



On Some Finite Difference Schemes for the Solutions of Parabolic Partial Differential Equations

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Abstract: This paper presents the comparison of three different and unique finite difference schemes used for finding the solutions of parabolic partial differential equations (PPDE). Knowing fully that the efficiency of a numerical schemes depends solely on their stability therefore, the schemes were compared based on their stability using von Neumann method. The implicit scheme and Dufort-Frankel schemes using von Neumann stability method are unconditionally stable, while the explicit scheme is conditionally stable. The schemes were also applied to solve a one dimensional parabolic partial differential equations (heat equation) numerically and their results compared for best in efficiency. The numerical experiments as seen in the tables presented and also the percentage errors, which proves that the implicit scheme is good compare to the other two schemes. Also, the implementation of the implicit scheme is faster than that of the explicit and Dufort-Frankel schemes. The results obtained in work also compliment and agrees with the results in literature.

Keywords: Finite Difference Schemes, Stability, Von Neumann Method, Accuracy, Heat Equation

1. Introduction

The time-dependent diffusion equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial t^2} \quad (1)$$

describing a damp diffusion in time is considered and it is called a second order parabolic partial differential equation. If the equation (1) above is given the following initial and boundary conditions

$$f = f(x) \quad (2)$$

and

$$f(0, t) = 0 = f(l, t) \quad (3)$$

respectively, the equation (1)-(3) becomes an initial-boundary value problem and the solution can be obtained using numerical methods. Numerical methods is a method of obtaining an approximate solution to partial differential

equations whose solution cannot be obtain analytically. There are different types of numerical methods but for the purpose of this work, we shall consider the explicit (FTCS); Implicit (BTCS) and Du Fort-Frankel schemes. The efficiency of the numerical schemes depends solely on their stability. Lot of researchers have worked on finite difference methods for parabolic partial differential equations, among them are; Crank J and Philis N. [2] worked on practical method for evaluating numerical of solution of partial differential equation of heat conduction type. Recktenwald G. W [4] discussed the three finite difference methods (FTCS; BTCS and Crank-Nicolson methods) to solve one dimensional boundary problem. Karatay I. and Bayramoglu. S [5] obtained the solution of time fractional heat equation using Crank-Nicolson method. Aswin V. S et al [6] described three different numerical schemes to approximate the solution of the convection-diffusion equation. Azad T. M. A. K and Andallah I. S. [7] studied stability analysis for two standard finite difference schemes, forward time and centered space and centered space (FTBSCS) and forward time and centered

space (FTCS) for convection-diffusion equation. Olusegun O. A et al [9] solved the one dimensional heat equation using the explicit scheme. Adak M. and Mandal N. R [13] solved the transient heat equation with convection boundary condition using explicit finite difference scheme. Adak. M [14] studied the effect of explicit and implicit schemes on one dimensional diffusion equation with dirichlet boundary condition. There are also some interesting texts for the subject, they are Williams F. Ames [10], Smith G. D [11] and Grewal B. S [12].

The main objectives of this paper is to compare the stability of the schemes using von Newmann method. Also, their results when applied to a parabolic equations are also compared.

2. Problem Definition and Methodology

The one dimensional heat equation (1) – (3) of length L rod is considered, where f = temperature, t = time, x = length and $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity, also k = thermal conductivity, c = heat capacity and ρ = density.

For the derivation of the scheme, we shall use the following derivative in [3, 11],

$$\frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j}}{\delta t} + O(\delta t) \text{ forward difference in time;}$$

$$\frac{\partial f}{\partial t} = \frac{f_{i,j} - f_{i,j-1}}{\delta t} + O(\delta t) \text{ Backward difference in time;}$$

$$\frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j-1}}{\delta t} + O(\delta t^2) \text{ central difference in time;}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{(\delta x)^2} + O(\delta x)^2 \text{ central difference in space.}$$

All derivatives in the equation (1) are approximated using Taylor's series expansion.

