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Improving Risk Assessment and Pricing with Dividend Barriers and Dependence Modelling: An Extension of the Cramer-Lundberg Model with Spearman Copulas

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Abstract: The compound Poisson risk model is a probabilistic model commonly used to evaluate the financial risk of an insurance company. This model assumes that claims arrive according to a Poisson process and that claim sizes follow an independent probability distribution. This paper presents an extension of this model, incorporating a dividend payment strategy with a constant threshold b. This extension allows for a better representation of the reality of insurance companies, which typically pay dividends to their shareholders. The traditional assumption of independence between claim sizes and interclaim intervals is also relaxed in this extension. This relaxation allows for recognition of the potential dependence between these variables, which can have a significant impact on the company's ruin probability. The Spearman copula is used to model the dependent structure between claim sizes and interclaim intervals. The Spearman copula is a function that measures the degree of dependence between two variables. It is used in many fields, including insurance, finance, and statistics. The study focuses on the Laplace transform of the adjusted penalty function. The adjusted penalty function is a function that allows for the determination of the company's ruin probability. The results of the study show that the dependence between claim sizes and interclaim intervals can have a significant impact on the company's ruin probability. In particular, positive dependence between these variables can increase the ruin probability.

Keywords: Gerber-Shiu Functions, Dependency, Integro-differential Equation, Laplace Transformation, Probability of Ruin

1. Introduction

Mathematical models continue to be developed every day in response to a better understanding of risks and their evolution with the simplifying assumption of independence between the random variables involved in risk modellinge (See for example references [10, 15, 18]). However, in certain practical contexts, this assumption should be relaxed because it is unsuitable and too restrictive. For example, in flood damage insurance, the occurrence of several floods in a short period of time can generate large amounts of damage and therefore large amounts of claims. In earthquake insurance, the opposite is true, because in a high-risk area, the longer the time between two earthquakes, the greater the impact of the second earthquake due to the accumulation of energy.

To compensate for this inadequacy due to the simplifying assumption of independence between the random variables involved in risk modelling, in many works, dependence between certain random variables is incorporated into the risk model, in particular the variables amount of claims and time between claims, thanks to the Farlie Gumbel Morgenstern copula see for example references [4, 5, 9, 11, 16]. Although this copula is commonly used in the literature, it has certain limitations. It fails to model tail dependencies (see references [2, 3]).

To remedy the inadequacy of the Farlie Gumbel Morgenstern copula, this article proposes a composite Poisson risk model that takes account of the reality of insurance companies. This model includes the dependence between the variables claim amounts and inter-claim periods via the Spearman copula, as well as a strategy of paying dividends to shareholders with a constant threshold b. In the risk model with a constant threshold dividend payment b, when the surplus process reaches the constant threshold barrier b set, bonuses are paid in full to shareholder. Denoting by $U_b\left(t\right)$ the surplus process in the presence of the dividend barrier of level b (with $U_b\left(0\right)=u$), the model follows the following dynamics:

$$dU_b\left(t\right)=\left\{\begin{array}{lll} cdt-dS\left(t\right) & if & U_b\left(t\right)<b, \\ -dS\left(t\right) & if & U_b\left(t\right)=b \end{array}\right. \quad (t\geq 0) \qquad (1)$$

where:

1. $U_b\left(t\right)$ is the surplus process in the presence of a threshold dividend barrier b (with $U_b\left(0\right)=u$ the initial surplus and $0<u\leq b$);
2. c is the constant rate of premium received by the insurer per unit time;
3. $S\left(t\right)=\sum\limits_{i=1}^{N(t)}X_i$ is the aggregate Poisson loss process composed of:
   a. $\{N\left(t\right),t\geq 0\}$ the total number of claims recorded up to time t which follows a Poisson process of intensity λ ¿ 0; (Note that $S\left(t\right)=0$ if $N\left(t\right)=0$);
   b. $\{X_i,i\geq 1\}$ a sequence of random variables representing the individual amounts of claims with common density function f and distribution function F and assumed to have an exponential distribution with parameter β.
4. The inter-claim times $\{V_i,i\geq 1\}$ form a sequence of random variables with exponential law of parameter λ, probability density function $k\left(t\right)$ $=\lambda e^{-\lambda t}$ and distribution function $K\left(t\right)=1-e^{-\lambda t}$;

The aim of this work is to determine the probability of ruin in the risk model defined by the relation (1). To achieve this, the rest of the article is structured as follows: In section (2), In section (1) and present the tail dependency structure. In section (3), wwe determine the integro-differential equation satisfied by the Gerber Shiu function in the risk model defined by relation (1), then The next step involves determining the Laplace transform of the Gerber Shiu function, which proves highly suitable for risk quantification. This is followed by the derivation of the probability of ruin within the risk model defined by relation (1).

