a property, then for any X there exists a set $Y = \{x \in X : P(x)\}$.

Once we give up the full Comprehension Schema, Russell's Paradox is no longer a threat; moreover, it provides this useful information: The set of all sets does not exist. (Otherwise, apply the Separation Schema to the property $x \notin x$.)

Figure 1. Quotation from (Hart, [8]).

Let's do what the extract tells us to do (apply the Separation Schema to the mentioned property):

$P(x)$: x does not belong to x .

If not belonging to itself is a property, then for any X there exists a set $Y = \{x \text{ belongs to } X : x \text{ does not belong to } x\}$.

The argument used in another source was that this violates the principles of Classical Logic, which is the logical system we use to deal with mathematical problems, so that this set cannot exist: In Classical Logic, we have only two possible truth-values, true and false (Bauer, [11]). If x belongs to X is true, then x does not belong to X is false. We must notice that we have X and x , so that, rigorously, we have different mathematical entities. When we read the last paragraph, we get the impression that X is a set and x is an element, so that we don't really get Russell's Paradox here.

We can always expand and decide that both X and x are sets of sets, however. In this case, X could have x as an element and x could also be thought of as something that could contain x . We would need to replace X with x to get Russell's Paradox here, first of all.

According to the mess we have just created, X and x are sets, so that we can get the statement by its foot, basically: If it is any X , then we can use x as well.

After the replacement, we get:

If not belonging to itself is a property, then for x there exists a set $Y = \{x \text{ belongs to } x : x \text{ does not belong to } x\}$.

Now, if x does belong to x , it is not true that x does not

belong to x because of the amount of truth-values that Classical Logic accepts and the Law of the Excluded Middle (either true or false, never both).

With this, we do have an empty set, so that nobody is in x , and therefore all sets belong to themselves. Such is not true. Not all sets are sets of sets. In this case, the axiom is wrong.

We do not have an impossibility: We are sure that nobody can obey the right side because everyone is obeying the left side. Were that an *and*, we would have successfully applied Excluded Middle and then gotten to the desired conclusion, but that is not an *and*.

It makes sense saying that this axiom is equivocated in the way it is presented. See: Say P is being blue. Say X is the set of the yellow things.

If being blue is a property, then, for the set of yellow things, there exists a set $Y = \{x \text{ belongs to the set of yellow things} : x \text{ is blue}\}$.

It is obviously badly written.

3. Conclusion

Russell's solution seems to impose additional rules to the formation of sets, but that is like the suggestion of the numberphiles to solve The Sorites: Count the grains!

We want to use sets as we originally defined them, if possible.

That is possible.

Zermelo and Fraenkel proposed a solution, according to some, but that would imply using the Law of the Excluded Middle and something that is not a conjunction.

The problem is that we don't really have a conjunction in the Russell's Paradox where they seem to think we do.

All that happens there is that our set would have no elements, since the condition to form the set can never be fulfilled.

When we are describing the condition to form the set, we have to use the axiom created by them, and that, written in a particular fashion, includes saying that x belongs to a set when it doesn't. Because it must belong for us to describe the set, then it cannot not belong, what makes the set of those who do not belong be empty. Notwithstanding, if that set is empty, all sets belong to themselves. All sets belong to the set of their subsets, but they cannot belong to themselves, all of them, because some of them at least won't be sets of sets, just for starters.

That particular way of describing a set can only be wrong then. And, in fact, we found a suitable counter-example, something unrelated, and exposed it in this paper.

With all this, our solution seems to reign in an absolute manner: it is natural, it does not use any law that cannot be applied for the situation at hand, and it really solves the problem.

Our solution is simple and preserves things as they are: We have is an enthymeme each, and every, time we form a set, something that is part of our Inner Reality, but that we don't explicitly say accompanies the creation of the set. We all create a set at a certain time and on a certain date, so that date

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