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Schrödinger Equation with Double- Cosine and Sine – Squared Potential by Darboux Transformation Method and Supersymmetry

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Abstract: On this paper the one-dimensional Schrödinger equation for the double cosine and sine- squared potential is considered. Here, we construct the first order Darboux transformation and the real valued condition of transformed potential for two corresponding equations. In that case we obtain the transformed of potential and wave function and finally, investigate the supersymmetry aspect of such corresponding equation. Also we show that the first order equation is satisfied by commutative and anti-commutative algebra with the α constant condition at different limit for the x.

Keywords: Double-Cosine Potential, Sine-Squared Potential Darboux Transformation, Supersymmetry, Shape Invariance Potential

1. Introduction

There are several methods to study for the integrability model. One of the methods that we focus here is Darboux transformation. It is well known that the Darboux transformation [1] is one of the major tools for the analysis of physical systems and for finding new solvable systems, using a linear differential operator, Darboux construct solutions of one ordinary differential equation in terms of another ordinary differential equation. It has been shown that the transformation method is useful in finding soliton solutions of the integrable systems [2-4] and constructing supersymmetric quantum mechanical systems [5-7]. Also, more general solvable cases were obtained by means of factorization methods [8] and via lie algebraic approaches [9-13]. Darboux transformation is known as one of the most powerful methods for finding solvable Schrödinger equations with constant mass, in the context of which it is also called supersymmetric factorization method [14]. On the other hand during the past few years there has been great interest in studying class of trigonometric potentials [15]. The solution of such equation may be found by mapping it into a Schrödinger-like equation. So, we take advantage from Darboux transformation and obtain the generalized form of double-cosine and sine-squared potential. The Darboux transformation has been extensively used in quantum mechanics in the search of isospectral potential for exactly Schrödinger equations of constant mass and position - dependent mass [16-21]. So, we take advantage from such transformation and obtain the effective potential, modified wave function and shape invariance condition and generators of supersymmetry algebra for the two corresponding potential. So, this paper is organized as follows: So, we first introduce the one-dimensional Schrödinger equation for the double-cosine and sine -squared potential and apply such transformation to these equations. In that case, we show that the corresponding potential change to new form of potential. Finally, we study the supersymmetry version and shape invariance condition for transformed double-cosine and sine-squared potential.

2. Double-Cosine Potential

First of all we are going to consider a single particle in double-cosine potential which is given by [22],

$$V(x) = \begin{cases} V_1 \cos(x) + V_2 \cos(2x) & 0 < x < 2\pi \\ \infty & x < 0 \ and \ x > 2\pi \end{cases} \tag{1}$$

Where Schrödinger like equation will be as,

$$\left[-\frac{d^2}{dx^2} + V_1\cos(x) + V_2\cos(2x)\right]\psi(x) = E\psi(x), \psi(0) = \psi(2\pi) = 0 \tag{2}$$

The maximum of the potential (1) occurs within the given interval between $x = 0$ and $x = 2\pi$ and has the value of $V_{max} = V_1 + V_2$, while the minimum occurs at $x = \pi$ with the value of $V_{max} = V_2 - V_1$. Clearly, if $V_1 = 0$ and $V_2 > 0$, the problem corresponds to Mathieus equation [23]. Assume the general solution of the differential equation (2) is satisfied the boundary conditions takes the form $\psi(x) = \sin\left(\frac{\pi}{2}\right)f(x)$. In that case, we use such condition and make the second order equation (2) in terms of $f(x)$, which is given by,

$$f''(x) + cot\left(\frac{x}{2}\right)f'(x) - \left[\frac{1}{4} + V_1\cos(x) + 2V_2cos^2(x) - V_2 - E\right]f(x) = 0 \tag{3}$$

Now we choose the following variable,

$$f(x) = exp\left(-\frac{V_1}{2}\right)g(y), y = cos\left(\frac{x}{2}\right) \tag{4}$$

