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Quaternions Algebra and Its Applications: An Overview

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Abstract: The real quaternions algebra was invented by W.R. Hamilton as an extension to the complex numbers. In this paper, we study various kinds of quaternions and investigate some of basic algebraic properties and geometric applications of them.

Keywords: Generalized Quaternion, Rotation, Split Quaternion, Quasi-Quaternion, Homothetic Motion

1. Introduction

In mathematics, the quaternions are a number system that extends the complex numbers. They find uses in both theoretical and applied mathematics, in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics, computer vision and crystallographic texture analysis [24]. Here, we study various kinds of quaternions and review some important basic algebraic properties and geometric applications of them.

2. Generalized Quaternions Algebra

A brief introduction to the generalized quaternions is provided in [25], the subject which have investigated in algebra [27]. Recently, we have studied the generalized quaternion and some of their algebraic properties [1]. It is shown that the set of all unit generalized quaternions with the group operation of quaternions multiplication is a Lie group of *3*-dimension. Their Lie algebra and properties of the bracket multiplication are looked for, e.g. a matrix corresponding to Hamilton operators, defined for the generalized quaternions, determines a Homothetic motion in $E^4_{\alpha\beta}$ [2]. We showed how the unit generalized quaternions can be used to describe the rotation in three and four dimensional space [4, 5].

A generalized quaternion q is an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \tag{1}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^2 = -\alpha, \quad \vec{j}^2 = -\beta, \quad \vec{k}^2 = -\alpha\beta,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \alpha\vec{j} = -\vec{i}\vec{k}, \quad \alpha, \beta \in \mathbb{R}.$$

The set of all generalized quaternions is denoted by $H_{\alpha\beta}$. A generalized quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. Therefore $H_{\alpha\beta}$ is form a 4-dimensional real space which contains the real axis $\mathbb{R}$ and a 3-dimensional real linear space $E^3_{\alpha\beta}$, so that, $H_{\alpha\beta} = \mathbb{R} \oplus E^3_{\alpha\beta}$.

Special cases:

1. $\alpha = \beta = 1$, is considered, then $H_{\alpha\beta}$ is the algebra of real quaternions.
2. $\alpha = 1, \beta = -1$, is considered, then $H_{\alpha\beta}$ is the algebra of split quaternions.
3. $\alpha = 1, \beta = 0$, is considered, then $H_{\alpha\beta}$ is the algebra of semi-quaternions.
4. $\alpha = -1, \beta = 0$, is considered, then $H_{\alpha\beta}$ is the algebra of split semi-quaternions.
5. $\alpha = 0, \beta = 0$, is considered, then $H_{\alpha\beta}$ is the algebra of quasi-quaternions(or 1/4 quaternions).

The multiplication rule for generalized quaternions is defined as

$$qp = S_q S_p - \langle \vec{V}_q, \vec{V}_p \rangle + S_q \vec{V}_p + S_p \vec{V}_q + \vec{V}_p \times \vec{V}_q \quad (2)$$

where

$S_q = a_0$, $S_p = b_0$, $\langle \vec{V}_q, \vec{V}_p \rangle = \alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3$,

$\vec{V}_q \times \vec{V}_p = \beta(a_2 b_3 - a_3 b_2)\vec{\mathrm{i}} + \alpha(a_3 b_1 - a_1 b_3)\vec{\mathrm{j}} + (a_1 b_2 - a_2 b_1)\vec{\mathrm{k}}$.

It could be written

$$qp = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}. \quad (3)$$

Obviously, quaternions multiplication is associative and distributive with respect to addition and subtraction, but the commutative law does not hold in general.

Corollary 1. $\mathrm{H}_{\alpha\beta}$ with addition and multiplication has all the properties of a number field without commutativity of the multiplication. Therefore it is called the skew field of quaternions.