2. Preliminaries

2.1. Probability of Ruin

The insurer's probability of ruin is the probability of ruin occurring either over a finite horizon or over an infinite horizon. In the latter case, it is referred to as the ultimate probability of risk.

Let τ be the moment of ruin of the insurance company. τ is defined by:

$$\tau=\inf\left\{t\geq 0,U_b\left(t\right)<0\right\}. \qquad (2)$$

In the scenario where the probability of ruin remains invariably zero, the convention dictates assigning $\tau=\infty$. Under this condition, the following inequality holds

$$U_b\left(t\right)\geq 0,\forall t\geq 0.$$

The probability of ultimate ruin is defined by:

$$\psi_b\left(u\right)=\mathbb{P}\left[\tau<\infty\left|U_b\left(0\right)=u\right.\right] \qquad (3)$$

2.2. Gerber-Shiu Discounted Penalty Function

The Gerber-Shiu expected penalty function, or Gerber-Shiu function, first appeared in 1998 in the work of Gerber and Shiu. Today, this function is of great research interest. Its analysis remains a central issue in both insurance and finance, as it is a valuable tool not only for studying the probability of ruin, but also for calculating pension and reinsurance premiums, pricing options, etc. It is defined by:

$$\phi_b\left(u\right)=\mathbb{E}\left[e^{-\delta\tau}w\left(U_b\left(\tau^-\right),\left|U_b\left(\tau\right)\right|\right)\mathbf{1}_{\{\tau<\infty\}}\left|U_b\left(0\right)=u\right.\right] \qquad (4)$$

where:

1. τ is the instant of ruin defined by the relation (2);
2. τ^- is the instant just before ruin;
3. δ is a force of interest;
4. The penalty function $w\left(x,y\right)$ is a positive function of the surplus just before ruin $U_b\left(\tau^-\right)$ and the deficit at ruin $\left|U_b\left(\tau\right)\right|$, $\forall\, x,y\geq 0$;
5. **1** is the indicator function which is 1 if event A occurs and 0 otherwise.

2.3. Dependency Model Based on Spearman's Copula

Copulas, introduced by Abe Sklar in 1959, are an innovative and relevant tool for introducing dependency between several random variables to introduce dependence between several random variables. Given the marginal distribution functions of several random variables, copulas can be used to establish their joint distribution function. Copulas are now a basic tool for modelling multivariate distributions in finance, insurance and hydrology. Reference works on copula theory include Joe (see [7]) and Nelsen (see [6]). In this paper, the dependence structure is provided by the Spearman copula. It is defined for any $(u,v) \in [0,1]^2$ and $\alpha \in [0,1]$ by:

$$C_\alpha (u,v) = (1-\alpha) C_I (u,v) + \alpha C_M (u,v), \tag{5}$$

Where:

$C_I (u,v) = uv$; $C_M (u,v) = \min (u,v)$; α is the dependence parameter.

Spearman's copula allows for introducing positive dependence as well as tail dependencies in many situations. It also encompasses independence when $\alpha = 0$. Using the formula (5), the joint distribution function of the random vector representing claim amounts and inter-arrival times (X,V) is given by (see [3]):

$$F_{X,V} (x,t) = C_\alpha (F_X (x), F_V (t)) = (1-\alpha) F_I (x,t) + \alpha F_M (x,t), \tag{6}$$

where F_X, F_V are the marginal distributions of the random variables X and V.

To preclude ruin as a certain event, the following net profit condition is assumed:

$$\mathbb{E} (cW - X) > 0. \tag{7}$$

The relationship (7) is equivalent to $c > \frac{\beta}{\lambda}$.