And we obtain,

$$(1-y^2)g''(y) + \left[y\left(1 + 4V_1(1-y^2)\right)\right]g'(y) - [1 - 2V_1 + 4V_2 - 4E - 4(V_1^2 + 8V_2)y^2(1-y^2)]g(y) = 0 \tag{5}$$

So, the exact solution for the E_0 and V_2 are,

$$g_0(x) = 1, E_0 = \frac{1}{4} - \frac{V_1}{2} + V_2, V_2 = -\frac{1}{8}V_1^2 \tag{6}$$

In order to change the equation (2) in form of known polynomial we need to choose the following variable,

$$g(y) = u(y)P(y) \tag{7}$$

So, one can rewrite the equation (5) as,

$$(1-y^2)P''(y) + \left[2(1-y^2)\frac{u'}{u} + y\left(1 + 4V_1(1-y^2)\right)\right]P'(y) + \left[(1-y^2)\frac{u''}{u} + y\left(1 + 4V_1(1-y^2)\right)\frac{u'}{u} - \{1 - 2V_1 + 4V_2 - 4E - 4(V_1^2 + 8V_2)y^2(1-y^2)\}\right]P(y) = 0 \tag{8}$$

Here, we consider the following associated - Legendre differential equation [24-26],

$$(1-y^2)P''^{\alpha,\beta}_{n,m}(y) - [\alpha - \beta + (\alpha + \beta + 2)y]P'^{\alpha,\beta}_{n,m}(y) + \left[n(\alpha + \beta + n + 1) - \frac{m(\alpha+\beta+m) + m(\alpha-\beta)y}{1-y^2}\right]P^{\alpha,\beta}_{n,m}(y) = 0 \tag{9}$$

Also, we compare the equations (8) and (9) to each other and obtain the wave function $u(y)$ and $g(y)$ as,

$$u(y) = e^{-V_1y^2}\left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}}(1-y^2)^{\frac{\alpha+\beta+3}{4}} \tag{10}$$

So, the general form of $g(y)$ and $g(x)$ functions will be following,

$$g(y) = e^{-V_1y^2}\left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}}(1-y^2)^{\frac{\alpha+\beta-1}{4}}P^{\alpha,\beta}_{n,m}, y = cos\left(\frac{x}{2}\right),$$

$$g(x) = e^{-V_1cos^2\left(\frac{x}{2}\right)}\left[\frac{1+cos\left(\frac{x}{2}\right)}{1-cos\left(\frac{x}{2}\right)}\right]^{\left(\frac{\beta-\alpha}{4}\right)} sin\left(\frac{x}{2}\right)^{\left(\frac{\alpha+\beta+3}{2}\right)} P^{\alpha,\beta}_{n,m}(x) \tag{11}$$

Also, we take advantage from comparing (8) and (9) and obtain the V_1, E and $f(x)$;

$$V_1 = \frac{1}{4}\left(m - \alpha - \beta + \frac{3}{2}\right), \tag{12}$$

$$E = \frac{1}{4}\left[(\beta-\alpha)^2 - \left(\alpha + \beta + \frac{7}{4}\right) - \frac{1}{32}\left(m - \alpha - \beta + \frac{3}{2}\right)^2 - n(\alpha + \beta + n + 1) + m\left(\alpha + \beta + m + \frac{1}{2}\right)\right] \tag{13}$$

And,

$$f(x) = e^{-\frac{1}{2}V_1\left(1+2cos^2\left(\frac{x}{2}\right)\right)}\left[\frac{1+cos\left(\frac{x}{2}\right)}{1-cos\left(\frac{x}{2}\right)}\right]^{\left(\frac{\beta-\alpha}{4}\right)} sin\left(\frac{x}{2}\right)^{\left(\frac{\alpha+\beta+3}{2}\right)} P_{n,m}^{\alpha,\beta}(x) \quad (14)$$

3. Trigonometric Sine-Squared Potential

The second example, we consider here is one-dimensional Schrödinger equation for the trigonometric sine-squared potential, which is given by,

$$\left[-\frac{d^2}{dx^2} + V_0 sin^2\left(\frac{x}{a}\right)\right] C(x) = EC(x) \quad (15)$$

Where, $C\left(\frac{\pi a}{2}\right) = C\left(\frac{-\pi a}{2}\right)$.