The conjugate of the quaternion $q = S_q + \vec{V}_q$ is denoted by $\overline{q}$ and defined as $\overline{q} = S_q - \vec{V}_q$. The norm of a quaternion $q = (a_0, a_1, a_2, a_3)$ is defined by $N_q = q\overline{q} = \overline{q}q = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2$ and we say that $q_0 = q/N_q$ is a unit generalized quaternion where $q \neq 0$. The set of unit generalized quaternions, G, with the group operation of quaternions multiplication is a Lie group of 3 -dimension. The scalar product of two generalized quaternions $q = S_q + \vec{V}_q$ and $p = S_p + \vec{V}_p$ is defined as

$$\langle q, p \rangle_s = S_q S_p + \langle \vec{V}_q, \vec{V}_p \rangle = S(q\overline{p}).$$

Also, using the scalar product we can defined an angle λ between two quaternions q, p to be such

$$\cos\lambda = \frac{S(q\overline{p})}{\sqrt{N_q}\sqrt{N_p}}. \quad (4)$$

Definition 1. A matrix A is called a quasi-orthogonal matrix if $A^T \varepsilon A = \varepsilon$ and $\det A = 1$ where

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix},$$

and $\alpha, \beta \in \mathrm{R}$. The set of all quasi-orthogonal matrices with the operation of matrix multiplication is called rotations group in 4-spaces $\mathrm{E}^4_{\alpha\beta}$ [3, 9, 17].

3. Real Quaternions Algebra

Quaternions algebra was introduced by Hamilton in 1843. In modern mathematical language, quaternions form a four-dimensional associative normed division algebra over the real numbers, and therefore also a domain. In fact, the quaternions were the first non-commutative division algebra to be discovered. The unit quaternions can be thought of as a choice of a group structure on the 3-sphere S^3 that gives the group Spin(3), which is isomorphic to SU(2) and also to the universal cover of $SO(3)$ [24].

A real quaternion is defined as

$$q = a_0 1 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \quad (5)$$

where a_0, a_1, a_2 and a_3 are real number and $1, \vec{i}, \vec{j}, \vec{k}$ of q may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$$

$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \ \vec{j}\vec{k} = \vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}, \ \vec{i}\vec{j}\vec{k} = -1.$$

The quaternion algebra H is the even subalgebra of the Clifford algebra of the 3-dimensional Euclidean space. The Clifford algebra $Cl(\mathrm{E}^n_p) = Cl_{n-p,p}$ for the n-dimensional non-degenerate vector space E^n_p having an orthonormal base $\{e_1, e_2, ..., e_n\}$ with the signature $(p, n-p)$ is the associative algebra generated by 1 and $\{e_i\}$ with satisfying the relations $e_i e_j + e_j e_i = 0$ for $i \neq j$ and $e_i^2 = \begin{cases} -1, & if\ i = 1,2,...,p \\ 1, & if\ i = p+1,...,n \end{cases}$.

The Clifford algebra $Cl_{n-p,p}$ has the basis $\{e_{i_1} e_{i_2} ... e_{i_k} : 1 \leq i_1 < i_2 < ... < i_k < n\}$ that is the division algebra of quaternions H is isomorphic with the even subalgebra $Cl^+_{3,0}$ of the Clifford algebra $Cl_{3,0}$ such that $Cl^+_{3,0}$ has the basis $\{1, e_2 e_3 \to \vec{j}, e_1 e_3 \to \vec{k}, e_1 e_2 \to \vec{i}\}$. The conjugate of the quaternion $q = S_q + \vec{V}_q$ is denoted by $\overline{q}$, and defined as $\overline{q} = S_q - \vec{V}_q$. The norm of a quaternion $q = (a_0, a_1, a_2, a_3)$ is defined by $q\overline{q} = \overline{q}q = a_0^2 + a_1^2 + a_2^2 + a_3^2$ and is denoted by N_q and say that $q_0 = q/N_q$ is unit quaternion where $q \neq 0$. Unit quaternions provide a convenient mathematical notation for representing orientations and rotations of objects in three dimensions. One can represent a quaternion $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ by a 2×2 complex matrix (with i' being the usual complex imaginary);

$$A = \begin{bmatrix} a_0 + i'a_1 & -i'a_1 + a_2 \\ -i'a_1 - a_2 & a_0 - i'a_3 \end{bmatrix}$$

or by a 4×4 real matrix

$$A=\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_2 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}. \quad (6)$$

The Euler's and De-Moivre's formulae for the matrix A are studied in [6]. It is shown that as the De Moivre's formula implies, there are uncountably many matrices of unit quaternion satisfying $A^n = I_4$ for $n > 2$.