In the risk model defined by the equation (1), the Gerber-Shiu function $\phi_b (u)$ takes the following form (see reference [2] or [3]):

$$\phi_b (u) = (1-\alpha) (I_{b,1} (u) + I_{b,2} (u)) + \alpha (I_{b,3} (u) + I_{b,4} (u)), \tag{8}$$

where

$$\begin{aligned}
I_{b,1} (u) &= \int_0^{\frac{b-u}{c}} \int_0^{u+ct} e^{-\delta t} \phi_b (u+ct-x) \, dF_I (x,t) + \int_0^{\frac{b-u}{c}} \int_{u+ct}^{\infty} e^{-\delta t} w (u+ct, x-u-ct) \, dF_I (x,t), \\
I_{b,2} (u) &= \int_{\frac{b-u}{c}}^{\infty} \int_0^{b} e^{-\delta t} \phi_b (b-x) \, dF_I (x,t) + \int_{\frac{b-u}{c}}^{\infty} \int_b^{\infty} e^{-\delta t} w (b, x-b) \, dF_I (x,t). \\
I_{b,3} (u) &= \int_0^{\frac{b-u}{c}} \int_0^{u+ct} e^{-\delta t} \phi_b (u+ct-x) \, dF_M (x,t) + \int_0^{\frac{b-u}{c}} \int_{u+ct}^{\infty} e^{-\delta t} w (u+ct, x-u-ct) \, dF_M (x,t), \\
I_{b,4} (u) &= \int_{\frac{b-u}{c}}^{\infty} \int_0^{b} e^{-\delta t} \phi_b (b-x) \, dF_M (x,t) + \int_{\frac{b-u}{c}}^{\infty} \int_b^{\infty} e^{-\delta t} w (b, x-b) \, dF_M (x,t).
\end{aligned}$$

Indeed, in order to derive the integro-differential equation satisfied by the Gerber-Shiu function $\phi_b (u)$ in the risk model defined by equation (1), the following approach has been adopted:

1. The first claim occurs at time t before the surplus process reaches the barrier b, ($t < \frac{b-u}{c}$). Its amount, x, satisfies $x < u + ct$.
2. The first claim occurs at time t before the surplus process has reached the barrier b, ($t < \frac{b-u}{c}$). Its amount, x, satisfies $u + ct < x$.
3. The first claim occurs at time t after the surplus process has reached the barrier b, ($t > \frac{b-u}{c}$). Its amount, x, satisfies $x < b$.
4. The first claim occurs at time t after the surplus process has reached the barrier b, ($t > \frac{b-u}{c}$). Its amount, x, satisfies $x > b$.

By conditioning on the time and the amount of the first claim and taking into account the various preceding scenarios, it can be shown that:

$$\begin{aligned}
I_{b,1} (u) &= \int_0^{\frac{b-u}{c}} \int_0^{u+ct} e^{-\delta t} \phi_b (u+ct-x) \, dF_I (x,t) + \int_0^{\frac{b-u}{c}} \int_{u+ct}^{\infty} e^{-\delta t} w (u+ct, x-u-ct) \, dF_I (x,t), \\
I_{b,2} (u) &= \int_{\frac{b-u}{c}}^{\infty} \int_0^{b} e^{-\delta t} \phi_b (b-x) \, dF_I (x,t) + \int_{\frac{b-u}{c}}^{\infty} \int_b^{\infty} e^{-\delta t} w (b, x-b) \, dF_I (x,t).
\end{aligned}$$

By defining $I_b (u) = I_{b,1} (u) + I_{b,2} (u)$, t can be shown that:

$$\begin{aligned}
I_b (u) &= \lambda \int_0^{\frac{b-u}{c}} \int_0^{u+ct} e^{-(\delta+\lambda)t} \phi_b (u+ct-x) f_X (x) \, dxdt + \lambda \int_0^{\frac{b-u}{c}} \int_{u+ct}^{\infty} e^{-\delta \tau} w (u+ct, x-u-ct) f_X (x) \, dxdt \\
&\quad + \lambda \int_{\frac{b-u}{c}}^{\infty} \int_0^{b} e^{-(\delta+\lambda)t} \phi_b (b-x) f_X (x) \, dxdt + \lambda \int_{\frac{b-u}{c}}^{\infty} \int_b^{\infty} e^{-\delta \tau} w (b, x-b) f_X (x) \, dxdt
\end{aligned} \tag{9}$$