This boundary condition leads us to consider following change of variable,

$$C(x) = cos\left(\frac{x}{a}\right) f(x) \quad (16)$$

So, one can rewrite equation (15) as,

$$f''(x) = \frac{2}{a} tan\left(\frac{x}{2}\right) f'(x) + \left[\frac{1}{a^2} - E + V_0 sin^2\left(\frac{x}{a}\right)\right] f(x) \quad (17)$$

By putting $x = a\, Arcsin(y)$ in (17), one can obtain,

$$f''(y) = \frac{3y}{1-y^2} f'(y) + \left(\frac{\omega}{1-y^2} - \mu\right) f(y) \quad (18)$$

Where $\omega = 1 + \mu - a^2$ and $\mu = V_0 a^2$. By choosing suitable variable and same as previous case we have,

$$f(y) = u(y)P(y) \quad (19)$$

We substitute equation (19) in (18), we obtain following equation,

$$(1-y^2)P''(y) + \left[2(1-y^2)\frac{u'}{u} - 3y\right] P'(y) + \left[(1-y^2)\frac{u''}{u} - 3y\frac{u'}{u} - \omega + \mu(1-y^2)\right] P(y) = 0 \quad (20)$$

Here, we compare equation (9) and (20) to each other, one can arrive the following expression for $u(y), f(y)$ and $f(x)$ respectively,

$$u(y) = \left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}} (1-y^2)^{\frac{\alpha+\beta-1}{4}} \quad (21)$$

$$f(y) = \left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}} (1-y^2)^{\frac{\alpha+\beta-1}{4}} p_{n,m}^{\alpha,\beta}(y)\, , y = sin\frac{x}{a}$$

And,

$$f(x) = \left(\frac{1+sin\left(\frac{x}{a}\right)}{1-sin\left(\frac{x}{a}\right)}\right)^{\frac{\beta-\alpha}{4}} cos\left(\frac{x}{a}\right)^{\frac{\alpha+\beta-1}{2}} p_{n,m}^{\alpha,\beta}(x) \quad (22)$$

On the other hand this comparing gives us opportunity to obtain the energy spectrum and wave function which are given by,

$$E = \frac{\hbar^2}{2\mu a^2}[n(\alpha+\beta+n)+1] \quad (23)$$

And,

$$C(x) = cos\left(\frac{x}{a}\right)\left(\frac{1+sin\left(\frac{x}{a}\right)}{1-sin\left(\frac{x}{a}\right)}\right)^{\frac{\beta-\alpha}{4}} cos\left(\frac{x}{a}\right)^{\frac{\alpha+\beta-1}{2}} p_{n,m}^{\alpha,\beta}(x) \quad (24)$$

The corresponding energy spectrum always is positive, so we have stable system.

4. Darboux Transformation and Double-Cosine Potential

Now we are going to apply the Darboux transformation to corresponding example such as double-cosine and trigonometric sine-squared potential. So, we simplify the equation (5) as,

$$Fg_{yy} + Gg_y - Vg = 0 \quad (25)$$

Where F and G and V are respectively,

$$F = (1-y^2)\, , G = y + 4V_1 y(1-y^2)$$

$$V = 1 - 2V_1 + 4V_2 - 4E - 4(V_1^2 + 8V_2)y^2(1-y^2) \quad (26)$$

In here we introduce the new variable as η which play important role in Darboux transformation. So, we can write the above equation with η variable which is given by,

$$ig_t + \eta g = 0\, , \eta = F\partial_{yy} + G\partial_y - V \quad (27)$$

The Darboux transformation help us to write the equation (25) and (27) as a new form with different potential as,

$$i\hat{g}_t + \hat{\eta} g = 0\, , \hat{\eta} = F\partial_{yy} + G\partial_y - \hat{V} \quad (28)$$

Where $V \neq \hat{V}$, implying $g(y) \neq \hat{g}(y)$. We introduce transformation operator Δ as,

$$\Delta(i\partial + \eta) = (i\partial + \hat{\eta})\Delta \quad (29)$$

Which are called Darboux transformation operator for the Hamiltonian η and $\hat{\eta}$, respectively.