In geometry and linear algebra, a rotation is a transformation in a plane or in space that describes the motion of a rigid body around a fixed point. There are at least eight methods used commonly to represent rotation, including: i) orthogonal matrices, ii) axis and angle, iii) Euler angles, iv) Gibbs vector, v) Pauli spin matrices, vi) Cayley-Klein parameters, vii) Euler or Rodrigues parameters, and viii) Hamilton's quaternions. But to use the unit quaternions is a more useful, natural, and elegant way to perceive rotations compared to other methods [4].

Theorem 1. All the rotation about lines through the origin in ordinary space form a group, homomorphic to the group of all unit quaternions.

With Cartesian point coordinates in 3-space, a rotation in 3-space about the origin can be represented by the orthogonal matrix

$$R=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

where $RR^T = I_3$ and $\det R = 1$. It is known that unit quaternions can represent rotations about the origin. Wittenburg gives the following conversion formulae. For any unit quaternion q, the entries of the rotation matrix are

$$r_{11} = 2(a_0^2 + a_1^2) - 1,\ r_{21} = 2(a_1 a_2 + a_0 a_3),\ r_{31} = 2(a_1 a_3 - a_0 a_2),$$
$$r_{12} = 2(a_1 a_2 - a_0 a_3),\ r_{22} = 2(a_0^2 + a_2^2) - 1,\ r_{32} = 2(a_2 a_3 + a_0 a_1),$$
$$r_{13} = 2(a_1 a_3 + a_0 a_2),\ r_{23} = 2(a_2 a_3 - a_0 a_1),\ r_{33} = 2(a_0^2 + a_3^2) - 1.$$

Example 1. For the unit real quaternion $q = \frac{1}{\sqrt{2}} + \frac{1}{2}(1,-1,0)$, the rotation matrix is

$$R_q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

the axis of this rotation is spanned by the vector $(1,-1,0)$, and the angle of rotation is $\phi = \frac{\pi}{2}$.

4. Split Quaternions Algebra

Split quaternions, or coquaternions are elements of a 4-dimensional associative algebra introduced by James Cockle in 1849. Like the quaternions introduced by Hamilton in 1843, they form a four dimensional real vector space equipped with a multiplicative operation. Unlike the quaternion algebra, the split quaternions contain zero divisors, nilpotent elements, and nontrivial idempotents. Manifolds endowed with coquaternion structures are studied in differential geometry and superstring theory [26]. Some algebraic properties of Hamilton operators are considered in [20] where split quaternions have been expressed in terms of 4×4 matrices by means of these operators. In addition, the homothetic motions has been considered with to aid of the Hamilton operators in three and four-dimensional semi-Euclidean spaces E_2^4 [21, 12]. De-Moivre's and Euler's formulae for matrices associated with split quaternions is studied in [7] and every power of these matrices are immediately obtained. Rotations in Minkowski 3-space can be stated with split quaternions, such as expressing Euclidean rotations using quaternions [20].

A split quaternion q is represented in the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \quad (7)$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^2 = -1,\ \vec{j}^2 = \vec{k}^2 = 1,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i},\ \ \vec{j}\vec{k} = -\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

The set of all split quaternions is denoted by H'. A split quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. Therefore H' is the even subalgebra of the Clifford algebra of 3-dimensional Lorentzian space.

The conjugate of the quaternion $q = S_q + \vec{V}_q$ is denoted by $\overline{q}$ and defined as $\overline{q} = S_q - \vec{V}_q$. The norm of a quaternion $q = (a_0, a_1, a_2, a_3)$ is defined by $N_q = q\overline{q} = \overline{q}q = a_0^2 + a_1^2 - a_2^2 - a_3^2$ and we say that $q_0 = q/N_q$ is a unit split quaternion where $q \neq 0$.