2.1. The Explicit Scheme

The explicit scheme is derived from equation (1) by replacing the first order derivative and second order derivative by forward difference in time and central difference in space respectively. The derivation is shown below:

$$\frac{f_{i,j+1} - f_{i,j}}{\delta t} = \alpha \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{(\delta x)^2} \quad (4)$$

$$f_{i,j+1} = f_{i,j} + \frac{\alpha \delta t}{(\delta x)^2} (f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) \quad (5)$$

Let $r = \frac{\alpha \delta t}{(\delta x)^2}$, then equation (5) becomes

$$f_{i,j+1} = r f_{i-1,j} + (1 - 2r) f_{i,j} + r f_{i+1,j} \quad (6)$$

2.1.1. Local Truncation Error

The local Truncation error of the explicit scheme has it principal part as

$$\left(\frac{1}{2} k \frac{\partial^2 f}{\partial x^2} - \frac{1}{12} h^2 \frac{\partial^4 f}{\partial x^4} \right)_{i,j}$$

Therefore, its local truncation error is $O(k) + O(h^2)$.

2.1.2. Stability Condition (Using von Newman Method)

von Newman stability method is the most widely used procedure for determining the stability of finite difference approximation (Lapidus Leon). The method introduces an initial line of errors as represented by Fourier series and consider the growth of these error as x increases. The stability of the explicit method using von Newmann method is shown as follows; the explicit scheme is given by

$$f_{i,j+1} = r f_{i-1,j} + (1 - 2r) f_{i,j} + r f_{i+1,j}$$

the equation above in error form is written as

$$e_{i,j+1} = r e_{i-1,j} + (1 - 2r) e_{i,j} + r e_{i+1,j} \quad (7)$$

let

$$e_{i,j} = \zeta^{\gamma r h} \zeta^{i \beta s k} = \xi^q \zeta^{i \beta s k} \text{ where } \xi^q = \zeta^{\gamma r h} \quad (8)$$

substituting equation (8) into equation (7) gives

$$\xi^{q+1} \zeta^{i \beta s k} = r \xi^q \zeta^{i \beta (s-1)k} + (1 - 2r) \xi^q \zeta^{i \beta s k} + r \xi^q \zeta^{i \beta (s+1)k} \quad (9)$$

using simple mathematical principle and on cancelation of common terms, equation (9) yields

$$\xi = r \zeta^{i \beta s k} + (1 + 2r) + r \zeta^{-i \beta k}$$

which can be written as

$$\xi = (1 - 2r) + r(\zeta^{i \beta k} + \zeta^{-i \beta k}) \quad (10)$$

using the following trigonometry identities

$$1 - \cos \beta k = 2 \sin^2 \left(\frac{\beta k}{2} \right)$$

and

$$2 \cos \beta k = \zeta^{i \beta k} + \zeta^{-i \beta k} \quad (11)$$

on substituting (11) into (10) we have

$$\xi = (1 - 2r) + r(2 \cos \beta k)$$

which can be written as

$$1 - 2r(1 - \cos \beta k) \quad (12)$$

using equation (11) in (12) we get

$$1 - 2r \left(2 \sin^2 \left(\frac{\beta k}{2} \right) \right)$$

from whence,

$$\xi = 1 - 4r \sin^2 \left(\frac{\beta k}{2} \right)$$

The necessary and sufficient condition for the error to be bounded, keeping to numerical stability is

$$|\xi| \leq 1$$

therefore, the condition for stability of the explicit scheme will be

$$|\xi| = \left| 1 - 4r \sin^2 \left(\frac{\beta k}{2} \right) \right| \leq 1 \quad (13)$$

considering equation (13) we have that

$$4r \sin^2 \left(\frac{\beta k}{2} \right) \leq 2 \quad (14)$$

since the term $4r \sin^2 \left(\frac{\beta k}{2} \right)$ has its range in $[0, 1]$ that is positive, the worst case is when $\sin^2 \left(\frac{\beta k}{2} \right) = 1$, such that the equation (14) becomes

$$r = \frac{1}{2} \quad (15)$$

similarly, the second part of equation (13) gives

$$4r \sin^2 \left(\frac{\beta k}{2} \right) \geq 0$$

from where

$$r \geq 0 \quad (16)$$

combining equation (15) and (16) gives $0 \leq r \leq \frac{1}{2}$. This shows that the explicit scheme is conditionally stable.