The operator Δ transforms any solution $f(y)$ into a new solution,

$$\hat{f}(y) = \Delta f(y) \quad (30)$$

Let Darboux transformation operator be the form of a linear, first- order differential operator,

$$\Delta = A + B\partial_y \quad (31)$$

Where we take special case as $A = B$. In order to find A or B, we consider the explicit form of Δ and $\hat{\Delta}$ in form of the Darboux transformation and apply it to the solution $g(y)$, so

$$\Delta(i\partial_t + \eta)g(y) = (i\partial_t + \hat{\eta})\Delta g(y) \tag{32}$$

Making linear independence of $g(y)$ and its partial derivatives, we collect their respective coefficients and equal them to zero, from which one can obtain the following value for the functions A and $\hat{V}$.

$$2F = (f)_y B \;\Rightarrow\; B = \frac{-1}{y}(1-y^2) \tag{33}$$

So, the Darboux transformation operator will be as,

$$\Delta = \frac{-1}{y}(1-y^2)\left(1+\partial_y\right) \; or \; \Delta = -cot\left(\frac{x}{2}\right)\left[sin\left(\frac{x}{2}\right) - 2\frac{d}{dx}\right] \tag{34}$$

The relation between V and $\hat{V}$ will be as,

$$\hat{V} = v + \frac{2}{y^2} - 2\frac{1+y^2}{y} + 4V_1 - y^2(16V_1 + 1) \tag{35}$$

Now, we achieve the generalized form of wave function which is corresponding to usual wave function $\hat{f}(x)$ as,

$$\hat{f}(y,t) = \Delta f(y,t) = \frac{-1}{y}(1-y^2)\left(1+\partial_y\right)f(y,t), y = sin\left(\frac{x}{2}\right)$$

$$\hat{f}(x,t) =$$
$$\frac{cos^2\left(\frac{x}{2}\right)}{sin\left(\frac{x}{2}\right)}\left\{\left[1 + \frac{1}{2}V_1\sin(2x) - \frac{\beta-\alpha}{sin\left(\frac{x}{2}\right)} + \frac{\alpha+\beta+3}{4}cot\left(\frac{x}{2}\right)\right]f(x,t) + e^{-\frac{1}{2}V_1\left(1+2cos^2\left(\frac{x}{2}\right)\right)}\left\{\left[\frac{1+cos\left(\frac{x}{2}\right)}{1-cos\left(\frac{x}{2}\right)}\right]^{\frac{\beta-\alpha}{4}} sin\left(\frac{x}{2}\right)^{\frac{\alpha+\beta+3}{2}} p'^{\alpha,\beta}_{n,m}(x)\right\}\right\} \tag{36}$$

5. Darboux Transformation and Trigonometric Sine-Squared Potential

The one-dimensional Schrödinger equation for the trigonometric sine-squared potential is given by,

$$C''(y) = \frac{3y}{1-y^2}C'(y) + \left(\frac{\omega}{1-y^2} - \mu\right)C(y) \tag{37}$$

Thus, the trigonometric sine-squared potential equation (37) is,

$$(1-y^2)C_{yy} - 3yC_y - \left(\omega - \mu(1-y^2)\right)C = 0 \tag{38}$$

By taking $F = (1-y^2)\,, G = -3y$ and the potential $V = \omega - \mu(1-y^2)$, we can rewrite the above equation as,

$$F\partial_{yy} + G\partial_y - V = 0 \tag{39}$$

And

$$iC_t + \eta C = 0\,, \eta = F\partial_{yy} + G\partial_y - V \tag{40}$$

In order to have same equation as (38) and (40) with different of potential, we have to write following equation,

$$i\hat{C}_t + \hat{\eta}C = 0\,, \hat{\eta} = F\partial_{yy} + G\partial_y - \hat{V} \tag{41}$$