We introduce the R-linear transformations representing left and right multiplication in H'. Let q be a split quaternion, then $\overset{+}{h}_q : H' \to H'$ and $\overset{-}{h}_q : H' \to H'$ are defined as follows:

$$\overset{+}{h}_q(x) = qx, \qquad \overset{-}{h}_q(x) = xq \qquad x \in H'.$$

Since these multiplications are linear maps from four dimensional vector space into itself, we can find a matrix representation of each.

The Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$, could be represented as the matrices;

$$\overset{+}{H}(q)=\begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \tag{8}$$

and

$$\overset{-}{H}(q)=\begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}. \tag{9}$$

Theorem 2. If q and p are two split quaternions, λ is a real number and $\overset{+}{H}$ and $\overset{-}{H}$ are operators as defined in equations (1) and (2), respectively, then the following identities hold:

1. $q=p \Leftrightarrow \overset{+}{H}(q)=\overset{+}{H}(p) \Leftrightarrow \overset{-}{H}(q)=\overset{-}{H}(p).$
2. $\overset{+}{H}(q+p)=\overset{+}{H}(q)+\overset{+}{H}(p), \quad \overset{-}{H}(q+p)=\overset{-}{H}(q)+\overset{-}{H}(p).$
3. $\overset{+}{H}(\lambda q)=\lambda \overset{+}{H}(q), \ \overset{-}{H}(\lambda q)=\lambda \overset{-}{H}(q).$
4. $\overset{+}{H}(qp)=\overset{+}{H}(q)\overset{+}{H}(p), \quad \overset{-}{H}(qp)=\overset{-}{H}(p)\overset{-}{H}(q).$
5. $\overset{+}{H}(q^{-1})=\left[\overset{+}{H}(q)\right]^{-1}, \quad \overset{-}{H}(q^{-1})=\left[\overset{-}{H}(q)\right]^{-1}, \quad (N_q)^2 \neq 0.$
6. $\overset{+}{H}(\overline{q})=\varepsilon\left[\overset{+}{H}(q)\right]^T \varepsilon, \ \overset{-}{H}(\overline{q})=\varepsilon\left[\overset{-}{H}(q)\right]^T \varepsilon, \ \varepsilon=\begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}.$
7. $\det\left[\overset{+}{H}(q)\right]=(N_q)^2, \quad \det\left[\overset{-}{H}(q)\right]=(N_q)^2.$
8. $\overset{+}{H}(q)=\mathrm{C}\left[\overset{+}{H}(q)\right]^T \mathrm{C}, \ \mathrm{C}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{C}^T=\mathrm{C}^{-1}=\mathrm{C}, \mathrm{C}^2=I_4.$

Theorem 3. Let q be a unit split quaternion. Matrices generated by operators H^+ and H^- are semi-orthogonal matrices, *i.e.*

$$\left[\overset{+}{H}(q)\right]^T \varepsilon \overset{+}{H}(q)\varepsilon=\overset{+}{H}(q)\varepsilon\left[\overset{+}{H}(q)\right]^T \varepsilon=I_4,$$

$$\left[\overset{-}{H}(q)\right]^T \varepsilon \overset{-}{H}(q)\varepsilon=\overset{-}{H}(q)\varepsilon\left[\overset{-}{H}(q)\right]^T \varepsilon=I_4, \ \varepsilon=\begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}.$$

Corollary 1. Let q be a unit split quaternion. Then Hamilton operators $\overset{+}{h}_q$ and $\overset{-}{h}_q$ represent rotations of x in E_2^4 [8].

5. Semi-Quaternions Algebra

A brief introduction of the semi-quaternions is given by Rosenfeld [25]. In our previous work [11], we also studied some algebraic properties of semi-quaternions algebra. By representing semi-quaternions as four-dimensional vectors and the multiplication of quarternions as matrix-by-vector product, in [15] we investigated properties of matrix associated with a semi-quaternion and examined De Moivre's formula for this matrix, from which the *n*th power of such a matrix can be determined. In [14], we showed the kinematic mapping corresponding to Hamilton operators in semi-Euclidean 4-space is same the kinematic mapping of Blaschke and Grünwald.