The notation of equation (9) can be simplified by introducing:

$$\omega(u) = \int_u^{\infty} w(u, x-u) f_X(x)\, dx; \tag{10}$$

$$\sigma_b(u) = \int_0^{u} \phi_b(u-x) f_X(x)\, dx + \omega(u) \tag{11}$$

The equation (9) becomes:

$$I_b(u) = \lambda \int_0^{\frac{b-u}{c}} e^{-(\delta+\lambda)t} \sigma_b(u+ct)\, dt + \lambda \int_{\frac{b-u}{c}}^{\infty} e^{-(\delta+\lambda)t} \sigma_b(b)\, dt \tag{12}$$

By setting $s = u + ct$, equation (12) becomes:

$$I_b(u) = \frac{\lambda}{c}\left[\int_u^{\frac{b-u}{c}} e^{-(\delta+\lambda)\frac{s-u}{c}} \sigma_b(s)\, ds + \int_{\frac{b-u}{c}}^{\infty} e^{-(\delta+\lambda)\frac{s-u}{c}} \sigma_b(b)\, ds\right] \tag{13}$$

The equation (13) can be expressed simply as:

$$I_b(u) = \frac{\lambda}{c} \int_u^{\infty} e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)} \sigma_b(s \wedge b)\, ds, \tag{14}$$

where $s \wedge b = \min(s, b)$.

Now, let's determine the integrals $I_{b,3}(u)$ and $I_{b,4}(u)$.

The support of copula C_M is $D = \left\{(u,v) \in [0,1]^2 : u = v\right\}$.

On the domain $[0,1]^2 \setminus D$, $\frac{\partial^2 C_M}{\partial u \partial v} = 0$.

On D, C_M follows a uniform distribution.

As the dependence structure between the claim amounts and inter-arrival times is described by copula C_M, they are monotonic, and almost surely there exists an increasing function l such that $X = l(V)$ (see Nelsen 2006, page 27). It follows that (see [3]):

$$l(t) = \frac{\lambda}{\beta} t \tag{15}$$

The joint distribution $F_{X,V}(x,t)$ of the random vector (X,V) is singular, with its support being the domain

$$D' = \left\{(x,t) : x = \frac{\lambda}{\beta} t\right\}. \tag{16}$$

Its distribution is $G(t) = F_M(l(t), t) = 1 - e^{-\lambda t}$ on the domain $D' = \left\{(x,t) : x = \frac{\lambda}{\beta} t\right\}$.

The integral $I_{b,3}(u)$ becomes:

$$\begin{aligned} I_{b,3}(u) &= \int_0^{\frac{b-u}{c}} \int_0^{u+ct} e^{-\delta t} \phi_b(u+ct-x)\, dF_M(x,t) + \int_0^{\frac{b-u}{c}} \int_{u+ct}^{\infty} e^{-\delta t} w(u+ct, x-u-ct)\, dF_M(x,t) \\ &= \int_K e^{-\delta t} \phi_b\left(u+ct-\frac{\lambda}{\beta}t\right) dG(t) + \int_J e^{-\delta t} \phi_b\left(u+ct-\frac{\lambda}{\beta}t\right) dG(t) \end{aligned} \tag{17}$$

where $K = \left\{t \in \mathbb{R}^+ : 0 \le t \le \frac{b-u}{c}, 0 \le x = \frac{\lambda}{\beta} t \le u + ct\right\} = \left[0, \frac{b-u}{c}\right]$,car $c > \frac{\lambda}{\beta}$ (solvency condition: $\mathbb{E}[cV - X] > 0$) and $u \ge 0$.

And $K = \left\{t \in \mathbb{R}^+ : 0 \le t \le \frac{b-u}{c}, x = \frac{\lambda}{\beta} t \ge u + ct\right\} = \emptyset$,as $c > \frac{\lambda}{\beta}$ and $u \ge 0$.