Where $V \neq \hat{V}$ and this imply$C \neq \hat{C}$. In order to obtain the modified potential $\hat{V}$ and corresponding wave function for equation (41), we introduce operator Δ which are called Darboux transformation. The general form of such Durboux transformation will be as,

$$\Delta = A + B\partial_y \tag{42}$$

For simplicity we suppose $A = B$. By using the following property of Darboux transformation,

$$\Delta(i\partial_t + \eta) = (i\partial_t + \hat{\eta}) \tag{43}$$

One can obtain the generalized form of wave function which is corresponding to usual wave function C as,

$$\hat{C}(y,t) = \Delta C(y,t) = \left(1+\partial_y\right)C(y,t), y = sin\frac{x}{a}$$

$$\hat{C}(x,t) = \left[1 - \frac{\alpha+\beta+1}{2}\left(\frac{tan\frac{x}{a}}{cos\frac{x}{a}}\right) + \frac{\beta-\alpha}{cos\frac{x}{a}}\right]C(x,t) + a\left[\frac{1+sin\frac{x}{a}}{1-sin\frac{x}{a}}\right]^{\left(\frac{\beta-\alpha}{4}\right)} cos\left(\frac{x}{a}\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} P'^{\alpha,\beta}_{n,m}(x) \tag{44}$$

In order to obtain the parameter A, we need to use the equation (38) and (43) in following expression,

$$\Delta\left(i\partial_t + F\partial_{yy} + G\partial_y - V\right)C = \left(i\partial_t + F\partial_{yy} + G\partial_y - \hat{V}\right)\Delta C \tag{45}$$

Making linear independence of C and its partial derivatives, we collect their respective coefficients and equal them to zero, so we can obtain A as,

$$A = \alpha\sqrt{F} = \alpha\sqrt{1-y^2} \tag{46}$$

And the modified potential is given by,

$$\hat{V} = a^2V - a^2E + 2tan^2\frac{x}{a} + 2sin\frac{x}{a} \tag{47}$$

Where $C(y,t) = e^{-\frac{iEt}{\hbar}}C(y)$.

6. Supersymmetry and Darboux Transformation

In what follows, we will prove that the formalism of supersymmetry for our generalized trigonometric Double-Cosine potential equation is equivalent to the Darboux transformation. So, here we introduce the following self-adjoint operator,

$$(i\partial_t + \theta)^* = i\theta + \theta \quad (48)$$

Taking the operation of conjugation on Darboux transformation (21), we obtain,

$$(i\theta_t + \eta)\Delta^* = \Delta^*(i\partial_t + \hat{\eta}) \quad (49)$$

Where the operator Δ^* ad joint to $\Delta = \frac{-1}{y}(1-y^2)(1+\partial_y)$ in double-cosine system is given by,

$$\Delta^* = \frac{-1}{y}(1-y^2)(1-\partial_y) \quad (50)$$

Eqs. (29) and (30) can be rewritten by single matrix equation,

$$\begin{bmatrix} i\partial_t + \eta & 0 \\ 0 & i\partial_t + \hat{\eta} \end{bmatrix} \begin{bmatrix} f \\ \hat{f} \end{bmatrix} = 0 \quad (51)$$

We assume that $H = diag(\eta, \hat{\eta})$ and $F = (f, \hat{f})^T$, so the above equation can be written as

$$[i\partial_t + H]F = 0 \quad (52)$$

Two supercharge operator Q and Q^* are defined by following matrices,

$$Q = \begin{bmatrix} 0 & 0 \\ \Delta & 0 \end{bmatrix}, Q^* = \begin{bmatrix} 0 & \Delta^* \\ 0 & 0 \end{bmatrix} \quad (53)$$