A semi-quaternion q is an expression of the form

$$q=a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k} \tag{10}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^2=-1, \ \ \vec{j}^2=\vec{k}^2=0,$$
$$\vec{i}\vec{j}=\vec{k}=-\vec{j}\vec{i}, \ \ \vec{j}\vec{k}=0=-\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i}=\vec{j}=-\vec{i}\vec{k}.$$

The set of all semi-quaternions is denoted by H_s. A semi-quaternion q can also be written as

$$q=z_1+\vec{i}\,z_2,$$

where $z_1=a_0+a_2\,\vec{j}, \ z_2=a_1+a_3\,\vec{j}.$

We express the basic operations in terms of $\vec{i}, \vec{j}, \vec{k}$. The addition becomes as

$$(a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k})+(b_0+b_1\vec{i}+b_2\vec{j}+b_3\vec{k})$$
$$=(a_0+b_0)+(a_1+b_1)\vec{i}+(a_2+b_2)\vec{j}+(a_3+b_3)\vec{k}$$

and the multiplication as

$$\begin{aligned}(a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k})&(b_0+b_1\vec{i}+b_2\vec{j}+b_3\vec{k}) \\ &=(a_0b_0-a_1b_1) \\ &+(a_1b_0+a_0b_1)\vec{i} \\ &+(a_2b_0+a_3b_1+a_0b_2-a_1b_3)\vec{j} \\ &+(a_3b_0-a_2b_1+a_1b_2+a_0b_3)\vec{k}.\end{aligned}$$

Also, this can be written as

$$=\begin{bmatrix} a_0 & -a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Given $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, a_0 is called the *scalar part* of q, denoted by

$$S(q) = a_0,$$

and $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is called the *vector part* of q, denoted by

$$\vec{V}(q) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

The *conjugate* of q is

$$\overline{q} = a_0 - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}.$$

The *norm* of q is

$$N_q = \overline{q}q = q\overline{q} = a_0^2 + a_1^2.$$

If $N_q = 1,$ then q is called a unit semi-quaternion.

The *inverse* of q with $N_q \neq 0,$ is

$$q^{-1} = \frac{1}{N_q}\overline{q}.$$

Clearly $qq^{-1} = 1 + 0\vec{i} + 0\vec{j} + 0\vec{k}.$ Note also that $\overline{qp} = \overline{p}\overline{q}$ and $(qp)^{-1} = p^{-1}q^{-1}.$

6. Split Semi-Quaternions Algebra

A brief introduction of the split semi-quaternions is provided in [19]. In our previous work [10, 16, 22], we also studied some algebraic properties of split semi-quaternions algebra.

A split semi-quaternion q is an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^2 = 1, \quad \vec{j}^2 = \vec{k}^2 = 0,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = 0 = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

The set of all split semi-quaternions is denoted by Hss. We express the basic operations in terms of $\vec{i}, \vec{j}, \vec{k}.$ The addition becomes as

$$\begin{aligned}(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) + (b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\ = (a_0 + b_0) + (a_1 + b_1)\vec{i} \\ + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}\end{aligned}$$

and the multiplication as

$$\begin{aligned}(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k})(b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\ = (a_0b_0 + a_1b_1) \\ + (a_1b_0 + a_0b_1)\vec{i} \\ + (a_2b_0 - a_3b_1 + a_0b_2 + a_1b_3)\vec{j} \\ + (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)\vec{k}.\end{aligned}$$

Also, this can be written as

$$= \begin{bmatrix} a_0 & a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Given $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, a_0 is called the *scalar part* of q, denoted by

$$S(q) = a_0,$$

and $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is called the *vector part* of q, denoted by

$$\vec{V}(q) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

The *conjugate* of q is

$$\overline{q} = a_0 - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}.$$

The *norm* of q is

$$N_q = \overline{q}q = q\overline{q} = a_0^2 - a_1^2.$$

If $N_q = 1,$ then q is called a unit split semi-quaternion.