Using equation (15), the integral $I_{b,3}(u)$ can be expressed as:

$$I_{b,3}(u) = \lambda \int_0^{\frac{b-u}{c}} e^{-(\delta+\lambda)t} \phi_b\left(u+ct-\frac{\lambda}{\beta}t\right) dt. \tag{18}$$

By analogy, it can be shown that:

$$\begin{aligned} I_{b,4}(u) &= \int_{\frac{b-u}{c}}^{\infty} \int_0^{b} e^{-\delta t} \phi_b(b-x)\, dF_M(x,t) + \int_{\frac{b-u}{c}}^{\infty} \int_b^{\infty} e^{-\delta t} w(b, x-b)\, dF_M(x,t) \\ &= \int_{K'} e^{-\delta t} \phi_b(b-x)\, dG(t) + \int_{J'} e^{-\delta t} w(b, x-b)\, dG(t), \end{aligned} \tag{19}$$

where $K' = \left\{t \in \mathbb{R}^+ : t \geq \frac{b-u}{c}, 0 \leq x = \frac{\lambda}{\beta}t \leq b\right\} = \left[\frac{b-u}{c}; \frac{b\beta}{\lambda}\right]$, as $c > \frac{\lambda}{\beta}$ (solvency condition: $\mathbb{E}\left[cV - X\right] > 0$) and $u \geq 0$.

And,

$J' = \left\{t \in \mathbb{R}^+ : t \geq \frac{b-u}{c}, x = \frac{\lambda}{\beta}t \geq u + ct\right\} = \left[\frac{b\beta}{\lambda}, +\infty\right[$,as $c > \frac{\lambda}{\beta}$ and $u \geq 0$.

The equation (19) takes the form

$$\begin{aligned} I_{b,4}\left(u\right) &= \int_{\frac{b-u}{c}}^{\frac{b\beta}{\lambda}} e^{-\delta t}\phi_b\left(b - \frac{\lambda}{\beta}t\right)dG\left(t\right) + \int_{\frac{b\beta}{\lambda}}^{\infty} e^{-\delta t}w\left(b, \frac{\lambda}{\beta}t - b\right)dG\left(t\right) \\ &= \lambda\int_{\frac{b-u}{c}}^{\frac{b\beta}{\lambda}} e^{-(\delta+\lambda)t}\phi_b\left(b - \frac{\lambda}{\beta}t\right)dt + \lambda\int_{\frac{b\beta}{\lambda}}^{\infty} e^{-(\delta+\lambda)t}w\left(b, \frac{\lambda}{\beta}t - b\right)dt \end{aligned} \tag{20}$$

Let's define $\tilde{I}_b\left(u\right) = I_{b,3}\left(u\right) + I_{b,4}\left(u\right)$. By using equations (18) and (20), It can be shown that:

$$\begin{aligned} \tilde{I}_b\left(u\right) &= \lambda\int_0^{\frac{b-u}{c}} e^{-(\delta+\lambda)t}\phi_b\left(u + ct - \frac{\lambda}{\beta}t\right)dt + \lambda\int_{\frac{b-u}{c}}^{\frac{b\beta}{\lambda}} e^{-(\delta+\lambda)t}\phi_b\left(b - x\right)dt \\ &+\lambda\int_{\frac{b\beta}{\lambda}}^{\infty} e^{-(\delta+\lambda)t}w\left(b, \frac{\lambda}{\beta}t - b\right)dt \end{aligned} \tag{21}$$

The equation (21) can be expressed as:

$$\tilde{I}_b\left(u\right) = \lambda\int_0^{\frac{b\beta}{\lambda}} e^{-(\delta+\lambda)t}\phi_b\left((u+ct)\wedge b\right)dt + \lambda\int_{\frac{b\beta}{\lambda}}^{\infty} e^{-(\delta+\lambda)t}w\left(b, \frac{\lambda}{\beta}t - b\right)dt \tag{22}$$

By making a change of variable in each of the integrals in equation (22) $s = u + ct$ and $s = \frac{\lambda}{\beta}t - b$, it can be shown that::

$$\tilde{I}_b\left(u\right) = \frac{\lambda}{c}\int_u^{u+\frac{b\beta c}{\lambda}} e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)}\phi_b\left(s\wedge b\right)ds + \beta\int_0^{\infty} e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w\left(b, s\right)ds \tag{23}$$

From equations (8), (14) and (23), the Gerber-Shiu function $\phi_b\left(u\right)$ can be expressed as:

$$\begin{aligned} \phi_b\left(u\right) &= (1-\alpha)\left(\frac{\lambda}{c}\int_u^{\infty} e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)}\sigma_b\left(s\wedge b\right)ds\right) + \alpha\left(\frac{\lambda}{c}\int_u^{u+\frac{b\beta c}{\lambda}} e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)}\phi_b\left(s\wedge b\right)ds\right. \\ &\left. + \beta\int_0^{\infty} e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w\left(b, s\right)ds\right) \end{aligned} \tag{24}$$

3. Main Results

This section presents the main results.

