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A Proposed New Non-Linear Programming Technique for Solving a Mixed Strategy Problem in Game Theory

Md. Golam Robbani*, Md. Asadujjaman, Md. Mehedi Hassan

Department of Mathematics, University of Dhaka, Dhaka, Bangladesh

Email address:
mdgolamrobbanisk@gmail.com (Md. Golam Robbani), asad132@du.ac.bd (Md. Asadujjaman),
mehedihassan116328@gmail.com (Md. Mehedi Hassan)

*Corresponding author

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Abstract: This Paper explores a new non-linear programming approach for determining mixed strategies in non-zero-sum games. Our approach leverages the power of non-linear optimization algorithms to solve the mixed strategy determination problem efficiently. We formulate the problem as a non-linear programming model, considering the individual player's utility functions and the strategic interdependencies among them. The proposed approach offers accurately represents strategic interactions by incorporating non-linear objective functions and constraints. The proposed non-linear programming technique offers several advantages for solving game theory problems. Firstly, it enables the consideration of complex and nonlinear relationships among players' strategies, allowing for more realistic and nuanced modeling. Secondly, the technique offers flexibility in incorporating various types of constraints, including capacity limitations, budget constraints, or regulatory requirements, enhancing the applicability to real-world scenarios. Lastly, NLP algorithms provide efficient and robust optimization procedures, ensuring reliable solutions within reasonable time frames. We use *MATLAB* to solve the Non-Linear programming problem which gives us more accurate results. To demonstrate the effectiveness of the proposed technique, it can be applied to diverse game theory problems, such as auctions, bargaining, pricing decisions, and resource allocation. The results obtained through this approach offer insights into optimal strategies, equilibrium outcomes, and potential trade-offs, facilitating informed decision-making in strategic environments.

Keywords: Equilibrium Concepts, Strategic Interactions, Nash Equilibrium, Pareto Optimal, Dominant Strategy, Mixed Strategy, Payoff Matrix, Saddle Point

1. Introduction

Game theory is a branch of mathematics that deals with the study of strategic decision-making in situations where multiple players are involved. It is often used to model and analyze complex interactions between different actors in various fields, including economics, politics, and biology. Game theory provides a framework for understanding how individuals or organizations make decisions, and how these decisions affect others. Game theory provides a valuable framework for analyzing strategic interactions among rational decision-makers. The successful resolution of game theory problems often relies on finding optimal strategies that maximize players' utilities or minimize their costs. Nonlinear programming (NLP) techniques have emerged as powerful tools for solving complex optimization problems with nonlinear constraints, making them well-suited for addressing game theory scenarios. This paper explores solution techniques for game theory problems, aiming to provide a comprehensive understanding of how to approach and solve such problems effectively. The thesis begins by introducing the fundamental concepts of game theory, including players, strategies, and payoffs. Different types of games, such as simultaneous and sequential games, are discussed, along with considerations of information asymmetry and other relevant characteristics that influence strategic decision-making. The focus then shifts to solution techniques. Equilibrium concepts, such as Nash equilibrium, subgame perfect equilibrium, and Bayesian Nash equilibrium, are discussed as key solution concepts in game theory. The thesis explores the conditions for achieving equilibrium and illustrates how equilibrium solutions provide stable outcomes in strategic interactions [1].

To solve a game theory problem, the first step is to identify the players, the strategies available to each player, and the payoffs associated with each strategy. Once these components have been defined, various solution concepts can be used to determine the optimal strategies for each player. Some of the most common solution concepts include Nash equilibrium, Dominant strategy equilibrium, Pareto efficiency, and Minimax strategy. There are many different methods for solving game theory problems, including algebraic methods, graphical methods, and computer simulations [2]. The appropriate method depends on the specific problem and the available resources. However, regardless of the method used, it is important to carefully analyze the problem and understand the assumptions underlying the solution concept being used.

2. What Is a Game

Game theory is a mathematical theory studying situations of conflict and cooperation between rational decision-makers (players). Game theory, a branch of applied mathematics that provides tools for analyzing situations in which parties, called players, make decisions that are interdependent. This interdependence causes each player to consider the other player's possible decisions, or strategies, in formulating strategy. [3] A solution to a game describes the optimal decisions of the players, who may have similar, opposed, or mixed interests, and the outcomes that may result from these decisions. Although game theory can be and has been used to analyze parlor games, its applications are much broader. In fact, game theory was originally developed by the Hungarian-born American mathematician John von Neumann and his Princeton University colleague Oskar Morgenstern, a German-born American economist, to solve problems in economics. Theory of Games and Economic Behavior (1944), von Neumann and Morgenstern asserted that the mathematics developed for the physical sciences, which describes the workings of a disinterested nature, was a poor model for economics. [4] They observed that economics is much like a game, wherein players anticipate each other's moves, and therefore requires a new kind of mathematics, which they called game theory.