The *trace* of q is

$$tr(q) = q + \overline{q} = 2a_0.$$

The *inverse* of q with $N_q \neq 0,$ is

$$q^{-1} = \frac{1}{N_q}\overline{q}.$$

Clearly $qq^{-1} = 1 + 0\vec{i} + 0\vec{j} + 0\vec{k}.$ Note also that $\overline{qp} = \overline{p}\overline{q}$ and $(qp)^{-1} = p^{-1}q^{-1}.$

The *scalar product* of two split semi-quaternions $q = S_q + \vec{V}_q$ and $p = S_p + \vec{V}_p$, is defined as

$$\langle q, p\rangle = S_qS_p + \vec{V}_q . \vec{V}_q = a_0b_0 - a_1b_1.$$

The algebra H_{ss} with this product has the 4-dimensional pseudo-Euclidean space structure ${}_2R^4$ with rank 2 semi-metric.

Theorem 4. The set H_{ss}^{1} of unit split semi-quaternions is a subgroup of the group

$$\mathrm{H}_{ss}^{0}=\mathrm{H}_{ss}-\{[0,(0,0,0)]\}.$$

Proof: The proof can be found in [16].

7. Quasi-Quaternions Algebra

A brief introduction of the quasi-quaternions (1/4-quaternion) is provided in [25]. In [19], we studied some algebraic properties of dual quaternions, which is similar to the 1/4-quaternion.

A quasi-quaternion q is an expression of the form

$$q=a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i},\vec{j},\vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^2=\vec{j}^2=\vec{k}^2=0,$$
$$\vec{i}\vec{j}=\vec{k}=-\vec{j}\vec{i},\quad \vec{j}\vec{k}=0=\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i}=0=\vec{i}\vec{k}.$$

The set of all quasi-quaternions is denoted by Hq. We express the basic operations in terms of $\vec{i},\vec{j},\vec{k}$. The addition becomes as

$$\begin{aligned}(a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k})+(b_0+b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\\ =(a_0+b_0)+(a_1+b_1)\vec{i}\\ +(a_2+b_2)\vec{j}+(a_3+b_3)\vec{k}\end{aligned}$$

and the multiplication as

$$\begin{aligned}(a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k})(b_0+b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\\ =(a_0b_0)\\ +(a_1b_0+a_0b_1)\vec{i}\\ +(a_2b_0+a_0b_2)\vec{j}\\ +(a_3b_0-a_2b_1+a_1b_2+a_0b_3)\vec{k}.\end{aligned}$$

Also, this can be written as

$$=\begin{bmatrix}a_0&0&0&0\\a_1&a_0&0&0\\a_2&0&a_0&0\\a_3&-a_2&a_1&a_0\end{bmatrix}\begin{bmatrix}b_0\\b_1\\b_2\\b_3\end{bmatrix}$$

Given $q=a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, a_0 is called the scalar part of q, denoted by

$$S(q)=a_0,$$

and $a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ is called the vector part of q, denoted by

$$\vec{V}(q)=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}.$$

The conjugate of q is

$$\overline{q}=a_0-a_1\vec{i}-a_2\vec{j}-a_3\vec{k}.$$

The norm of q is

$$N_q=\overline{q}q=q\overline{q}=a_0^2.$$

If $N_q=1$, then q is called a unit quasi-quaternion.

The trace of q is

$$tr(q)=q+\overline{q}=2a_0.$$

The inverse of q with $N_q\neq 0$, is

$$q^{-1}=\frac{1}{N_q}\overline{q}.$$

Clearly $qq^{-1}=1+0\vec{i}+0\vec{j}+0\vec{k}$. Note also that $\overline{qp}=\overline{p}\overline{q}$ and $(qp)^{-1}=p^{-1}q^{-1}$.

8. Conclusion

We studied real, split, semi-quaternions, split semi-quaternions, quasi-quaternions algebras and review some important basic algebraic properties and geometric applications of them.

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