Theorem 3.1. The ultimate probability of ruin $\psi_b\left(u\right)$ has the expression:

$$\psi_b\left(u\right) = \frac{\lambda}{c\beta}\left(1-\alpha\right)e^{-\frac{(c\beta-\lambda)+\lambda\alpha}{c}u},\ 0 \leq u \leq b. \tag{25}$$

To prove the theorem, consider the following lemma.

Henceforth, the Laplace transform of a function $f(x)$ will be denoted as

$$\widehat{f}(s) = \int_0^{\infty} e^{-sx}f\left(x\right)dx$$

Lemma 3.1. The Laplace transform of the ultimate probability of failure has the expression:

$$\widehat{\psi}_b\left(s\right) = \frac{cs\left(s+\beta\right)\psi_b\left(0\right) + \lambda\alpha\left(s+\beta\right)e^{-b\beta}\left(\psi_b\left(b\right) - 1\right) - \lambda\left(1-\alpha\right)s}{s^2\left(cs + c\beta - \lambda\left(1-\alpha\right)\right)} \tag{26}$$

For the proof of the lemma (3.1), consider the following lemma.
Let $\mathcal{D} = \frac{d}{du}(\cdot)$ and $\mathcal{I}$ denote the differentiation and identity operators respectively.
Lemma 3.2. The Gerber-Shiu function $\phi(u)$ satisfies the following integral-differential equation:

$$\begin{aligned}\left(\mathcal{D} - \frac{\delta+\lambda-\lambda\alpha}{c}\mathcal{I}\right)\phi_b(u) &= -\frac{\lambda}{c}(1-\alpha)\sigma_b(u) + \frac{\lambda\alpha}{c}e^{-\left(\frac{\delta+\lambda}{\lambda}\right)b\beta c}\phi_b(b) \\ &\quad -\alpha\beta\left(\frac{\delta+\lambda}{c}\right)\int_0^\infty e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w(b,s)\,ds, \qquad (27)\end{aligned}$$

Proof [Proof of the lemma (3.2)] Let us derive the function $\phi_b(u)$ in the relation (24) with respect to u.

$$\begin{aligned}\phi_b'(u) &= \frac{\lambda}{c}(1-\alpha)\left(\frac{\delta+\lambda}{c}\right)\int_u^\infty e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)}\sigma_b(s\wedge b)\,ds + \frac{\lambda\alpha}{c}\left(\frac{\delta+\lambda}{c}\right)\int_u^{u+\frac{b\beta c}{\lambda}} e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)}\phi_b(s\wedge b)\,ds \\ &\quad -\frac{\lambda}{c}(1-\alpha)\sigma_b(u) - \frac{\lambda\alpha}{c}\phi_b(u) + \frac{\lambda\alpha}{c}e^{-\left(\frac{\delta+\lambda}{\lambda}\right)b\beta c}\phi_b(b) \qquad (28)\end{aligned}$$

Using the formulas (24) and (28), let's calculate: $g(u) = \left(\mathcal{D} - \frac{\delta+\lambda-\lambda\alpha}{c}\mathcal{I}\right)\phi_b(u)$.

$$g(u) = -\frac{\lambda}{c}(1-\alpha)\sigma_b(u) + \frac{\lambda\alpha}{c}e^{-\left(\frac{\delta+\lambda}{\lambda}\right)b\beta}\phi_b(b) - \beta\alpha\left(\frac{\delta+\lambda}{c}\right)\int_0^\infty e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w(b,s)\,ds \qquad (29)$$

This gives the formula (27).
Consider the following result.
Lemma 3.3. The Gerber-Shiu function $\phi_b(u)$ has the Laplace transform $\widehat{\phi_b}(s)$ defined by:

$$\widehat{\phi_b}(s) = \frac{\phi_b(0) + \frac{\lambda\alpha}{sc}e^{-\left(\frac{\delta+\lambda}{\lambda}\right)b\beta}\phi_b(b) - \beta\left(\frac{\delta+\lambda}{sc}\right)\int_0^\infty e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w(b,s)\,ds - \frac{\lambda}{c}(1-\alpha)\widehat{\omega}(s)}{s - \frac{\delta+\lambda-\lambda\alpha}{c} + \frac{\lambda\beta(1-\alpha)}{c(s+\beta)}} \qquad (30)$$