3. Some Definitions

Game: A game is a formal description of a strategic situation.

Game theory: Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

Mixed strategy: A mixed strategy is an active randomization, with given probabilities, that determines the player's decision. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies.

Nash equilibrium: A Nash equilibrium, also called strategic equilibrium, is a list of strategies, one for each player, which has the property that no player can unilaterally change his strategy and get a better payoff. [6]

Payoff: A payoff is a number, also called utility, that reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff incorporates the player's attitude towards risk.

Player: A player is an agent who makes decisions in a game.

Rationality: A player is said to be rational if he seeks to play in a manner which maximizes his own payoff. It is often assumed that the rationality of all players is common knowledge.

Strategic form: A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

Strategy: In a game in strategic form, a strategy is one of the given possible actions of a player.

Dominant Strategy: A strategy p for the row player strictly dominates strategy p

$$if, \pi R(p,q) > \pi R(p,q), q \in S_c$$

"The row player is better off playing p rather than p no matter what the column player does"
weakly dominant with '≥' instead of '>'.

Nash Equilibrium: A strategy pair $(p*, q*)$ is a Nash Equilibrium if

$$\pi R(p*, q*) \geq \pi R(p, q*), \forall p \in S_r$$

$$\pi C(p*, q*) \geq \pi C(p*, q), \forall q \in S_c$$

"no player could have done better in hindsight by just changing their own strategy (assuming their opponent's strategy does not change)" [7].

Pareto Optimal: A strategy pair (p∗, q∗) is a Pareto Optimal (Socially Optimal) if

$$\pi R(p*, q*) + \pi C(p*, q*) \geq \pi R(p,q) + \pi C(p,q), \forall p \in S_r, q \in S_c$$

"The strategy pair with the largest combined payoff" [7].
S_r- Strategy for row Player
S_c- Strategy for column Player

4. Solving Procedure a Game Theory Problem

To solve a real-life problem by game theory, you would typically follow these steps:

1) *Identify the problem*: Begin by identifying the problem that you want to solve. This could be a situation where two or more parties are involved and each party's

decision affects the outcome of the problem.

2) *Define the players*: Identify the parties involved in the problem and define them as the players. Players can be individuals, organizations, or groups of people.
3) *Define the strategy space*: Define the possible strategies available to each player. These strategies can be actions or decisions that a player can make to achieve a desired outcome.
4) *Define the payoffs*: Define the payoffs for each player associated with each strategy. Payoffs represent the rewards or penalties for each player associated with each strategy.
5) *Determine the Nash equilibrium*: Use the game theory concept of Nash equilibrium to identify the strategy combination that is most likely to be played by the players. This is the strategy combination where no player has the incentive to change their strategy given the strategies of the other players.
6) *Analyze the results*: Analyze the results of the Nash equilibrium to determine if it is a desirable outcome or if it requires further action.

By following these steps, you can use game theory to solve real-life problems involving strategic interactions between multiple parties. [8]

5. Solving Methods

According to the strategy, we can solve a game by following Process

For Pure strategy

1) Saddle point Method
2) Dominance Method

For Mixed strategy

1) Linear Programming method
2) By the Oddment matrix method
3) Non-linear method (proposed)

6. Proposed Non-Linear Programming Method

6.1. Algorithm Proposed Non-Linear Programming Method

Solving matrix game theory problems using nonlinear programming involves the following steps:

1) *Define the variables*: Let X and Y be the set of strategies of the two players in the game.
2) *Write the objective function*: The objective function represents the payoff of each player. For example, if the payoff of two players are $f_1(X,Y)$ and $f_2(X,Y)$ respectively and every player wants to maximize their payoff. Then the objective function can be written as $Z = f_1(X,Y) + f_2(X,Y)$, where $f_1(X,Y)$ and $f_2(X,Y)$ are the payoff function. Now Z is a quadratic programming problem.
3) *Write the constraints*: The constraints represent the limitations on the strategies of the players. For example, if the players can only choose between two strategies, then the constraints can be written as $\sum X(i) = 1$ and $\sum Y(i) = 1$.
4) *Solve the quadratic programming problem*: Solve the quadratic programming problem using a suitable method such as the interior point method or the active set method. The solution will give the optimal values of x and y that maximize the objective function.
5) *Interpret the results*: The optimal values of x and y represent the optimal strategies for each player. The corresponding objective function value represents the maximum payoff for each player.