2.2. Dufort-Frankel Scheme

The derivation of the Dufort-Frankel approximation is simply by replacing the first and second order derivatives in equation (1) by central difference in time and central difference in space, resulting to the following

$$\frac{f_{i,j+1} - f_{i,j-1}}{2\delta t} = a \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{(\delta x)^2} \quad (17)$$

also $f_{i,j}$ on the R. H. S is replaced with time average of previous and current time values that is $(j-1)$ and $(j+1)$ to get

$$\frac{f_{i,j+1} - f_{i,j-1}}{2\delta t} = \frac{a}{(\delta x)^2} [f_{i-1,j} - 2 \left(\frac{f_{i,j-1} + f_{i,j+1}}{2} \right) + f_{i+1,j}] \quad (18)$$

which can be written as

$$f_{i,j+1} - f_{i,j-1} = \frac{2a\delta t}{(\delta x)^2} (f_{i-1,j} - f_{i,j-1} - f_{i,j+1} + f_{i+1,j}) \quad (19)$$

Which further gives

$$f_{i,j+1} + 2rf_{i,j+1} = f_{i,j-1} - 2rf_{i,j-1} + 2rf_{i-1,j} + 2rf_{i+1,j} \quad (20)$$

where $r = \frac{a\delta t}{(\delta x)^2}$ equation (20) can be written as

$$(1 + 2r)f_{i,j+1} = (1 - 2r)f_{i,j-1} + 2r(f_{i-1,j} + f_{i+1,j}) \quad (21)$$

Equation (21) is called the Dufort-Frankel finite difference approximation. It can also be written more explicitly as

$$f_{i+1,j} = \frac{(1-2r)}{(1+2r)} f_{i,j-1} + \left(\frac{2r}{1+2r} \right) (f_{i-1,j} + f_{i+1,j}) \quad (22)$$

2.2.1. Local Truncation Error

The local truncation error of the Dufort-Frankel scheme

has it principal part as $2h \left(-\frac{k^2}{12} \frac{\partial^4 f}{\partial x^4} + \frac{h^2}{6} \frac{\partial^3 f}{\partial x^3} + \frac{h^2}{k^2} \frac{\partial^2 f}{\partial x^2} \right)_{i,j}$ with it local truncation error as

$$O(h^2 + k^2 + \frac{h^2}{k^2})$$

2.2.2. Stability Condition (Using von Neumann)

The Dufort-Frankel scheme is given by equation (21), rewriting equation (21) in error form gives;

$$(1 + 2r)e_{i,j+1} = (1 - 2r)e_{i,j-1} + 2r(e_{i-1,j} + e_{i+1,j}) \quad (23)$$

using the same procedure as in the stability criterion for the explicit scheme, we have

$$(1 + 2r)\xi^{q+1}\zeta^{i\beta sk} = (1 - 2r)\xi^{q-1}\zeta^{i\beta sk} + 2r(\xi^q\zeta^{i\beta(s-1)k} + \xi^q\zeta^{i\beta(s+1)k}) \quad (24)$$

in equation (24), we have substituted equation (8) into (22). On cancelation of common terms we get

$$(1 + 2r)\xi = (1 - 2r)\xi^{-1} + 2r(\zeta^{-i\beta k} + \zeta^{i\beta k}) \quad (25)$$

using trigonometric identities we get

$$(1 + 2r)\xi = (1 - 2r)\xi^{-1} + 2r(2\cos\beta k) \quad (26)$$

solving further we get

$$\xi^2(1 + 2r) = (1 - 2r) + 2r\xi(2\cos\beta k) \quad (27)$$

solving equation (27) and using the fact that $\cos^2\beta k = 1 - \sin^2\beta k$, we get

$$\xi = \frac{2r\cos\beta k \pm \sqrt{4r^2(1 - \sin^2\beta k) + 1 - 4r^2}}{1 + 2r}$$

Which gives

$$\xi = \frac{2r\cos\beta k \pm \sqrt{4r^2\sin^2\beta k}}{1 + 2r} \quad (28)$$

Considering the term in square root for the following $r \leq \frac{1}{2}, r \geq \frac{1}{2}, |2r\sin\beta k| \leq 1$ and $|2\sin\beta k| \geq 1$, we have that the Dufort-Frankel approximation is unconditionally stable.

2.3. Implicit Scheme (BTCS)

The implicit scheme is derived by replacing the first order derivative by forward difference and the second order derivative by central difference with the $j + 1$. The procedure is as follows:

$$\frac{f_{i,j+1} - f_{i,j}}{\delta t} = a \frac{f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}}{(\delta x)^2}$$

$$f_{i,j+1} - f_{i,j} = \frac{a\delta t}{(\delta x)^2} (f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1})$$

let $r = \frac{a\delta t}{(\delta x)^2}$ we have

$$-f_{i,j} = rf_{i-1,j+1} - (1 + 2r)f_{i,j+1} + f_{i+1,j+1} \quad (29)$$

Equation (29) is the implicit approximation. The equation leads to a tridiagonal system.