Proof [Proof of the lemma (3.3)] Let's apply the Laplace transformation to both members of the integro-differential equation (27), it can be shown that:

$$\int_0^\infty e^{-su}\left(\mathcal{D} - \frac{\delta+\lambda-\lambda\alpha}{c}\mathcal{I}\right)\phi_b(u)\,du = s\widehat{\phi}_b(s) - \phi_b(0) - \frac{\delta+\lambda-\lambda\alpha}{c}\widehat{\phi}_b(s) \qquad (31)$$

and

$$\begin{aligned}&\int_0^\infty e^{-su}\left[-\frac{\lambda}{c}(1-\alpha)\sigma_b(u) + \frac{\lambda\alpha}{c}e^{-\left(\frac{\delta+\lambda}{\lambda}\right)b\beta}\phi_b(b) - \beta\alpha\left(\frac{\delta+\lambda}{c}\right)\int_0^\infty e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w(b,s)\,ds\right]du \\ &= -\frac{\lambda}{c}(1-\alpha)\widehat{\sigma}_b(s) + \frac{\lambda\alpha}{sc}e^{-\left(\frac{\delta+\lambda}{\lambda}\right)b\beta c}\phi_b(b) - \beta\alpha\left(\frac{\delta+\lambda}{sc}\right)\int_0^\infty e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w(b,s)\,ds. \qquad (32)\end{aligned}$$

From the formula (10), it can be shown that $\sigma_b(u) = \int_0^u \phi_b(u-x)f_X(x)\,dx + \omega(u) = f_X * \phi_b(u) + \omega(u)$ and therefore $\widehat{\sigma}_b(s) = \widehat{f_X}(s)\widehat{\phi}_b(s) + \widehat{\omega}(s) = \frac{\beta}{s+\beta}\widehat{\phi}_b(s) + \widehat{\omega}(s)$.
From the formulas (31) and (32), the following formula is derived:

$$\hat{\phi}_b(s) = \frac{\phi_b(0) + \frac{\lambda\alpha}{sc}e^{-\left(\frac{\delta+\lambda}{\lambda}\right)b\beta}\phi_b(b) - \alpha\beta\left(\frac{\delta+\lambda}{sc}\right)\int_0^\infty e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}w(b,s)\,ds - \frac{\lambda}{c}(1-\alpha)\widehat{\omega}(s)}{s - \frac{\delta+\lambda-\lambda\alpha}{c} + \frac{\lambda\beta(1-\alpha)}{c(s+\beta)}}.$$

The probability of ruin, denoted as ψ,, is obtained when the force of interest $\delta = 0$ nd the penalty function $w(x,y) = 1$. The Gerber-Shiu function $\phi_b(u)$ then reduces to the ultimate probability of ruin $\psi_b(u)$.

Proof [Proof of the lemma (3.1)] Substituting $\widehat{\omega}(s)$ by $\frac{1}{s+\beta}$ and substituting $\int_0^\infty e^{-\frac{\beta}{\lambda}(\delta+\lambda)(s+b)}ds$ by $\frac{e^{-\frac{\beta}{\lambda}(\delta+\lambda)b}}{\frac{\beta}{\lambda}(\delta+\lambda)}$ in the formula (30), it can be shown that:

$$\widehat{\psi_b}(s) = \frac{\psi_b(0) + \frac{\lambda\alpha}{sc}e^{-b\beta}(\psi_b(b)-1) - \frac{\lambda(1-\alpha)}{c(s+\beta)}}{s - \frac{\lambda(1-\alpha)}{c} + \frac{\lambda\beta(1-\alpha)}{c(s+\beta)}}. \quad (33)$$

Mutiplying the numerator and denominator in the formula (33) buy $cs(s+\beta)$ gives the formula (26).

Proof [Proof of the theorem (3.1)] Examination of formula (26) reveals a denominator that is evidently a polynomial of degree 3 in s, indicating the presence of poles.

$$\begin{aligned} R_0 &= 0, \\ R_1 &= -\frac{(c\beta-\lambda)+\lambda\alpha}{c}. \end{aligned} \quad (34)$$

Its numerator is a polynomial of degree 2 in s.