Consider a game

Table 1. *A game for two player.*

	Left	Right
Up	$A_{1,1}$, $B_{1,1}$	$A_{1,2}$, $B_{1,2}$
Down	$A_{2,1}$, $B_{2,1}$	$A_{2,2}$, $B_{2,2}$

Assume that Player 1 chooses "Up" with probability p, $0 < p < 1$ and "Down" with probability $1 - p$. In the same way, Player 2 chooses "Left" with probability q, $0 < q < 1$ and "Right" with probability $1 - q$.

The payoff of player 1 is $f_1 = (p\ 1-p) * \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} * \begin{pmatrix} q \\ 1-q \end{pmatrix}$

The payoff of player 2 is $f_2 = (p\ 1-p) * \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} * \begin{pmatrix} q \\ 1-q \end{pmatrix}$

Objective function $= f_1 + f_2$

Now. We try to solve the quadratic programming problem using a suitable method such as Lagrange multipliers Method. The solution will give the optimal values of x and y that maximize the objective function.

A Numerical Example

Table 2. *A game for two player.*

	Left	Right
Up	3,2	1,1
Down	0,0	2,3

Let us look at this third case first. Assume that Player 1 chooses "Up" with probability *x*, $0 < x < 1$ and "Down" with probability $1 - x$. In the same way, Player 2 chooses "Left" with probability *y*, $0 < q < 1$ and "Right" with probability $1 - y$.

The payoff of player 1 is $f_1 = (x\ 1-x) * \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} * \begin{pmatrix} y \\ 1-y \end{pmatrix} = 4xy - x - 2y + 2$

The payoff of player 2 is $f_2 = (x\ 1-x) * \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} * \begin{pmatrix} y \\ 1-y \end{pmatrix} = 3xy - 2x - 3y + 3$

Objective function $= f_1 + f_2 = 7xy - 3x - 5y + 5$

Now. We try to solve the quadratic programming problem using a suitable method such as Lagrange multipliers Method. The idea is to introduce a constraint function that represents the constraints, and then find the values of x, y, and the

Lagrange multiplier that maximize the expression subject to the constraints.

Let $g(x,y) = x^2 + y^2 - 1$ be the constraint function, which represents the fact that x and y must lie on the unit circle centered at the origin. We can then form the Lagrangian function as follows:

$$L(x,y,\lambda) = 7xy - 3x - 5y + 5 + \lambda * (x^2 + y^2 - 1)$$

We take partial derivatives of L with respect to x, y, and λ, and set them equal to 0:

$$\frac{\partial L}{\partial x} = 7y - 3 + 2\lambda x = 0$$

$$\frac{\partial L}{\partial y} = 7x - 5 + 2\lambda y = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

Solving for x, y, and λ, we get: $x = \frac{5}{7}$, $y = \frac{3}{7}, \lambda = -\frac{56}{49}$

Note that these values of x and y are the same as in the unconstrained problem we solved earlier. However, the Lagrange multiplier λ is a new variable that arises from the constraints.

We can check that these values satisfy the constraints by verifying that

$$g(x,y) = x^2 + y^2 - 1 = 0:$$

$$x^2 + y^2 - 1 = \left(\frac{5}{7}\right)^2 + \left(\frac{3}{7}\right)^2 - 1 = \frac{25}{49} + \frac{9}{49} - 1 = 0$$

Therefore, the maximum value of the expression $7xy - 3x - 5y + 5$ subject to the constraints $0 < x < 1$ and $0 < y < 1$ occurs when $x = \frac{5}{7}$ and $y = \frac{3}{7}$.

We can solve this nonlinear programming problem using a nonlinear solver, such as *MATLAB*.

```
fun1 = @(x) 4*x(1)*x(2)-x(1)-2*x(2)+2
fun2 = @(x) 3*x(1)*x(2)-2*x(1)-3*x(2)+3
fun = @(x) -(7*x(1)*x(2)-3*x(1)-5*x(2)+5)
x0 = [1/2,1/2];
A = [1 0; 0 1];
b = [1; 1];
lb = [0; 0];
ub = [1; 1];
Aeq = [1,1];
beq = 1;
[x, xval] = fmincon(fun,x0,[],[],Aeq,beq,lb,ub);
fprintf('Player 1 chooses Up with probability: %f\n', x(1))
fprintf('Player 1 chooses Down with probability: %f\n', 1-x(1))
fprintf('Player 2 chooses Left with probability: %f\n', x(2))
fprintf('Player 2 chooses Right with probability: %f\n', 1-x(2))
fprintf('The Payoff player 1 is: %f\n', fun1(x))
fprintf('The Payoff player 1 is: %f\n', fun2(x))
```

This code finds the optimal strategies for Player 1 and Player 2, and the corresponding optimal payoff.