2.3.1. Local Truncation Error

The local truncation error of the implicit scheme is given by $O(k, h^2)$.

2.3.2. Stability Condition (Using von Neumann)

The stability of the implicit scheme using von Neumann method is given by

$$\xi = \frac{1}{1 + 4r \sin^2 \frac{\beta k}{2}} \tag{30}$$

for all $r > 0$, and all β . It is observed that $0 < \xi \leq 1$. Showing that the scheme is unconditionally stable.

3. Numerical Examples

This section presents some numerical examples on the comparison of the finite difference schemes.

Example 1.

Consider the following mathematical model

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}, [0,1] \tag{31}$$

subject to the initial condition

$$f(x, 100) = 100 \tag{32}$$

and boundary conditions

$$f(0, t) = 0 = f(1, t), t > 0 \tag{33}$$

of the temperature distribution in a rod of length $L = 1m$ with its end point at 0^0 .

and initial point at 100^0C . Given that the analytical solution of the model is

$$f(x, t) = \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{1}{n} \sin n \pi x \exp(-n^2 \pi^2 t) \tag{34}$$

If $\delta x = 0.1$ and $r = \frac{1}{2}$, then the problem using explicit, Dufort-Frankel and Implicit schemes and compare the numerical solutions with the exact solutions at $x = 0.4$.

Solution:

Solving first with the explicit scheme, we use equation (6) to obtain the values for $1 \leq i \leq 9$ and the various steps from $0 \leq j \leq 9$ and the results are presented in the following table, see [1] for few steps on the solvings.

Table 1. Result of example 1 using explicit scheme.

x	i	$f_{ij=0}$	$f_{ij=1}$	$f_{ij=2}$	$f_{ij=3}$	$f_{ij=4}$	$f_{ij=5}$	$f_{ij=6}$	$f_{ij=7}$	$f_{ij=8}$
0.1	1	50.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	50.0000
0.2	2	50.0000	75.0000	100.0000	100.0000	100.0000	100.0000	100.0000	75.0000	50.0000
0.3	3	37.5000	75.0000	87.5000	100.0000	100.0000	100.0000	87.5000	75.0000	37.5000
0.4	4	37.5000	62.5000	87.5000	93.7500	100.0000	93.7500	87.5000	62.5000	37.5000
0.5	5	31.2500	62.5000	78.1250	93.7500	93.7500	93.7500	78.1250	62.5000	31.2500
0.6	6	31.2500	54.6875	78.1250	85.9300	93.7500	85.9300	78.1250	54.6975	31.3500
0.7	7	27.3438	54.6875	70.3088	85.9375	85.9300	85.9375	70.3088	54.6875	27.3438
0.8	8	27.3438	48.8263	70.3125	78.1194	85.9375	78.1194	70.3125	48.8263	27.3438
0.9	9	24.4132	48.8282	63.4729	78.1250	78.1194	78.1250	63.4729	48.8282	24.4132

Table 2. Comparison of the explicit scheme with the exact solutions.

t	Dufort-Frankel Scheme	Exact solutions	Percentage error
0.025	93.7500	99.9900	6.24
0.03	93.7500	99.5300	5.81
0.035	85.9300	97.8500	12.20
0.04	85.9375	95.1800	9.71

Table 3. Results of example 1 using Implicit scheme.