Decomposing $\widehat{\psi_b}(s)$ into simple elements gives:

$$\widehat{\psi_b}(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-R_1}. \quad (35)$$

The formula (35) is equivalent to

$$\widehat{\psi_b}(s) = \frac{c(A+C)s^2 + c(B-AR_1)s - cBR_1}{s^2(cs+c\beta-\lambda(1-\alpha))}. \quad (36)$$

By identifying the formulas (26) and (36), the following formulas are obtained:

$$\begin{aligned} A + C &= \psi_b(0) && (37) \\ B - AR_1 &= \beta\psi_b(0) + \frac{\lambda\alpha}{c}e^{-b\beta}(\psi_b(b)-1) \\ &\quad -\frac{\lambda}{c}(1-\alpha) && (38) \\ B &= \frac{\alpha\beta\lambda}{cR_1}e^{-b\beta}(1-\psi_b(b)) && (39) \end{aligned}$$

By inverting the Laplace transform of the formula (35) it can be shown that:

$$\psi_b(u) = A + Bu + Ce^{R_1 u} \quad (40)$$

Since $\lim_{u\to+\infty}\psi_b(u) = 0$ (see [19]) and $R_1 < 0$, it can be shown that

$$A = 0 \quad (41)$$
$$B = 0 \quad (42)$$

From the formulas (39) and (42), it can be shown that

$$\psi_b(b) = 1. \quad (43)$$

From the formulas (38), (39) and (42), it can be shown that

$$A = \frac{\lambda}{cR_1}(1-\alpha) - \frac{\beta}{R_1}\psi_b(0). \quad (44)$$

From the formulas (41) and (44), it can be shown that

$$\psi_b(0) = \frac{\lambda}{c\beta}(1-\alpha) \quad (45)$$

From the formulas (37), (41) and (45), it can be shown that

$$C = \frac{\lambda}{c\beta}(1-\alpha) \quad (46)$$

From the formulas (40),(42), (46) and (41) $\psi_b(u)$can be written as:

$$\psi_b(u) = \frac{\lambda}{c\beta}(1-\alpha)e^{R_1 u}. \quad (47)$$

The derivation of formula (3.1) proceeds from the application of formulas (34) and (47).

Example 3.1. With parameters fixed at $c = 0.5$; $\lambda = 0.3$; $\beta = 1$; and $b = 10$, MATLAB is employed to generate curves depicting probabilities of failure corresponding to diverse values of the dependency parameter α.

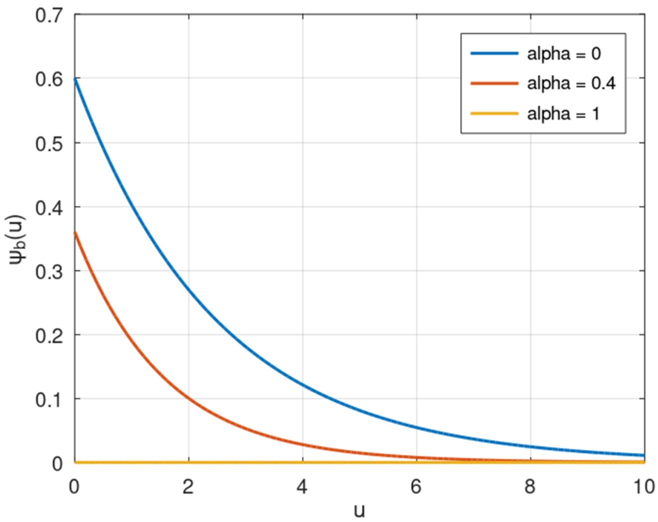

Figure 1. *Curve of $\psi_b(u)$ as a function of u for various values of α.*

The probability of ruin $\psi_b(u)$ is the decreasing function of the dependence parameter α.

4. Conclusion

This article investigates the Gerber-Shiu penalty function within the framework of a compound fish-risk model employing a shareholder dividend strategy, a constant threshold (b), and a Spearman's copula-based dependence between claim amounts and inter-claim times. The probability of ultimate ruin is a special case within this context. An integral-differential equation for the Gerber-Shiu function and its Laplace transform has been established. Through this analysis, the Laplace transform of the probability of ruin and an explicit formula for the probabilities of ruin were subsequently derived.

Conflicts of Interest

The authors declare no conflicts of interest.

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