The output is:

Player 1 chooses Up with probability: 0.642857
Player 1 chooses Down with probability: 0.357143
Player 2 chooses Left with probability: 0.357143
Player 2 chooses Right with probability: 0.642857
The Payoff player 1 is: 1.561224
The Payoff player 1 is: 1.561224

6.2. Generalized Form

Table 3. *A game for Generalized Form.*

	Left	Right
Up	$A_{1,1}$, $B_{1,1}$	$A_{1,2}$, $B_{1,2}$
Down	$A_{2,1}$, $B_{2,1}$	$A_{2,2}$, $B_{2,2}$

Assume that Player 1 chooses "Up" with probability $x(1)$, and "Down" with probability $x(2)$. In the same way, Player 2 chooses "Left" with probability $y(1)$, and "Right" with probability $y(2)$.

The payoff of player 1 is $f_1 = \big(x(1)x(2)\big)*\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} *\begin{pmatrix} y(1) \\ y(2) \end{pmatrix}$

$$= A_{1,1} * x(1) * y(1) + A_{1,2} * x(1) * y(2) + A_{2,1} * x(2) * y(1) + A_{2,2} * x(2) * y(2)$$

$$=\sum_{i=1}^{2} \sum_{j=1}^{2} A_{i,j} * x(i) * y(j)$$

The payoff of player 2 is $f_2 = \big(x(1)x(2)\big)*\begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} *\begin{pmatrix} y(1) \\ y(2) \end{pmatrix}$

$$= B_{1,1} * x(1) * y(1) + B_{1,2} * x(1) * y(2) + B_{2,1} * x(2) * y(1) + B_{2,2} * x(2) * y(2)$$

$$=\sum_{i=1}^{2} \sum_{j=1}^{2} B_{i,j} * x(i) * y(j)$$

Objective function = $f_1 + f_2$

Now. We try to solve the quadratic programming problem using a suitable method. The solution will give the optimal values of x and y that maximize the objective function.

A Numerical Example

Table 4. *A game for Two Players.*

	Left	Right
Up	3,2	1,1
Down	0,0	2,3

Let us look at this third case first. Assume that Player 1 chooses "Up" with probability $x(1)$, and "Down" with probability $x(2)$. In the same way, Player 2 chooses "Left" with probability $y(1)$, and "Right" with probability $y(2)$.

The payoff of player 1 is $f_1 = \big(x(1)x(2)\big) * \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} * \begin{pmatrix} x(3) \\ x(4) \end{pmatrix}$

$$= 3 * x(1) * y(1) + 1 * x(1) * y(2) + 0 * x(2) * y(1) + 2 * x(2) * y(2)$$

The payoff of player 2 is $f_2 = \left(x(1)x(2)\right) * \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} * \begin{pmatrix} x(3) \\ x(4) \end{pmatrix}$

$$= 2 * x(1) * y(1) + 1 * x(1) * y(2) + 0 * x(2) * y(1) + 3 * x(2) * y(2)$$

$$\text{Objective function } = f_1 + f_2$$

$$= 5 * x(1) * y(1) + 2 * x(1) * y(2) + 0 * x(2) * y(1) + 5 * x(2) * y(2)$$

We can solve this nonlinear programming problem using a nonlinear solver, such as *MATLAB*.

```
fun1 = @(x) 3*x(1)*x(3)+x(1)*x(4)+2*x(2)*x(4)
fun2 = @(x) 2*x(1)*x(3)+x(1)*x(4)+3*x(2)*x(4)
fun = @(x) -(5*x(1)*x(3)+2*x(1)*x(4)+5*x(2)*x(4))
x0 = [.5,.5,.5,.5];
%A = [1 1 0 0; 0 0 1 1];
%b = [1; 1];
lb = [0; 0; 0; 0];
ub = [1; 1; 1; 1];
Aeq = [0 0 1 1; 1 1 0 0];
beq = [1; 1];
x = fmincon(fun,x0,[],[],Aeq,beq,lb,ub);
fprintf('Player 1 chooses Up with probability: %f\n', x(1))
fprintf('Player 1 chooses Down with probability: %f\n', x(2))
fprintf('Player 2 chooses Left with probability: %f\n', x(3))
fprintf('Player 2 chooses Right with probability: %f\n', x(4))
fprintf('The Payoff player 1 is: %f\n', fun1(x))
fprintf('The Payoff player 1 is: %f\n', fun2(x))
```

This code finds the optimal strategies for Player 1 and Player 2, and the corresponding optimal payoff.