x	i	$f_{ij=0}$	$f_{ij=1}$	$f_{ij=2}$	$f_{ij=3}$	$f_{ij=4}$	$f_{ij=5}$	$f_{ij=6}$	$f_{ij=7}$	$f_{ij=8}$
0.1	1	73.2044	92.8177	98.0663	99.4475	99.7238	99.4475	98.0663	92.8177	73.2044
0.2	2	57.7310	84.5151	94.6942	98.1289	98.9263	98.1289	94.6942	84.5151	57.7310
0.3	3	48.0988	76.9331	90.6036	96.0927	97.5095	96.0927	90.6036	76.9331	48.0988
0.4	4	41.6630	70.4545	86.2889	93.4940	95.5918	93.4940	86.2889	70.4545	41.6630
0.5	5	37.0823	65.0030	82.0209	90.5028	93.0023	90.5028	82.0209	65.0030	37.0823
0.6	6	33.6393	60.3926	77.9252	87.2663	90.1343	87.2663	77.9252	60.3926	33.6393
0.7	7	30.9298	56.4406	74.0476	83.8992	87.0168	83.8992	74.0476	56.4406	30.9298
0.8	8	28.7143	52.9976	70.3947	80.4862	83.7515	80.4862	70.3947	52.9976	28.7143
0.9	9	26.8444	49.9490	66.9563	77.0870	80.4192	77.0870	66.9563	49.9490	26.8444

Table 4. Comparison of the Implicit scheme with the exact solutions.

t	Implicit Scheme	Exact solutions	Percentage error
0.025	99.7238	99.99	0.27
0.03	98.9263	99.53	0.61
0.035	97.5095	97.85	0.35
0.04	95.5018	95.18	-0.34

Example 2.

Solve the following heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}, [0,1] \tag{35}$$

subject to the initial condition

$$f(x, t) = \sin \pi x, [0,1] \tag{36}$$

and boundary conditions

$$f(0, t) = 0 = f(1, t), t > 0 \quad (37)$$

Using the explicit, Dufort-Frankel and the implicit schemes. Carry out the computation for two levels taking $h = \frac{1}{3}$ and $r = \frac{1}{4}$

Solution:

from the problem above, we have that $\frac{1}{4} = r$, then we have that $f_{1,0} = \frac{\sqrt{3}}{2}$ and $f_{2,0} = \frac{\sqrt{3}}{2}$, on solving using equation (6), at $i = 1, j = 0$ we have that

$$f_{1,1} = \frac{1}{4}(f_{1,0} + f_{2,0}) = f_{1,1} = 0.65$$

and $f_{2,1} = 0.65$, similarly, at $i = 1, j = 1$, we have $f_{1,2} = 0.49$ and $f_{2,2} = 0.49$ also, using Dufort-Frankel scheme, we have that at $i = 1, j = 0, f_{1,1} = f_{2,1} = 0.65$ and at $i = 1, j = 1$ we have

$$f_{1,2} = \frac{1}{3}\left(\frac{\sqrt{3}}{2} + 0.65\right) = 0.5$$

also, $f_{2,2} = 0.5$, lastly, using implicit scheme, we have the following equations to solve

$$\frac{3}{2}f_{1,1} - \frac{1}{4}f_{2,1} = \frac{\sqrt{3}}{2}$$

and

$$\frac{-1}{4}f_{1,1} + \frac{3}{2}f_{2,1} - \frac{1}{4}f_{3,1} = \frac{\sqrt{3}}{2}$$

4. Discussion

Table 1 presents the results of both explicit and Dufort-Frankel scheme, from our calculation at $r = \frac{1}{2}$ in the explicit scheme, it results into Bender-Schmidt scheme, also, the Dufort-Frankel scheme also work like the explicit scheme at $r = \frac{1}{2}$, hence the same result is presented for the two scheme at $r = \frac{1}{2}$. Table 2 shows the comparison of the numerical solutions of the explicit schemes and the exact solutions, the percentage errors are also presented. Tables 3 and 4, shows the results of the implicit scheme and the comparison of its numerical solutions with the exact solutions. From tables 2 and 4, it can be observed that the implicit scheme is good compare to the other two schemes. Also, the implementation of the implicit scheme is faster than that of the explicit and Dufort-Frankel schemes.

5. Conclusion

From the results, it is observed clearly that the implicit scheme is efficient and fast in implementation than the other two schemes. Also, the implicit scheme and Dufort-Frankel schemes as seen using von Neumann stability method are unconditionally stable, while the explicit scheme is conditionally stable. This can be seen in example 2, where both schemes perform better than the explicit scheme and also the implicit scheme performs better than the Dufort-Frankel

scheme. The results of the methods agree with existing findings in literature, see Omowo B. J and Abhulimen C. E [3], Olusegun O. A, Hoe Y. S, Ogunbode E. B [9] and Adak M. [14] that the implicit scheme has no restriction for the value of its mesh ratio and that smaller time steps produces more accurate results.

Competing Interests

Authors have declared that no competing interests exist.

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