Where Δ and Δ^* are the operator given by Eqs. (36) and (50), respectively. One can show that the Hamiltonian H satisfies the following expressions,

$$\{Q, Q\} = \{Q^*, Q^*\} = 0$$

$$[Q, i\partial_t + H] = [i\partial_t + H, Q]$$

$$[Q^*, i\partial_t + H] = [i\partial_t + H, Q^*] \quad (54)$$

Considering the complementing relations of the supersymmetry algebra; the anti-commutators $\{Q, Q\}$ and $\{Q^*, Q^*\}$. We obtain the operators $R = Q^*Q$ and $\hat{R} = QQ^*$ and consider the relations of them with our Hamiltonian η and $\hat{\eta}$ So, one obtain the R and $\hat{R}$ as follow,

$$R = |\alpha|^2[F(1-\partial_{yy}) - (F)_y(\partial_y + 1)] \quad (55)$$

And

$$\hat{R} = |\alpha|^2\left[F(1-\partial_{yy}) - (F)_y(\partial_y + 1) - \frac{1}{2}(F)_{yy} + \frac{F_y}{2F}\right] \quad (56)$$

Where, the index y will be derivative with respect to$y$. In order to have shape invariance and supersymmetric algebra we need to obtain, $\hat{R} - R$. If such value be constant and zero there is some supersymmetry partner for such systems. Otherwise we need to apply some condition in $\hat{R} - R$ to have constant value. So, we will arrive at following equation for the $\hat{R} - R$

$$\hat{R} - R = |\alpha|^2\left[1 + sin\left(\frac{x}{2}\right)\right] \quad (57)$$

By using the condition $\Psi(0) = \Psi(2\pi) = 0$ and, $x \in [0,2\pi]$ the value of $\hat{R} - R$ be zero or function of α, and we have supersymmetry for the Double-Cosine potential in case of α constant. So, in generally we can say that there is shape invariance for usual and generalized potential in above mentioned condition. The shape invariance for the potential is $\hat{V} = V + constant$.

In second example we consider sine-squared potential, so Δ and Δ^* will be as,

$$\Delta = \alpha\sqrt{1-y^2}(1+\partial_y) \quad (58)$$

And

$$\Delta^* = \alpha\sqrt{1-y^2}(1-\partial_y) \quad (59)$$

For the sine-squared potential also we consider information from previous section such as equations (51-54) and $R = Q^*Q$ and $\hat{R} = QQ^*$, one can obtain R and $\hat{R}$ as(55) and (56).

Otherwise we need to apply some condition in $\hat{R} - R$ to have constant value. So, one can obtain the following equation for the$\hat{R} - R$

$$\hat{R} - R = |\alpha|^2\left(1 - \frac{y}{1-y^2}\right) \quad (60)$$

We mention here that if we want to supersymmetry algebra we need to have also the following commutation relation, and also anti-commutation relation between Q and Q^+.

$$\{Q, Q^+\} = H, \{Q, Q\} = \{Q^+, Q^+\} = 0 \quad (61)$$

If we look at the equation (61) we need to apply the condition $\hat{R} - R$ be zero or constant, in the corresponding condition $\hat{R} - R$ be zero or function of α (α is constant). So, we have supersymmetry system, and it means that two potentials are satisfied by the shape invariance condition.

7. Conclusion

In this paper, the Double-Cosine potential equation was studied. The first-order Darboux transformation applied to the corresponding equation. In order to relate between supersymmetry and Darboux transformation we discussed the supersymmetry algebra and its commutation and anti-commutation relations. It shown that for the satisfying such anti-commutation supercharges the $\hat{R} - R$ must be constant. Also, we applied this condition on the $\hat{R} - R$ and shown that in the interval $[0,2\pi]$, α must be constant. This condition completely guarantees relation between supersymmetry and Darboux transformation. This result play important role for any solvable, non-solvable and quasi-solvable systems.

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