The output is:

Player 1 chooses Up with probability: 0.624999
Player 1 chooses Down with probability: 0.375001
Player 2 chooses Left with probability: 0.375000
Player 2 chooses Right with probability: 0.625000
The Payoff player 1 is: 1.562500
The Payoff player 1 is: 1.562500

Solution by others technic:

There exists a Nash equilibrium in mixed strategies.
Player 1 plays the mixed strategy of (Up, Down) = (0.75,0.25).
Player 1 has an expected payout = 1.5.
Player 2 plays the mixed strategy of (Left, Right) = (0.25,0.75).
Player 2 has an expected payout = 1.5.

7. Comparison Between Non-Linear Method and Others

When it comes to solving game theory problems, various methods can be employed, including non-linear methods, as well as other techniques such as analytical methods, iterative algorithms, and numerical simulations. Let's compare the non-linear method with other approaches commonly used in game theory problem-solving:

1) *Non-linear methods*: Non-linear methods involve solving game theory problems by formulating them as non-linear optimization problems. These methods consider the players' decisions as variables and aim to optimize a certain objective function, taking into account the strategic interactions and constraints within the game. Non-linear programming solvers, such as Newton's method or gradient-based methods, can be used to find the optimal solutions. Non-linear methods offer flexibility in handling complex game structures and can accommodate various forms of strategic interactions. [9]
2) *Analytical methods*: Analytical methods involve using mathematical tools and techniques to analyze and solve game theory problems. These methods often rely on algebraic manipulations, optimization techniques, and equilibrium concepts to determine the optimal strategies and outcomes. Analytical methods are particularly useful for simple games with known structures and can provide precise and exact solutions. Examples include finding dominant strategies, Nash equilibria, or performing iterated elimination of dominated strategies. [10]
3) *Iterative algorithms*: Iterative algorithms are used to solve game theory problems through repeated iterations and adjustments of strategies. These methods often involve players updating their strategies based on the strategies and payoffs of other players until a stable solution is reached. Examples include the replicator dynamics, fictitious play, or reinforcement learning algorithms. Iterative algorithms are suitable for dynamic games or scenarios where players can adapt their strategies over time. [11]

The choice of method depends on the specific characteristics of the game, its complexity, and the objectives of the analysis. Non-linear methods are beneficial when dealing with complex games with non-linear relationships between strategies and payoffs. Analytical methods are suitable for simple games with known structures. Iterative algorithms are helpful for dynamic games or scenarios involving adaptation. Numerical simulations are useful for approximating outcomes and exploring a wide range of possibilities.

8. Conclusion

In conclusion, the application of nonlinear programming techniques in solving game theory problems offers a valuable and effective approach. By formulating games as mathematical models and employing NLP algorithms, this technique enables the identification of optimal strategies that maximize players' utilities or minimize their costs. Nonlinear programming provides a framework to handle the complexity and nonlinear relationships inherent in game theory scenarios. The technique allows for the consideration of various constraints, capturing real-world limitations and requirements. This flexibility enhances the applicability of the approach across diverse domains, such as auctions, bargaining, pricing decisions, and resource allocation. Moreover, NLP algorithms offer efficient and robust optimization procedures, ensuring reliable solutions within reasonable time frames. The iterative nature of the technique allows for refining strategies and exploring the decision space, leading to convergence towards optimal solutions. [12] By considering the interdependencies among players' strategies and their impact on utilities, the approach provides valuable insights into equilibrium outcomes, optimal strategies, and potential trade-offs. The proposed nonlinear programming technique contributes to the advancement of strategic decision-making and analysis in game theory. It facilitates more realistic and nuanced modeling of complex interactions, empowering decision-makers to make informed choices. Furthermore, the technique holds promise for addressing future challenges and opportunities in game theory, as it can be extended and adapted to accommodate evolving scenarios and constraints.

In summary, the utilization of nonlinear programming techniques in game theory problem-solving offers a powerful and versatile methodology. This approach provides a systematic framework to formulate games, optimize strategies, and derive meaningful insights. By leveraging the strengths of NLP algorithms, it contributes to the understanding and resolution of strategic interactions, ultimately enhancing decision-making processes in various domains.

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